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Differential equations for modular forms of level three

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1 Introduction

It is known that elliptic modular forms satisfy nonlinear differential equations of third order. Historically, it was Jacobi who first studied a differential equation whose solutions are written in terms of theta constants. He showed that the equation

$$\left(y^2y''' - 15yy'y'' + 30y'^3\right)^2 + 32\left(yy'' - 3y'^2\right)^3 = -\pi^2 y^{10}\left(yy'' - 3y'^2\right)^2,$$

has solutions $y = \theta_2, \theta_3, \theta_4$ ([7]). The study of Jacobi's equationis is complicated, but Halphen found a simple system of differential equations ([5])

$$\begin{cases} X' + Y' = 2XY, \\ Y' + Z' = 2YZ, \\ Z' + X' = 2ZX, \end{cases}$$

whose solutions are written in terms of logarithmic derivatives of theta constants ([5], [8]). Although Halphen's equations are not integrable in the classical sence (there exists no algebraic Hamiltonian), Halphen's equations can be solved exactly. Halphen's equations may be considered as fundamental equations for modular forms of level two. J. Chazy also found a differential equation for modular forms of level one ([2]):

$$y''' = 2yy'' - 3(y')^2. \tag{1}$$

The aim of this paper is to deduce Halphen type equations for modular forms of level three:

$$\begin{cases} W' + X' + Y' = WX + XY + YW, \\ W' + Y' + Z' = WY + YZ + ZW, \\ W' + X' + Z' = WX + XZ + ZW, \\ X' + Y' + Z' = XY + YZ + ZX, \\ e^{\frac{4}{3}\pi i}(XZ + YW) + e^{\frac{2}{3}\pi i}(XW + YZ) + (XY + ZW) = 0. \end{cases} \tag{2}$$

A special solution of (2) is given by

$$\begin{aligned} W &= 3\frac{\partial}{\partial\tau}\log\eta\left(\frac{\tau}{3}\right) - \frac{\partial}{\partial\tau}\log\eta(\tau), \\ X &= 3\frac{\partial}{\partial\tau}\log\eta\left(3\tau\right) - \frac{\partial}{\partial\tau}\log\eta(\tau), \\ Y &= 3\frac{\partial}{\partial\tau}\log\eta\left(\frac{\tau+2}{3}\right) - \frac{\partial}{\partial\tau}\log\eta(\tau), \\ Z &= 3\frac{\partial}{\partial\tau}\log\eta\left(\frac{\tau+1}{3}\right) - \frac{\partial}{\partial\tau}\log\eta(\tau). \end{aligned} \tag{3}$$

The system (2) is invariant under the following $SL(2,\mathbb{C})$-action.

$$\widetilde{X_j}(\tau) = \frac{1}{(c\tau+d)^2}X_j\left(\frac{a\tau+b}{c\tau+d}\right) - \frac{c}{c\tau+d},$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$. Since (2) is equivalent to an equation of third order, generic solutions are given by the $SL(2,\mathbb{C})$-orbit of (3). In section 4, we will study the whole solution space of (2). The solution space of (2) decomposes into one generic orbit and several special orbits. Any solution in special orbits is a rational function.

In section 2, we will study the Hesse pencil, following [1] and [3]. Since the Hesse pencil is an elliptic modular surface for the modular group of level three, the period of the Hesse pencil is a modular form of level three. In section 3, we will deduce Halphen type equations of level three from the Picard-Fuchs equation of the Hesse pencil. A method for constructing Halphen type equations was found by Jacobi ([7]) and is generalized in [9]. Recently this method has been applied to replicable functions by Harnad and McKay ([6]).

Halphen's equation is a specialization of self-dual Einstein equation ([4]) and other Halphen type equations also have many applications in mathematical physics ([9]). It would be interesting to find an application of (2) in mathematical physics. It seems that nonlinear differential equations defining modular forms are closely related to self-dual equations.

Chazy considered the equation (1) in his study on classification of Painlevé type equations of third order ([2]). Although there is no singularity in (1), solutions of (1) have natural boundaries. Other Halphen type equations also do not have the Painlevé property. Halphen type equations may be significant examples for considering the meaning of the Painlevé property.

2 Theta functions and the Hesse Pencil

In this section we will review the classical work ([1], [3]) on the Hesse pencil

$$x_0^3 + x_1^3 + x_2^3 - 3ax_0x_1x_2 = 0. \tag{4}$$

We take Jacobi's elliptic theta function

$$\theta_1(z) = \sum_{n=-\infty}^{\infty} e^{\pi i(n-1/2)^2\tau + 2\pi i(n-1/2)(z-1/2)},$$

for $\Im\tau > 0$. The Hesse pencil (4) is uniformized by

$$x_j(z) = (-\omega)^j\,\theta_1\left(z - \frac{j}{3}\right)\theta_1\left(z - \frac{j+\tau}{3}\right)\theta_1\left(z - \frac{j+2\tau}{3}\right),$$

where $\omega = e^{\frac{2}{3}\pi i}$.

Proposition 2.1. *(1) Consider the complex torus*

$$E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau).$$

The image of the embedding

$$\begin{aligned} \varphi : E_\tau &\longrightarrow \mathbb{P}^2(\mathbb{C}) \\ \varphi(z) &= (x_0(z) : x_1(z) : x_2(z)) \end{aligned}$$

is the Hesse pencil (4).

(2) The modulus a satisfies

$$a - 1 = 9\frac{\eta(3\tau)^3}{\eta(\frac{\tau}{3})^3}, \tag{5}$$

$$a - \omega = \sqrt{3}e^{-\pi i/3}\frac{\eta(\frac{\tau+2}{3})^3}{\eta(\frac{\tau}{3})^3}, \tag{6}$$

$$a - \omega^2 = \sqrt{3}e^{\pi i/12}\frac{\eta(\frac{\tau+1}{3})^3}{\eta(\frac{\tau}{3})^3}. \tag{7}$$

Here $\eta(\tau)$ is the Dedekind η-function

$$\eta(\tau) = q^{1/12}\prod_{n=1}^{\infty}\left(1 - q^{2n}\right), \qquad q = e^{\pi i\tau}.$$

We take inhomogeneous coordinates $x = x_0/x_2$ and $y = x_1/x_2$. The elliptic integrals

$$\kappa = \int_0^{-1} \frac{dx}{y^2 - ax},$$
$$\kappa' = \int_0^{-1} \frac{dx}{y^2 - a\omega x},$$

give periods of the Hesse pencil. We have

$$\kappa = -\frac{2\pi}{3\sqrt{3}} \frac{\eta\left(\frac{\tau}{3}\right)^3}{\eta(\tau)}. \tag{8}$$

The analytic parameter τ is

$$\tau = \frac{\omega\kappa' - \kappa}{\kappa}.$$

κ and κ' satisfy the Picard-Fuchs equation

$$(1-a^3)\frac{d^2\kappa}{da^2} - 3a^2\frac{d\kappa}{da} - a\kappa = 0. \tag{9}$$

By $t = a^3$, (9) is changed into

$$t(1-t)\frac{d^2\kappa}{dt^2} + \left(\frac{2}{3} - \frac{5}{3}t\right)\frac{d\kappa}{dt} - \frac{1}{9}\kappa = 0.$$

Therefore the solutions of (9) near $a = 1$ are given by

$${}_2F_1(\frac{1}{3}, \frac{1}{3}, 1; 1-t), \quad \sum_{n=0}^{\infty} \left\{ \frac{\Gamma\left(n+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right) n!} \right\}^2 \left(2\psi(n+1) - 2\psi(n+\frac{1}{3}) - \log(1-t)\right)(1-t)^n,$$

where $\psi(x) = 2\frac{d}{dx}\log\Gamma(x)$.

Since

$$\kappa(1) = \int_0^{-1} \frac{dx}{y^2 - x} = -\frac{2\pi}{3\sqrt{3}},$$

we have

$$\kappa(a) = -\frac{2\pi}{3\sqrt{3}} {}_2F_1\left(\frac{1}{3}, \frac{1}{3}, 1; 1-a^3\right),$$

near $a = 1$.

3 The Picard-Fuchs equation

In the following, we will deduce (2) by projecting the Picard-Fuchs equation to the period domain of the Hesse pencil. The independent variable of the Picard-Fuchs equation is an algebraic parameter a . We will obtain a nonlinear differential equation whose independent variable is an analytic parameter τ. This method was found by Jacobi ([7], [9]).

At first we will show the logarithmic derivatives of elliptic periods satisfy (2). We set $\delta = (a^3 - 1)\frac{d}{da}$. Then (9) is

$$\delta^2 \kappa + a(a^3 - 1)\kappa = 0.$$

We will change the independent variable a into $\tau = \dfrac{\omega\kappa' - \kappa}{\kappa}$.

Since κ and κ' are solutions of (9),

$$\delta(\kappa\delta(\kappa') - \kappa'\delta(\kappa)) = 0.$$

Therefore

$$\delta(\tau) = \omega\frac{\kappa\delta(\kappa') - \kappa'\delta(\kappa)}{\kappa^2} = \frac{c}{\kappa^2},$$

for a constant c.

Hence we have

$$\frac{da}{d\tau} = \frac{(a^3 - 1)\kappa^2}{c}.$$

Changing the independent variable a for τ, (9) turns into

$$\frac{c}{\kappa^2}\frac{d}{d\tau}\left(\frac{c}{\kappa^2}\frac{d}{d\tau}\kappa\right) + a(a^3 - 1)\kappa = 0. \tag{10}$$

We set

$$A = \kappa, \quad B = (a - 1)\kappa, \quad C = (a - \omega)\kappa, \quad D = (a - \omega^2)\kappa,$$

and

$$W = \frac{A_\tau}{A}, \quad X = \frac{B_\tau}{B}, \quad Y = \frac{C_\tau}{C}, \quad Z = \frac{D_\tau}{D},$$

where $A_\tau = \dfrac{dA}{d\tau}$. ¿From (10), we have

$$\left(\frac{A_\tau}{A}\right)_\tau = \frac{A_\tau}{A} - \frac{1}{3c^2}(B + C + D)BCD. \tag{11}$$

We have

$$\begin{aligned} X - W &= \frac{A_\tau}{A} - \frac{B_\tau}{B} = \frac{a_\tau}{a-1} \\ &= \frac{1}{c}(a-\omega)(a-\omega^2)\kappa^2 \\ &= \frac{1}{c}CD. \end{aligned} \tag{12}$$

In the same way,

$$Y - W = \frac{1}{c}DB, \tag{13}$$

$$Z - W = \frac{1}{c}BC. \tag{14}$$

We can eliminate A, B, C and D from (11).

$$\begin{aligned} W_\tau &= W^2 - \frac{1}{3}\left((X-W)(Y-W) + (Y-W)(Z-W) + (Z-W)(X-W)\right) \\ &= \frac{1}{3}\left(-XY - YZ - ZX + 2XW + 2YW + 2ZW\right). \end{aligned} \tag{15}$$

¿From (12), we have

$$\begin{aligned} X_\tau &= W_\tau + \frac{1}{c}\left(CD\right)_\tau \\ &= \frac{1}{3}\left(2XY - YZ + 2ZX + 2XW - YW - ZW\right). \end{aligned} \tag{16}$$

In the same way, we have

$$Y_\tau = \frac{1}{3}\left(2XY + 2YZ - ZX - XW + 2YW - ZW\right), \tag{17}$$

$$Z_\tau = \frac{1}{3}\left(-XY + 2YZ + 2ZX - XW - YW + 2ZW\right). \tag{18}$$

The equations (15), (16), (17) and (18) are equivalent to the system

$$\begin{cases} W_\tau + X_\tau + Y_\tau = WX + XY + YW, \\ W_\tau + Y_\tau + Z_\tau = WY + YZ + ZW, \\ W_\tau + X_\tau + Z_\tau = WX + XZ + ZW, \\ X_\tau + Y_\tau + Z_\tau = XY + YZ + ZX. \end{cases}$$

Since W, X, Y and Z are functions of κ, κ_τ and a, there should be an algebraic relation between them. By (12) and (14) , we have

$$\begin{aligned} X - Z &= \frac{1}{c}\left(D - B\right)C \\ &= \frac{\kappa}{c}(1-\omega^2)C. \end{aligned}$$

Therefore

$$(X-Z)(Y-W)=\frac{\kappa}{c^2}(1-\omega^2)BCD.$$

In the same way

$$(X-Y)(Z-W)=\frac{\kappa}{c^2}(1-\omega)BCD.$$

Therefore we have

$$\begin{aligned} F(W,X,Y,Z): &= (X-Z)(Y-W)+\omega^2(X-Y)(Z-W) \\ &= \omega^2(XZ+YW)+\omega(XW+YZ)+(XY+ZW)=0. \end{aligned}$$

$F(W,X,Y,Z)$ is a relative invariant under even permutations of W,X,Y and Z.

By direct calculation

$$\frac{d}{d\tau}F(W,X,Y,Z)=3(W+X+Y+Z)F(W,X,Y,Z).$$

We obtain the main theorem:

Theorem 3.1. *The system of nonlinear differential equations* (2) *has a special solution* (3).

Proof. By (5), (6), (7) and (8), we have

$$\begin{aligned} A &= \frac{2\pi}{3\sqrt{3}}e^{\frac{5}{4}\pi i}\frac{\eta\left(\frac{\tau}{3}\right)^3}{\eta(\tau)}, \\ B &= 2\sqrt{3}\pi e^{\frac{5}{4}\pi i}\frac{\eta\left(3\tau\right)^3}{\eta(\tau)}, \\ C &= \frac{2\pi}{3}e^{\frac{11}{12}\pi i}\frac{\eta\left(\frac{\tau+2}{3}\right)^3}{\eta(\tau)}, \\ D &= \frac{2\pi}{3}e^{\frac{4}{3}\pi i}\frac{\eta\left(\frac{\tau+1}{3}\right)^3}{\eta(\tau)}. \end{aligned}$$

Taking logarithmic derivatives of the above functions, we obtain Theorem 3.1. □

¿From now on, we will study the solution space of (2). (2) is essentially a system of differential equations of third order. We will show that (2) has solutions with three parameters, which give generic solutions of (2).

For any function $f(\tau)$, we define an action of $A = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2,\mathbb{C})$ on $f(\tau)$:

$$(\rho(A) \cdot f)(\tau) = \frac{1}{(r\tau+s)^2} f(A \cdot \tau) - \frac{r}{r\tau+s}. \tag{19}$$

Here

$$A \cdot \tau = \frac{p\tau+q}{r\tau+s}.$$

The action ρ is a right action:

$$\rho(AB) \cdot f = \rho(B) \cdot (\rho(A) \cdot f).$$

Proposition 3.2. *If $W(\tau), X(\tau), Y(\tau)$ and $Z(\tau)$ satisfy (2), then $\rho(A) \cdot W, \rho(A) \cdot X$, $\rho(A) \cdot Y$ and $\rho(A) \cdot Z$ also satisfy (2) for any $A = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2,\mathbb{C})$.*

Proof. We have

$$\frac{d}{d\tau}(\rho(A) \cdot f) = \frac{1}{(r\tau+s)^4} \frac{df}{d\tau}\left(\frac{p\tau+q}{r\tau+s}\right) - \frac{2r}{(r\tau+s)^3} f(A \cdot \tau) + \frac{r^2}{(r\tau+s)^2}.$$

Using the equation above, we can easily prove this proposition. □

4 The Space of initial values

We will determine the solution space of the equations (2). Generic solutions of (2) are given by Proposition 3.2. The group

$$\Gamma(3) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z});\ p \equiv 1, q \equiv 0, r \equiv 0, s \equiv 1 \pmod 3 \right\}$$

is called the modular group of level three.

By the transformation law of the η-function, the special solution (3) is invariant under the action ρ for any $A \in \Gamma(3)$. Therefore effective actions on (3) are

$$\Gamma(3) \backslash SL(2,\mathbb{C}).$$

Moreover the action $\Gamma(3) \backslash SL(2,\mathbb{Z}) \simeq A_4$ induces even permutations of W_0, X_0, Y_0 and Z_0.

We will study the solution space of the equations (2) using the solution space of Chazy's equation ([8]). At first, we will show that (2) is a Galois extension of Chazy's equation. Chazy's equation (1) is equivalent to

$$\begin{cases} \dfrac{d}{d\tau}h_1 = h_2, \\ \dfrac{d}{d\tau}h_2 = 6h_3, \\ \dfrac{d}{d\tau}h_3 = 4h_1h_3 - h_2^2, \end{cases} \tag{20}$$

where $h_1 = 2y$. (20) has a special solution

$$h_1 = 6\frac{d}{d\tau}\log\eta(\tau),$$

and has the same $SL(2,\mathbb{C})$-symmetry as (2) ([8]).

We take the elementary symmetric polynomials of W, X, Y, Z:

$$\begin{aligned} P_1 &= W + X + Y + Z, \\ P_2 &= WX + WY + WZ + XY + YZ + XZ, \\ P_3 &= WXY + WXZ + WYZ + XYZ, \\ P_4 &= WXYZ. \end{aligned}$$

Let $\bar{F}$ be the complex conjugate of F:

$$\bar{F}(W,X,Y,Z) = \omega(XZ + YW) + \omega^2(XW + YZ) + (XY + ZW).$$

Since

$$F\bar{F} = 12P_4 + P_2^2 - 3P_1P_3,$$

we have

$$P_4 = \frac{3P_1P_3 - P_2^2}{12}.$$

By (2), P_1, P_2 and P_3 satisfy the following equations

$$\begin{cases} \dfrac{d}{d\tau}P_1 = \dfrac{2}{3}P_2, \\ \dfrac{d}{d\tau}P_2 = 3P_3, \\ \dfrac{d}{d\tau}P_3 = 4P_4 + 2P_1P_3 - \dfrac{2}{3}P_2^2 = 3P_1P_3 - P_2^2. \end{cases} \tag{21}$$

If we set

$$h_1 = \frac{3}{4}P_1, \qquad h_2 = \frac{1}{2}P_2, \qquad h_3 = \frac{1}{4}P_3,$$

(20) is changed to (21). Hence (21) is equivalent to Chazy's equation.

The Galois group of the algebraic equation

$$\lambda^4 - P_1\lambda^3 + P_2\lambda^2 - P_3\lambda + \left(\frac{3P_1P_3 - P_2^2}{12}\right) = 0 \tag{22}$$

is the alternating group A_4. $A_4 \simeq \Gamma(3)\backslash SL(2,\mathbb{Z})$ and the action of $SL(2,\mathbb{Z})$ defined by (19) induce the Galois action on (22).

We will solve (2) for distinct initial values, which gives the generic part of the solution space.

Theorem 4.1. *Assume that holomorphic functions W, X, Y and Z satisfy (2) near $t \in \mathbb{C}$ and have initial values*

$$W(t) = k_0, \quad X(t) = k_1, \quad Y(t) = k_2, \quad Z(t) = k_3,$$

where the complex numbers k_0, k_1, k_2, k_3 are distinct from each other and $F(k_0, k_1, k_2, k_3) = 0$. Then there exists a matrix $A = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2,\mathbb{C})$ such that

$$W = \rho(A)\cdot W_0, \quad X = \rho(A)\cdot X_0, \quad W = \rho(A)\cdot Y_0, \quad W = \rho(A)\cdot Z_0.$$

Proof. At first, we will reduce (2) to Chazy's equation. We consider the initial value problem of (20), whose initial value is

$$h_1(t) = \frac{3}{4}P_1(t), \quad h_2(t) = \frac{1}{2}P_2(t), \quad h_3(t) = \frac{1}{4}P_3(t).$$

By the proposition 4.1 in [8], if the discriminant of the equation

$$\lambda^3 - h_1(t)\lambda^2 + h_2(t)\lambda - h_3(t) = 0 \tag{23}$$

is not zero, the solution of Chazy's equation is written in terms of the η-function. By direct calculation, the discriminant of (23) is

$$-\frac{1}{64}\left(9P_1(t)^2P_2(t)^2 - 32P_2(t)^3 - 27P_1(t)^3P_3(t) + 108P_1(t)P_2(t)P_3(t) - 108P_3(t)^2\right),$$

which is the same as the discriminant of (22) up to a constant factor. Since the k_j's are distinct, the discriminant of (22) is not zero.

Hence there exists a matrix $A = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2,\mathbb{C})$ such that

$$P_1(\tau) = \frac{8}{(r\tau+s)^2}h\left(A\cdot\tau\right) - \frac{4r}{r\tau+s},$$

where

$$h(\tau) = \frac{d}{d\tau} \log \eta(\tau).$$

We can also write P_2 and P_3 in terms of the η-function by (21):

$$P_2(\tau) = \frac{12}{(r\tau + s)^4} h'(A \cdot \tau) - \frac{24r}{(r\tau + s)^3} h(A \cdot \tau) + \frac{6r^2}{(r\tau + s)^2},$$
$$P_3(\tau) = \frac{4}{(r\tau + s)^6} h''(A \cdot \tau) - \frac{24r}{(r\tau + s)^5} h'(A \cdot \tau) + \frac{24r^2}{(r\tau + s)^4} h(A \cdot \tau) - \frac{4r^3}{(r\tau + s)^3}.$$

We will find solutions of (22), which give solutions of (2). At first we will consider the case $p = 1, q = 0, r = 0, s = 1$. Since

$$W_0 + X_0 + Y_0 + Z_0 = 8h(\tau)$$

by (13), W_0, X_0, Y_0, Z_0 are the solutions of (22) in the case $p = 1, q = 0, r = 0, s = 1$. We will set

$$P_1^{(0)} = 8h(\tau),$$
$$P_2^{(0)} = \frac{12}{(r\tau + s)^4} h'(\tau),$$
$$P_3^{(0)} = \frac{4}{(r\tau + s)^6} h''(\tau),$$

and

$$\alpha = \frac{1}{(r\tau + s)^2},$$
$$\beta = \frac{r}{r\tau + s}.$$

Then

$$P_1 = \alpha P_1^{(0)}(A \cdot \tau) - 4\beta,$$
$$P_2 = \alpha^2 P_2^{(0)}(A \cdot \tau) - 3\alpha\beta P_1^{(0)}(A \cdot \tau) + 6\beta^2,$$
$$P_3 = \alpha^3 P_3^{(0)}(A \cdot \tau) - 2\alpha^2\beta P_2^{(0)}(A \cdot \tau) + 3\alpha\beta^2 P_1^{(0)}(A \cdot \tau) - 4\beta^3.$$

By the relation between solutions and coefficients, the solution of (22) can be written in terms of W_0, X_0, Y_0 and Z_0:

$$\alpha W_0(A \cdot \tau) - \beta = \rho(A) \cdot W_0(\tau),$$
$$\alpha X_0(A \cdot \tau) - \beta = \rho(A) \cdot X_0(\tau),$$
$$\alpha Y_0(A \cdot \tau) - \beta = \rho(A) \cdot Y_0(\tau),$$
$$\alpha Z_0(A \cdot \tau) - \beta = \rho(A) \cdot Z_0(\tau).$$

As a set, these four functions have the initial value $\{k_0, k_1, k_2, k_3\}$ at $\tau = t$. Therefore we should take a suitable Galois action $g \in A_4$ in order that

$$\begin{aligned} k_0 &= \rho(A) \cdot W_0(t), \\ k_1 &= \rho(A) \cdot X_0(t), \\ k_2 &= \rho(A) \cdot Y_0(t), \\ k_3 &= \rho(A) \cdot Z_0(t). \end{aligned}$$

Since the Galois action is deduced from the action ρ by the isomorphism

$$A_4 \simeq \Gamma(3)\backslash SL(2, \mathbb{Z}),$$

we can make the initial values coincide with the corresponding functions by changing A into gA for $g \in \Gamma(3)\backslash SL(2, \mathbb{Z})$. □

If $X = Y$, then $Y = Z$ or $X = W$ because $F = (Y - Z)(X - W) - \omega(X - Y)(Z - W)$. In general, if any two of W, X, Y, Z coincide, some three of W, X, Y, Z coincide. When $X = Y = Z$, (2) is reduced to the following equations:

$$\begin{cases} X_\tau = X^2, \\ W_\tau + 2X_\tau = 2WX + X^2. \end{cases}$$

Hence

$$\begin{aligned} X &= -\frac{p}{p\tau + q}, \\ W &= -\frac{p}{p\tau + q} + \frac{r}{(p\tau + q)^2}. \end{aligned}$$

We may consider $(p : q) \in \mathbb{P}^1(\mathbb{C})$ and $r \in O(2)$.

Thus we can solve the initial value problem of (2) for all case. This gives the whole space of meromorphic solutions of (2). Summing up our result, we obtain the following theorem:

Theorem 4.2. *The generic part of the solution space of* (2) *is isomorphic to* $\Gamma(3)\backslash SL(2, \mathbb{C})$, *and the degenerate part is a union of four line bundles* $O(2)$, *which intersect along the zero section. This intersection corresponds to the case* $k_0 = k_1 = k_2 = k_3$.

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