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Almost Global Existence of Solutions for the Quadratic Semilinear Klein-Gordon Equation in One Space Dimension

By

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Dedicated to Professor Hikosaburo Komatsu on his sixtieth birthday

1. Introduction

In the paper we consider the life span of solutions of the Cauchy problem for the nonlinear Klein-Gordon equation:

$$(1.1) \qquad (\partial_t^2 - \Delta + 1)u = F(u, u', u''), \qquad t > 0, \ x \in \boldsymbol{R}^n$$

$$(1.2) \qquad u(0,x) = \varepsilon u_0(x), \quad \partial_t u(0,x) = \varepsilon u_1(x), \qquad x \in \boldsymbol{R}^n.$$

Here, ε is a small positive parameter and F is a C^∞ function of u and its first and second partial derivatives satisfying the following properties: For some $p \in \boldsymbol{N}$ with $p \geq 2$,

$$(1.3) \qquad F(u, u', u'') = O(|u|^p + |u'|^p + |u''|^p) \text{ near } (u, u', u'') = (0, 0, 0)$$

and F depends linearly on u'', where we denote by u' the first derivatives of u and by u'' the second derivatives of u except $\partial^2 u/\partial t^2$. From this property (1.3) it follows that the equation (1.1) has the trivial solution $u \equiv 0$. We are interested in studying solutions of the equation (1.1) near the trivial solution.

There are many results concerning the global existence and estimates of existence time T_ε of the solution for (1.1)–(1.2). Klainerman [10] and Shatah [18] showed, using totally different techniques, that when $n \geq 3$ and $p \geq 2$, $T_\varepsilon = \infty$ for small ε, that is, the solution exists globally in time for small ε. In order to prove the global existence of the solution for (1.1)–(1.2), in [10], Klainerman used the invariant Sobolev space with respect to the generators of the Lorentz group. On the other hand, in [18], Shatah extended Poincaré's theory of normal forms for the ordinary differential equations to the case of nonlinear Klein-Gordon equations (earlier in [20] Simon used this kind of technique for the construction of the wave operator, but he did not consider the Cauchy problem with the initial data given at $t = 0$). Hörmander [8] refined Klainerman's technique to obtain new time decay estimates of solutions for inhomogeneous linear Klein-Gordon equations (see also Bachelot [1], Sideris [19] and Georgiev

[6]) and showed that when $n=2$ and $p=2$, $\lim_{\varepsilon\to 0}\varepsilon\log T_\varepsilon=\infty$ and that when $n=1$, and $p=2$, $\lim_{\varepsilon\to 0}\varepsilon^2 T_\varepsilon=\infty$. In [8], in addition, he presented two conjectures that when $n=2$ and $p=2$, $T_\varepsilon=\infty$ for small ε and that when $n=1$ and $p=2$, $\liminf_{\varepsilon\to 0}\varepsilon^2\log T_\varepsilon>0$. Recently his conjecture for $n=2$ was solved by Ozawa, Tsutaya, and Tsutsumi [15] and [16] (for partial results on the Hörmander conjecture of $n=2$, see [7], [12], and [21]). They used the normal form technique of Shatah [18] and the decay estimates due to Georgiev [6] in order to prove that $T_\varepsilon=\infty$ for small ε when $n=2$ and $p=2$. However, when $n=1$, much is not yet known about the global existence or the estimate of the life span of the solution for (1.1)–(1.2). When $n=1$, it is known that there exist global solutions for small ε if $p\geq 4$ (see [11] and [17]) or if $p=3$ and F is written as a linear combination of some special cubic polynomials of u, u' and u'' (see [9] and [14]).

We study here the lower estimate of the life span of the solution for (1.1)–(1.2) with quadratic nonlinearlity in one space dimension when the nonlinearity F does not depend on u'', that is, when (1.1) is a semilinear equation. Before we state our main result in the present paper, we give several notations.

Notations. We put

$$\partial_t=\partial/\partial_t,\qquad \partial_x=\partial/\partial_x,\qquad L=x\partial_t+t\partial_x,\qquad \partial=(\partial_t,\partial_x).$$

Let $\Gamma=(\Gamma_j)_{j=1}^3$ denote the generators of the Poincaré group $(\partial_t,\partial_x,L)$. For a multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, we put

$$\Gamma^\alpha=\Gamma_1^{\alpha_1}\Gamma_2^{\alpha_2}\Gamma_3^{\alpha_3}.$$

For $1\leq p\leq\infty$, let L^p denote the standard L^p space on $\boldsymbol{R}$. We define the weighted Sobolev space $H^{m,s}$ on $\boldsymbol{R}$ as follows:

$$H^{m,s}=\{v\in L^2;(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}v\in L^2\}$$

with the norm

$$\|v\|_{H^{m,s}}=\|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}v\|_{L^2}.$$

We put $H^m\equiv H^{m,0}$. Let $\omega=(1-\partial_x^2)^{1/2}$ and $\langle p\rangle=(1+p^2)^{1/2}$ for $p\in\boldsymbol{R}$. For $f\in\mathscr{S}(\boldsymbol{R})$ and $K\in\mathscr{S}'(\boldsymbol{R}\times\boldsymbol{R})$, we define their Fourier transforms by

$$\hat{f}(p)=\int e^{-ixp}f(x)dx,$$

$$\hat{K}(p,q)=\iint e^{-i(yp+zq)}K(y,z)dydz.$$

For $k \in N$, we define the norms by

$$\|u(t)\| = \|u(t,\cdot)\|_{L^2},$$
$$|u(t)|_\infty = \|u(t,\cdot)\|_{L^\infty},$$
$$\|u(t)\|_k = \sum_{|\alpha|\le k} \|\Gamma^\alpha u(t)\|,$$
$$\|u(t)\|_{k,\omega^{-1}} = \sum_{|\alpha|\le k} \|\omega^{-1}\Gamma^\alpha u(t)\|,$$
$$|u(t)|_k = \sum_{|\alpha|\le k} |\Gamma^\alpha u(t)|_\infty,$$
$$\|u\|_{k,T} = \sup_{t\in[0,T]} \|u(t)\|_k,$$
$$|u|_{k,T} = \sup_{t\in[0,T],x\in\boldsymbol{R}} (1+t+|x|)^{1/2} \sum_{|\alpha|\le k} |\Gamma^\alpha u(t,x)|,$$
$$|||u|||_{k,T} = |u|_{k,T} + \|u\|_{k+6,T} + \|\partial u\|_{k+6,T}.$$

We give an elementary remark that $(\partial_t, \partial_x, L)$ generates a Lie algebra with following commutation relations:

$$(1.4) \qquad [L,\partial_t] = -\partial_x, \qquad [L,\partial_x] = -\partial_t, \qquad [\partial_t,\partial_x] = 0.$$

We denote by C constants in the estimates, which may change from line to line.

In the present paper, we consider the special case that F does not depend on the second derivatives of u. Now we state the main theorem concerning the lower estimate of life span of the solution for the Cauchy problem (1.1)–(1.2) with F independent of u'' under (1.3) with $p = 2$, which implies to almost global existence of the solution for the one dimensional semilinear case with quadratic nonlinearity.

Main Theorem. *Assume that F depends only on u, $\partial_t u$ and $\partial_x u$ and satisfies*

$$F(u,\partial_t u,\partial_x u) = O(|u|^2 + |\partial_t u|^2 + |\partial_x u|^2) \ \textit{near}\ (u,\partial_t u,\partial_x u) = (0,0,0).$$

Let $k \ge 11$, $u_0 \in H^{k+7,k+6}$, $u_1 \in H^{k+6,k+6}$. If we set, for the local solution u of (1.1)–(1.2),

$$T_\varepsilon = \sup\{T > 0;\ |||u|||_{k,T} < \infty\},$$

then there exist three positive constants ε_0, A and B such that

$$(1.5) \qquad T_\varepsilon > A\exp(B\varepsilon^{-2}) \qquad \textit{for}\ 0 < \varepsilon \le \varepsilon_0.$$

Here, constants ε_0, A and B depend only on k, $\|u_0\|_{H^{k+7,k+6}}$ and $\|u_1\|_{H^{k+6,k+6}}$.

Remarks. (i) In fact, in the Main Theorem, the initial values $u(0,x)$ and $\partial_t u(0,x)$ do not neccesarily tend to 0 at the same order as $\varepsilon \to 0$. That is, the following assumption is sufficient for the proof of the Main Theorem:

$$\|u(0,x)\|_{H^{k+7,k+6}} + \|\partial_t u(0,x)\|_{H^{k+6,k+6}} \le \varepsilon.$$

(ii) The Main Theorem affirmatively answers the Hörmander second conjecture for the semilinear case. But, our proof of the Main Theorem is not applicable to the quasilinear case, that is, the case that F includes the second derivatives of u.

(iii) Yordanov [22] has recently proved that if $F = (\partial_t u)^2 \partial_x u + au^3 + bu^2$ for $a, b \in \boldsymbol{R}$ and the initial data satisfy a certain positivity condition, the solution of (1.1)–(1.2) blows up in finite time and $T_\varepsilon \le A \exp(B\varepsilon^{-2})$ for some $A, B > 0$. Accordingly, the lower estimate (1.5) is in general optimal.

We give an outline of the proof of the Main theorem here. The crucial part of proof is to establish a priori estimates of the solution for (1.1)–(1.2), with which we extend the local solution for (1.1)–(1.2) to the desired time that we claim in the Main Theorem. The unique existence of local solutions for (1.1)–(1.2) follows from the standard contraction argument (see, e.g., [11] and [17]). The a priori estimates of solutions obtained directly from the original equation (1.1) are not useful enough to prove our result because the nonlinearity of (1.1) is quadratic. Here, we use the argument of normal forms due to Shatah [18] to transform the quadratic nonlinearity into the cubic one. Still, the cubic nonlinearity in the one dimensional case does not lead to sufficient decay estimates if we use the usual $L^p - L^q$ estimate. Instead of the usual $L^p - L^q$ estimate, we use the decay estimate for the inhomogeneous linear Klein-Gordon equation due to Georgiev [6] to derive the good decay estimate of solutions. We will also use the argument of normal forms in the present paper to establish not only the decay estimate but also the energy estimate, since the known energy estimates are not applicable to the proof of the Main Theorem. Therefore, our main task is to establish a precise estimate of the cubic nonlinearity appearing in the normal form argument. For this purpose, by using a result concerning the Fourier multiplier due to Coifman and Meyer (see Theorem 5 in [4]), we verify the regularity of the quadratic integral operators which appears in the argument of normal forms. The rest part of the proof of the Main Theorem proceeds almost in the usual way.

The plan to prove the Main Theorem in the present paper is as follows. In Section 2 we first get the proper transformation. Then we verify its regularity and establish some inequalities which we will need in the next section. In Section 3 we prove the Main Theorem.

In this paper we prove the above result only for the case that F is a quadratic form because we can similarly prove the same result when F is a function of u and its first derivatives and satisfies the condition (1.3) for $p = 2$.

2. Existence of the transformation and its regularity

In this section, following Shatah [18] (see also Simon [20]), we obtain the transformation which converts F to cubic nonlinearity. Then we need to show that the transformation is regular. The regularity is important to establish a priori decay estimates and energy estimates of the solution for (1.1)–(1.2).

First we obtain the transformation. Let the quadratic nonlinearity be written as

$$\begin{aligned}(2.1)\qquad F(u,\partial_t u,\partial_x u) =& a_1u^2+2a_2u\partial_x u+a_3(\partial_x u)^2\\ &+2b_1u\partial_t u+2b_2\partial_x u\partial_t u+c_1(\partial_t u)^2\end{aligned}$$

with real constants. Following Shatah [18], we introduce the new unknown function $v(t,x)$:

$$(2.2)\qquad v=u+[u,B_{11},u]+[u,B_{12},\partial_t u]+[\partial_t u,B_{21},u]+[\partial_t u,B_{22},\partial_t u],$$

where each B_{ij} is thought of as a distribution and the representation of the quadratic form is given by

$$(2.3)\qquad [f,B_{ij},g](x)=\iint f(y)B_{ij}(x-y,x-z)g(z)dydz.$$

After some calculations, we obtain

$$\begin{aligned}(2.4)\qquad \partial_x v =& \partial_x u+[u,B_{11},\partial_x u]+[\partial_x u,B_{11},u]+[u,B_{12},\partial_{tx}^2 u]+[\partial_x u,B_{12},\partial_t u]\\ &+[\partial_t u,B_{21},\partial_x u]+[\partial_{tx}^2 u,B_{21},u]+[\partial_t u,B_{22},\partial_{tx}^2 u]+[\partial_{tx}^2 u,B_{22},\partial_t u],\end{aligned}$$

$$\begin{aligned}(2.5)\qquad \partial_t v =& \partial_t u+[u,B_{11},\partial_t u]+[\partial_t u,B_{11},u]+[\partial_t u,B_{12},\partial_t u]\\ &+[u,B_{12},\partial_x^2 u-u]+[\partial_t u,B_{21},\partial_t u]+[\partial_x^2 u-u,B_{21},u]\\ &+[\partial_t u,B_{22},\partial_x^2 u-u]+[\partial_x^2 u-u,B_{22},\partial_t u]+[u,B_{12},F]\\ &+[F,B_{21},u]+[\partial_t u,B_{22},F]+[F,B_{22},\partial_t u],\end{aligned}$$

$$\begin{aligned}(2.6)\qquad (\partial_t^2-\partial_x^2+1)v =& F(u,\partial_t u,\partial_x u)\\ &+[u,-(2\partial_1\partial_2+1)B_{11}+2(\partial_1^2-1)(\partial_2^2-1)B_{22},u]\\ &+[u,-(2\partial_1\partial_2+1)B_{12}+2(\partial_1^2-1)B_{21},\partial_t u]\\ &+[\partial_t u,-(2\partial_1\partial_2+1)B_{21}+2(\partial_2^2-1)B_{12},u]\\ &+[\partial_t u,-(2\partial_1\partial_2+1)B_{22}+2B_{11},\partial_t u]+R,\end{aligned}$$

where we put $\partial_1 B_{ij}=\partial_y B_{ij}(y,z)$ and $\partial_2 B_{ij}=\partial_z B_{ij}(y,z)$ and $R=R(u,u',u'',\partial_t^2 u)$

consists of terms of degree at least 3 as follows:

$$R = [u, B_{11}, F] + [F, B_{11}, u] + [\partial_t u, B_{21}, F] + [F, B_{12}, \partial_t u] + [u, B_{12}, \partial_t F] + [\partial_t F, B_{21}, u] + [\partial_t u, B_{22}, \partial_t F] + [\partial_t F, B_{22}, \partial_t u] + 2[\partial_t u, B_{12}, F] + 2[F, B_{21}, \partial_t u] + 2[\partial_x^2 u - u, B_{22}, F] + 2[F, B_{22}, \partial_x^2 u - u] + 2[F, B_{22}, F]. \tag{2.7}$$

We choose the distributions B_{ij} so that all quadratic terms in (2.6) cancel out:

$$0 = F(u, \partial_t u, \partial_x u) + [u, -(2\partial_1\partial_2 + 1)B_{11} + 2(\partial_1^2 - 1)(\partial_2^2 - 1)B_{22}, u] + [u, -(2\partial_1\partial_2 + 1)B_{12} + 2(\partial_1^2 - 1)B_{21}, \partial_t u] + [\partial_t u, -(2\partial_1\partial_2 + 1)B_{21} + 2(\partial_2^2 - 1)B_{12}, u] + [\partial_t u, -(2\partial_1\partial_2 + 1)B_{22} + B_{11}, \partial_t u]. \tag{2.8}$$

Taking the Fourier transform of (2.8), we obtain by (2.1)

$$\hat{B}_{11}(p, q) = \frac{1}{\det}\{-2(a_3 + c_1)p^2q^2 + 2ia_2pq(p + q) + (2a_1 + a_3)pq - 2c_1(p^2 + q^2) - ia_2(p + q) - (a_1 + 2c_1)\}, \tag{2.9}$$

$$\hat{B}_{12}(p, q) = \frac{1}{\det}\{4ib_2p^2q + 2b_1p(p + q) - ib_2(p - 2q) + b_1\}, \tag{2.10}$$

$$\hat{B}_{21}(p, q) = \hat{B}_{12}(q, p), \tag{2.11}$$

$$\hat{B}_{22}(p, q) = \frac{1}{\det}\{2(a_3 + c_1)pq - 2ia_2(p + q) - (2a_1 + c_1)\}, \tag{2.12}$$

where $\det = 4(p^2 + pq + q^2) + 3$. Formally, the function v given by the transformation (2.2) with (2.9)–(2.12) satisfies the following equation with cubic nonlinearity:

$$(\partial_t^2 - \partial_x^2 + 1)v = R(u, u', u'', \partial_t^2 u). \tag{2.13}$$

We will next prove several lemmas and propositions concerning the regularity of the transformation that we have now derived above. The first two propositions are easy to prove (see [15]) and are given without proofs.

Proposition 2.1. *Let $K \in L^1(\boldsymbol{R} \times \boldsymbol{R})$. Then,*

$$\|[f, K, g]\|_{L^r} \leq \|K\|_{L^1(\boldsymbol{R}\times\boldsymbol{R})}\|f\|_{L^p}\|g\|_{L^q},$$

where $r^{-1} = p^{-1} + q^{-1}$ with $1 \leq p, q, r \leq \infty$.

Proposition 2.2. *For any multi-index α, we have*

$$\Gamma^\alpha[f,K,g] = \sum_{|\beta|+|\gamma|\le|\alpha|,0\le m\le|\beta|,0\le n\le|\gamma|} C^\alpha_{\beta,\gamma,m,n}[\Gamma^\beta f, y^m z^n K, \Gamma^\gamma g],$$

where $C^\alpha_{\beta,\gamma,m,n}$ are constants depending on multi-indices α,β,γ and nonnegative integers m,n.

We next state the key lemma to deriving the decay estimate of u through the normal form (2.13).

Lemma 2.1. *Let a and b be 0 or 1, and let m and n be arbitrary nonnegative integers. Then the integral kernels B_{ij} satisfy*

$$(1+y^2)^a(1+z^2)^b\omega_y^{-2}\omega_z^{-2}y^m z^n B_{ij}(y,z) \in L^1(\boldsymbol{R}\times\boldsymbol{R}).$$

Proof. We verify here only the statement for B_{11}. Let $K_{m,n} = \omega_y^{-2}\omega_z^{-2}y^m z^n B_{11}(y,z)$. We first prove that $\hat{K}_{m,n} \in \bigcap_{k\ge 0} W^{k,2}(\boldsymbol{R}\times\boldsymbol{R})$. Note that

$$\begin{aligned}
\det &= 4(p^2+pq+q^2)+3 \ge \frac{3}{2}(\langle p\rangle^2+\langle q\rangle^2)\\
&\ge 3\langle p\rangle\langle q\rangle,\\
\det{}^k &\ge C_k(\langle p\rangle^{2k}+\langle q\rangle^{2k})\\
&\ge C_k\langle p\rangle^l\langle q\rangle^{2k-l} \qquad \text{for } 0\le l\le 2k,\\
|\partial_p^k\langle p\rangle^{-2}| &\le C_k\langle p\rangle^{-k-2} \le C_k\langle p\rangle^{-2},\\
|\hat{B}_{11}| &= \frac{1}{\det}|cp^2q^2+dpq(p+q)+(\text{a polynomial of degree 2})|\\
&\le C\langle p\rangle\langle q\rangle,\\
|\partial_p\hat{B}_{11}| &= \frac{1}{\det^2}|4c(p^2q^3+2pq^4)+(\text{a polynomial of degree 4})|\\
&\le C(\langle p\rangle+\langle q\rangle) \le C\langle p\rangle\langle q\rangle,\\
|\partial_p^k\partial_q^l\hat{B}_{11}| &= \frac{1}{\det^{2^{k+l}}}|\text{a polynomial of degree } (2^{k+l+1}+2-k-l))|\\
&\le C_{k,l} \le C_{k,l}\langle p\rangle\langle q\rangle \qquad \text{for } k+l\ge 2.
\end{aligned}$$

From these inequalities, it follows that

$$\begin{aligned}
|\partial_p^k\partial_q^l\hat{K}_{m,n}(p,q)| &= |\partial_p^k\partial_q^l\langle p\rangle^{-2}\langle q\rangle^{-2}\partial_p^m\partial_q^n\hat{B}_{11}(p,q)|\\
&\le C_{k,l}^{m,n}\langle p\rangle^{-1}\langle q\rangle^{-1}.
\end{aligned}$$

Therefore $\hat{K}_{m,n} \in \bigcap_{k\ge 0} W^{k,2}(\boldsymbol{R}\times\boldsymbol{R})$.

Now we see that $(1+y^2)^{a+1}(1+z^2)^{b+1}K_{m,n}(y,z)$ is in $L^2(\boldsymbol{R}\times\boldsymbol{R})$, since we have

$$(1+y^2)^{a+1}(1+z^2)^{b+1}K_{m,n}(y,z)$$
$$=\frac{1}{(2\pi)^2}\iint e^{i(yp+qz)}(1-\partial_p^2)^{a+1}(1-\partial_q^2)^{b+1}\hat{K}_{m,n}(p,q)dpdq.$$

Thus $(1+y^2)^a(1+z^2)^b K_{m,n}\in L^1(\boldsymbol{R}\times\boldsymbol{R})$. ∎

The following lemma is used to prove the next proposition which we will need in order to obtain the energy estimate of u through the normal form (2.13). We give it without proof (see Proposition 4.1 in [2], Theorem 5 in [4]).

Lemma 2.2. *Let*

$$|\partial_p^i\partial_q^j\hat{K}(p,q)|\le C_{i,j}(p^2+q^2)^{-(i+j)/2} \tag{2.14}$$

for all i and $j\in\boldsymbol{N}$ such that $0\le i+j\le 1$. Then

$$\|[f,K,g]\|_{L^r}\le C_{p,q}\|f\|_{L^p}\|g\|_{L^q}$$

where $r^{-1}=p^{-1}+q^{-1}, p,q>1$ and $r<\infty$.

The following proposition plays an important role when we derive the energy estimate through the normal form (2.13).

Proposition 2.3. *Let k be an integer larger than or equal to 4. For $f,g\in\mathscr{S}(\boldsymbol{R})$, we have the following inequalities*:

$$\|[f,B_{11},g]\|_k\le C(\|f\|_k|g|_{[k/2]}+|f|_{[k/2]}\|g\|_k),$$
$$\|[f,B_{12},g]\|_k\le C(\|f\|_k|g|_{[k/2]}+|f|_{[k/2]}\|g\|_{k,\omega^{-1}}),$$
$$\|[f,B_{21},g]\|_k\le C(\|f\|_{k,\omega^{-1}}|g|_{[k/2]}+|f|_{[k/2]}\|g\|_k),$$
$$\|[f,B_{22},g]\|_k\le C(\|f\|_{k,\omega^{-1}}|g|_{[k/2]}+|f|_{[k/2]}\|g\|_{k,\omega^{-1}}),$$

where we denote by $[s]$ the largest integer that is not larger than s. Here the constants appearing in these inequalities do not depend on f and g.

Proof. We prove here only the first inequality for B_{11}, since we can prove the others similarly.

Let $|\alpha|\le k$. We have by Proposition 2.2

$$\Gamma^\alpha[f,B_{11},g]=\sum C^\alpha_{\beta,\gamma,m,n}[\Gamma^\beta f,y^m z^n B_{11},\Gamma^\gamma g].$$

Therefore we have

$$\|\Gamma^{\alpha}[f, B_{11}, g]\| \leq C \sum \|[\Gamma^{\beta} f, y^m z^n B_{11}, \Gamma^{\gamma} g]\|. \tag{2.15}$$

We first consider the case that $m = n = 0$ in the right hand side. We set $\hat{K}_i$ as follows:

$$\hat{K}_1 = \frac{p^2 q^2}{\det},$$
$$\hat{K}_2 = \frac{pq(p+q)}{\det},$$
$$\hat{K}_3 = \frac{(\text{a polynomial of degree 2})}{\det}$$

so that

$$\hat{B}_{11} = d\hat{K}_1 + e\hat{K}_2 + \hat{K}_3$$

for some constants d and e. Since $\hat{K}_3$ satisfies (2.14), we have

$$\|[\Gamma^{\beta} f, K_3, \Gamma^{\gamma} g]\| \leq C \begin{cases} \|\Gamma^{\beta} f\| |\Gamma^{\gamma} g|_{\infty}, \\ |\Gamma^{\beta} f|_{\infty} \|\Gamma^{\gamma} g\|. \end{cases} \tag{2.16}$$

We remark that

$$[f, K, g](x) = \iint e^{ix(p+q)} \hat{K}(p, q) \hat{f}(p) \hat{g}(q) dp dq, \tag{2.17}$$

$$\widehat{\partial_x f}(p) = ip\hat{f}(p). \tag{2.18}$$

By (2.17) and (2.18), we have

$$\begin{aligned} [\Gamma^{\beta} f, K_1, \Gamma^{\gamma} g] &= -[\partial_x^2 \Gamma^{\beta} f, K_{02}, \Gamma^{\gamma} g] \\ &= -[\partial_x \Gamma^{\beta} f, K_{11}, \partial_x \Gamma^{\gamma} g] \\ &= -[\Gamma^{\beta} f, K_{20}, \partial_x^2 \Gamma^{\gamma} g], \end{aligned} \tag{2.19}$$

$$\begin{aligned} [\Gamma^{\beta} f, K_2, \Gamma^{\gamma} g] &= -i[\partial_x \Gamma^{\beta} f, K_{11} + K_{02}, \Gamma^{\gamma} g] \\ &= -i[\Gamma^{\beta} f, K_{20} + K_{11}, \partial_x \Gamma^{\gamma} g], \end{aligned} \tag{2.20}$$

where we define K_{ij} by

$$\hat{K}_{ij} = \frac{p^i q^j}{\det}.$$

Since $\hat{K}_{20}$, $\hat{K}_{11}$, $\hat{K}_{02}$ and their sums satisfy (2.14), we have from (2.19) and (2.20)

$$\text{(2.21)} \qquad \|[\Gamma^\beta f, K_1, \Gamma^\gamma g]\| \le C \begin{cases} |\partial_x^2 \Gamma^\beta f|_\infty \|\Gamma^\gamma g\|, \\ |\partial_x \Gamma^\beta f|_\infty \|\partial_x \Gamma^\gamma g\|, \\ |\Gamma^\beta f|_\infty \|\partial_x^2 \Gamma^\gamma g\|, \\ \|\partial_x^2 \Gamma^\beta f\| |\Gamma^\gamma g|_\infty, \\ \|\partial_x \Gamma^\beta f\| |\partial_x \Gamma^\gamma g|_\infty, \\ \|\Gamma^\beta f\| |\partial_x^2 \Gamma^\gamma g|_\infty, \end{cases}$$

$$\text{(2.22)} \qquad \|[\Gamma^\beta f, K_2, \Gamma^\gamma g]\| \le C \begin{cases} |\partial_x \Gamma^\beta f|_\infty \|\Gamma^\gamma g\|, \\ |\Gamma^\beta f|_\infty \|\partial_x \Gamma^\gamma g\|, \\ \|\partial_x \Gamma^\beta f\| |\Gamma^\gamma g|_\infty, \\ \|\Gamma^\beta f\| |\partial_x \Gamma^\gamma g|_\infty. \end{cases}$$

Now we note that at least one of the following six cases always holds:

(a) $|\beta| + 2 \le k, |\gamma| \le [k/2]$, (d) $|\beta| + 2 \le [k/2], |\gamma| \le k$,

(b) $|\beta| + 1 \le k, |\gamma| + 1 \le [k/2]$, (e) $|\beta| + 1 \le [k/2], |\gamma| + 1 \le k$,

(c) $|\beta| \le k, |\gamma| + 2 \le [k/2]$, (f) $|\beta| \le [k/2], |\gamma| + 2 \le k$.

Accordingly, from (2.16), (2.21) and (2.22) we obtain

$$\text{(2.23)} \qquad \sum_{|\beta|+|\gamma| \le |\alpha|} \|[\Gamma^\beta f, B_{11}, \Gamma^\gamma g]\| \le C(\|f\|_k |g|_{[k/2]} + |f|_{[k/2]} \|g\|_k) \quad (|\alpha| \le k).$$

We next consider the case that $m = 1$ and $n = 0$. We set $\hat{K}_i$ as follows:

$$\hat{K}_4 = \frac{pq(pq^2 + 2q^3)}{\det^2},$$

$$\hat{K}_5 = \frac{(\text{a polynomial of degree } 4)}{\det^2},$$

so that

$$\widehat{yB}_{11} = i\partial_p \hat{B}_{11} = d\hat{K}_4 + \hat{K}_5$$

for some constant d. It is easy to see that $\hat{K}_4$ and $\hat{K}_5$ satisfy the same inequalities as $\hat{K}_2$ and $\hat{K}_3$, respectively. Thus we have

$$\text{(2.24)} \qquad \sum_{|\beta|+|\gamma| \le |\alpha|} \|[\Gamma^\beta f, yB_{11}, \Gamma^\gamma g]\| \le C(\|f\|_k |g|_{[k/2]} + |f|_{[k/2]} \|g\|_k) \quad (|\alpha| \le k).$$

Similarly, we can obtain the same inequality when $m = 0$ and $n = 1$.

We finally consider the case that $m+n \geq 2$. In this case we have by induction

$$(y^m z^n B_{11})^\frown = i^{m+n} \partial_p^m \partial_q^n \hat{B}_{11}$$
$$= \frac{(\text{a polynomial of degree } 2^{m+n+1} + 2 - m - n)}{\det^{2^{m+n}}},$$

which satisfies (2.14). Therefore, we have

(2.25) $$\sum_{|\beta|+|\gamma| \leq |\alpha|} \|[\Gamma^\beta f, y^m z^n B_{11}, \Gamma^\gamma g]\| \leq C(\|f\|_k |g|_{[k/2]} + |f|_{[k/2]} \|g\|_k) \quad (|\alpha| \leq k).$$

Combining inequalities (2.15), (2.23), (2.24) and (2.25), we obtain the desired inequality. ■

We conclude this section by proving the following proposition.

Proposition 2.4. *If* $f, g \in \mathscr{S}(\boldsymbol{R})$, *then*

(2.26) $$\|\partial_x f\|_{k,\omega^{-1}} \leq C\|f\|_k,$$

(2.27) $$\|fg\|_k \leq C(\|f\|_k |g|_{[k/2]} + |f|_{[k/2]} \|g\|_k),$$

(2.28) $$\|fg\|_{k,\omega^{-1}} \leq C(\|f\|_{k,\omega^{-1}} |g|_{[k/2]+1} + |f|_{[k/2]+1} \|g\|_{k,\omega^{-1}}),$$

(2.29) $$|fg|_k \leq C|f|_k |g|_k.$$

Here the constants appearing in these inequalities do not depend on f and g.

Proof. We first prove (2.26). From the relations (1.4), we have

$$\Gamma^\alpha \partial_x = \partial_x \Gamma^\alpha + \sum_{|\beta| \leq |\alpha|} C_\beta^\alpha \Gamma^\beta.$$

Therefore we have

$$\|\omega^{-1} \Gamma^\alpha \partial_x f\| \leq \|\omega^{-1} \partial_x \Gamma^\alpha f\| + C \sum_{|\beta| \leq |\alpha|} \|\omega^{-1} \Gamma^\beta f\|$$
$$\leq \|\Gamma^\alpha f\| + C \sum_{|\beta| \leq |\alpha|} \|\Gamma^\beta f\|$$
$$\leq C\|f\|_k \quad \text{for } |\alpha| \leq k.$$

Summing up this over $|\alpha| \leq k$, we obtain (2.26).

We next prove (2.27). Note that

(2.30) $$\Gamma^\alpha(fg) = \sum_{|\beta|+|\gamma| \leq |\alpha|} C_{\beta,\gamma}^\alpha \Gamma^\beta f \Gamma^\gamma g.$$

Hölder's inequality and the fact that only one term of (2.30) will have derivatives Γ^{β} and Γ^{γ} of order greater than $[k/2]$ in the right hand side prove this inequality.

Finally we prove (2.28). We note that

$$\|\omega^{-1}(fg)\| = \|fg\|_{H^{-1}} = \sup_{\varphi \in H^1, \|\varphi\|_{H^1}=1} (fg, \varphi).$$

Since we have

$$\begin{aligned}(fg, \varphi) = (f, g\varphi) &\leq \|f\|_{H^{-1}} \|g\varphi\|_{H^1} \\ &\leq \|f\|_{H^{-1}} \|g\|_{W^{1,\infty}} \|\varphi\|_{H^1},\end{aligned}$$

we obtain

$$\|\omega^{-1}(fg)\| \leq \|\omega^{-1}f\| \|g\|_{W^{1,\infty}}.$$

From the fact that only one term will have derivatives Γ^{β} and Γ^{γ} of order greater than $[k/2]$ in the right hand side of (2.30), it follows that

$$\begin{aligned}&\|fg\|_{k,\omega^{-1}} \\ &\leq C\left(\sum_{|\beta| \leq k} \|\omega^{-1}\Gamma^{\beta}f\| \sum_{|\gamma| \leq [k/2]} \|\Gamma^{\gamma}g\|_{W^{1,\infty}} + \sum_{|\beta| \leq [k/2]} \|\Gamma^{\beta}f\|_{W^{1,\infty}} \sum_{|\gamma| \leq k} \|\omega^{-1}\Gamma^{\gamma}g\|\right) \\ &\leq C(\|f\|_{k,\omega^{-1}} |g|_{[k/2]+1} + |f|_{[k/2]+1} \|g\|_{k,\omega^{-1}}).\end{aligned}$$

The inequality (2.29) is obtained similarly as (2.27). ■

3. Proof of the Main Theorem

In this section we describe the proof of the Main Theorem. The proof consists of the classical local existence theorem and a priori estimates of the solution. We divide our proof into the following four steps:

Step 1. Local existence

Proposition 3.1. *Given the initial data $u_0 \in H^2(\boldsymbol{R})$, $u_1 \in H^1(\boldsymbol{R})$ and a positive constant ε, there exist a finite time interval $[0, T]$ with $T > 0$ and a unique solution $u(t, x)$ of (1.1)–(1.2) with $n = 1$ and (2.1) such that*

$$u \in \bigcap_{j=0}^{2} C^j([0, T]; H^{2-j}).$$

T depends only on $\|u_0\|_{H^2}$, $\|u_1\|_{H^1}$, ε and F. In addition, if $u_0 \in H^{m,m-1}$, $u_1 \in$

$H^{m-1,m-1}$ *for an integer* m *with* $m \geq 2$, *then the above solution* u *satisfies*

$$\Gamma^\alpha u, \partial_t \Gamma^\alpha u, \partial_x \Gamma^\alpha u \in C([0, T]; L^2) \tag{3.1}$$

for any multi-indices α *with* $|\alpha| \leq m - 1$.

The proof of Proposition 3.1 is standard and will be omitted here (see, e.g., [11] and [17]).

Setp 2. A priori decay estimate

In this step we show the a priori decay estimate of the solution to (1.1)–(1.2) with (2.1). The following time decay estimate due to Georgiev [6] is essential.

Lemma 3.1. *Let* $u(t, x)$ *be a solution of the inhomogeneous linear Klein-Gordon equation:*

$$(\partial_t^2 - \partial_x^2 + 1)u = f(t, x), \qquad t > 0, \ x \in \boldsymbol{R}.$$

Then we have

$$\begin{aligned}(1 + t + |x|)^{1/2}|u(t, x)| \leq C\sum_{j=0}^{\infty} \sum_{|\alpha|\leq 3} \sup_{s\in(0,t)} \varphi_j(s)\|(1 + s + |y|)\Gamma^\alpha f(s, y)\|_{L^2(\boldsymbol{R}_y)} \\ + C\sum_{j=0}^{\infty} \sum_{|\alpha|\leq 4} \|(1 + |y|)^{1/2}\varphi_j(|y|)\Gamma^\alpha u(0, y)\|_{L^2(\boldsymbol{R}_y)},\end{aligned}$$

where $\{\varphi_j\}_{j=0}^{\infty}$ *is a Littlewood-Paley partition of unity, i.e.,*

$$\sum_{j=0}^{\infty} \varphi_j(s) = 1, s \geq 0; \varphi_j \in C_0^\infty(\boldsymbol{R}), \varphi_j \geq 0 \qquad \text{for all } j \geq 0;$$

$$\operatorname{supp} \varphi_j = [2^{j-1}, 2^{j+1}] \qquad \text{for } j \geq 1, \operatorname{supp} \varphi_0 \cap [0, \infty) = [0, 2].$$

Now we combine this lemma with Propositions 2.1, 2.2 and Lemma 2.1 so that the following decay estimate will be derived.

Lemma 3.2. *Assume that the solution* u *of* (1.1)–(1.2) *with* (2.1) *exists on* $[0, T]$ *with* $T > 0$. *Then* u *satisfies*

$$|u|_{k,T} \leq C[\varepsilon + |||u|||_{k,T}^2 + \log(2 + T)(|||u|||_{k,T}^3 + |||u|||_{k,T}^4)]$$

if ε *is sufficiently small. Here,* C *depends only on* k, $\|u_0\|_{H^{k+7,k+6}}$ *and* $\|u_1\|_{H^{k+6,k+6}}$.

Proof. We first evaluate the solution v of (2.13) before estimating u. We note that the following commutation relation holds:

$$[\partial_t^2 - \partial_x^2 + 1, \Gamma] = 0.$$

It follows from (2.13) and Lemma 3.1 that

$$(3.2)\qquad \begin{aligned}&(1+t+|x|)^{1/2}|\Gamma^{\alpha}v(t,x)|\\ &\le C\sum_{j=0}^{\infty}\sum_{|\lambda|\le 3}\sup_{s\in(0,t)}\varphi_j(s)\|(1+s+|y|)\Gamma^{\alpha+\lambda}R(s,y)\|_{L^2(\boldsymbol{R}_y)}\\ &\quad+C\sum_{j=0}^{\infty}\sum_{m=0}^{4}\|(1+|y|)^{1/2}\varphi_j(|y|)\partial_x^m\Gamma^{\alpha}v(0,y)\|_{L^2(\boldsymbol{R}_y)}\\ &\quad+C\sum_{j=0}^{\infty}\sum_{|\lambda|\le 3}\|(1+|y|)^{3/2}\varphi_j(|y|)\Gamma^{\alpha+\lambda}\partial_t v(0,y)\|_{L^2(\boldsymbol{R}_y)}.\end{aligned}$$

If $u(0,\cdot)\in H^{k+7,k+6}$, $\partial_t u(0,\cdot)\in H^{k+6,k+6}$, we easily see by (2.2), (2.5), Propositions 2.1, 2.2 and Lemma 2.1 that for small ε,

$$(3.3)\qquad \begin{aligned}&\sum_{j=0}^{\infty}\sum_{m=0}^{4}\|(1+|y|)^{1/2}\varphi_j(|y|)\partial_x^m\Gamma^{\alpha}v(0,y)\|_{L^2(\boldsymbol{R}_y)}\\ &+\sum_{j=0}^{\infty}\sum_{|\lambda|\le 3}\|(1+|y|)^{3/2}\varphi_j(|y|)\Gamma^{\alpha+\lambda}\partial_t v(0,y)\|_{L^2(\boldsymbol{R}_y)}\le C\varepsilon,\qquad |\alpha|\le k.\end{aligned}$$

We next estimate R. We evaluate only the eleventh term on the right hand side of (2.7), since the rest terms of (2.7) are similarly estimated.

By Proposition 2.2, we have

$$(3.4)\quad \begin{aligned}&\|(1+s+|x|)\Gamma^{\alpha+\lambda}[\partial_x^2u-u,B_{22},F]\|\\ &\le C\sum_{\substack{|\beta|+|\gamma|\le|\alpha+\lambda|\\ 0\le m\le|\beta|,0\le n\le\gamma}}\|(1+s+|x|)[\Gamma^{\beta}(\partial_x^2u-u),y^mz^{\gamma}B_{22},\Gamma^{\gamma}F]\|\\ &\le C\sum_{\substack{|\beta|+|\gamma|\le|\alpha+\lambda|\\ 0\le m\le|\beta|,0\le n\le\gamma}}\|(1+s+|x|)[\omega^2\Gamma^{\beta}(\partial_x^2u-u),\omega_y^{-2}\omega_z^{-2}y^mz^nB_{22},\omega^2\Gamma^{\gamma}F]\|.\end{aligned}$$

Noting that

$$(3.5)\qquad \begin{aligned}&1+s+|x|\le C(1+s+|z|)(1+|x-z|^2)\\ &1+s+|x|\le C(1+s+|y|)^{1/2}(1+s+|z|)^{1/2}(1+|x-y|^2)(1+|x-z|^2),\end{aligned}$$

for $s \geq 0$, we obtain by (3.4), Proposition 2.1 and Lemma 2.1

(3.6)
$$\begin{aligned}
&\|(1+s+|x|)\Gamma^{\alpha+\lambda}[\partial_x^2 u - u, B_{22}, F]\| \\
&\leq C \sum_{\substack{|\beta|+|\gamma|\leq k+3 \\ 0\leq|\beta|\leq[k/2]+1}} |(1+s+|x|)^{1/2}\omega^2\Gamma^\beta(\partial_x^2 u - u)|_\infty \|(1+s+|x|)^{1/2}\omega^2\Gamma^\gamma F\| \\
&\quad + C \sum_{\substack{|\beta|+|\gamma|\leq k+3 \\ 0\leq|\gamma|\leq[k/2]+2}} \|\omega^2\Gamma^\beta(\partial_x^2 u - u)\| |(1+s+|x|)\omega^2\Gamma^\gamma F|_\infty \\
&\leq C|u|_{[k/2]+5,T}^2(\|u\|_{k+6,T} + \|\partial u\|_{k+6,T}) \\
&\leq C|||u|||_{k,T}^3 \qquad \text{for } 0\leq s\leq T, |\alpha|\leq k.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
(3.7) \qquad \|(1+s+|x|)\Gamma^{\alpha+\lambda}[F, B_{22}, F]\| &\leq C|u|_{[k/2]+5,T}^3(\|u\|_{k+6,T} + \|\partial u\|_{k+6,T}) \\
&\leq C|||u|||_{k,T}^4 \qquad \text{for } 0\leq s\leq T, |\alpha|\leq k.
\end{aligned}$$

Since we have

$$\sum_{j=0}^{\infty} \sup_{s\in(0,T)} \varphi_j(s) \leq C\log(2+T)$$

for $T > 0$, inequalities (3.2), (3.3), (3.6) and (3.7) yield

$$(3.8) \qquad (1+t+|x|)^{1/2}|\Gamma^\alpha v(t,x)| \leq C[\varepsilon + \log(2+T)(|||u|||_{k,T}^3 + |||u|||_{k,T}^4)]$$

for $0\leq t\leq T$, $|\alpha|\leq k$.

We next obtain the estimate of u through (2.2) and (3.8). We evaluate only the last term on the right hand side of (2.2) here, since the rest terms are similarly estimated. Using Propositions 2.1, 2.2, Lemma 2.1 and the Sobolev imbedding theorem, we have

$$\begin{aligned}
(3.9)\ &(1+t+|x|)^{1/2}|\Gamma^\alpha[\partial_t u, B_{22}, \partial_t u]| \\
&\leq C \sum_{\substack{|\beta|+|\gamma|\leq k \\ 0\leq m\leq|\beta|, 0\leq n\leq|\gamma|}} |(1+t+|x|)^{1/2}[\omega^2\Gamma^\beta\partial_t u, \omega_y^{-2}\omega_z^{-2}y^m z^n B_{22}, \omega^2\Gamma^\gamma\partial_t u]|_\infty \\
&\leq C \sum_{|\beta|\leq[k/2]} |(1+t+|x|)^{1/2}\omega^2\Gamma^\beta\partial_t u|_\infty \times \sum_{|\gamma|\leq k} |\omega^2\Gamma^\gamma\partial_t u|_\infty \\
&\leq C|u|_{[k/2]+3,T} \times \sum_{|\gamma|\leq k} \|\omega^2\Gamma^\gamma\partial_t u\|_{H^1} \\
&\leq C|u|_{[k/2]+3,T}\|u\|_{k+4,T}, \qquad \text{for } 0\leq t\leq T, |\alpha|\leq k,
\end{aligned}$$

where we have used (3.5). Thus, combining (2.2), (3.8) and (3.9), we obtain the desired time decay estimate. ■

Step 3. A priori energy estimate

In this step, we show the following lemma concerning the a priori energy estimate of the solution to (1.1)–(1.2) with (2.1).

Lemma 3.3. *Assume that the solution u of* (1.1)–(1.2) *with* (2.1) *exists on* $[0, T]$ *with* $T > 0$. *Then u satisfies for sufficiently small* ε

$$\|u(t)\|_{k+6} + \|\partial u(t)\|_{k+6}$$
$$\le C[\varepsilon + |||u|||_{k,T}^2 + |||u|||_{k,T}^3 + \log(1+T)(|||u|||_{k,T}^3 + |||u|||_{k,T}^4)], \qquad t \in [0, T].$$

Here, C depends only on k, $\|u_0\|_{H^{k+7,k+6}}$ *and* $\|u_1\|_{H^{k+6,k+6}}$.

Proof. We first evaluate v as in the proof of the decay estimate.

We apply Γ^α to (2.2) and multiply the resulting equation by $\partial_t \Gamma^\alpha v$ to obtain

$$\|\Gamma^\alpha v(t)\| + \|\partial \Gamma^\alpha v(t)\| \le C\left(\|\Gamma^\alpha v(0)\| + \|\partial \Gamma^\alpha v(0)\| + \int_0^t \|\Gamma^\alpha R(s)\| ds\right).$$

Summing up this inequality over $|\alpha| \le k+6$, we have

$$\|v(t)\|_{k+6} + \|\partial v(t)\|_{k+6} \le C\left(\|v(0)\|_{k+6} + \|\partial v(0)\|_{k+6} + \int_0^t \|R(s)\|_{k+6} ds\right), \tag{3.10}$$

where we have used the fact that there exists a positive constant C independent of v such that

$$C^{-1}(\|v\|_{k+6} + \|\partial v\|_{k+6}) \le \|v\|_{k+6} + \sum_{|\alpha| \le k+6} \|\partial \Gamma^\alpha v\| \le C(\|v\|_{k+6} + \|\partial v\|_{k+6}).$$

We first estimate $\|R\|_{k+6}$. We note that

$$\begin{aligned} \partial_t F = &-2b_1 u^2 - 2b_2 u \partial_x u + 2(a_1 - c_1) u \partial_t u + 2a_1 \partial_t u \partial_x u + 2b_1 (\partial_t u)^2 \\ &+ 2b_1 u \partial_x^2 u + 2a_2 u \partial_{tx}^2 u + 2a_3 \partial_x u \partial_{tx}^2 u + 2b_2 \partial_t u \partial_{tx}^2 u + 2b_2 \partial_x u \partial_x^2 u \\ &+ 2c_1 \partial_t u \partial_x^2 u + 2b_1 u F + 2b_2 \partial_x u F + 2c_1 \partial_t u F \end{aligned}$$

and that

$$\|F(t)\|_{k+6} \le C(\|u(t)\|_{k+6} + \|\partial u(t)\|_{k+6}) |u(t)|_{[k/2]+4}, \tag{3.11}$$

$$|F(t)|_m \le C|u(t)|_{m+1}^2, \tag{3.12}$$

$$(3.13) \qquad \|\partial_t F(t)\|_{k+6,\omega^{-1}} \le C(\|u(t)\|_{k+6} + \|\partial u(t)\|_{k+6})(|u(t)|_{[k/2]+6} + |u(t)|^2_{[k/2]+4}),$$

$$(3.14) \qquad |\partial_t F(t)|_m \le C(|u(t)|^2_{m+2} + |u(t)|^3_{m+1}),$$

which are derived from (2.27)–(2.29). We can now obtain $\|\cdot\|_{k+6}$ estimates for all terms of (2.7) by Proposition 2.3, (2.26) and (3.11)–(3.14). We have, for example,

$$\begin{aligned}(3.15) \quad & \|[\partial_x^2 u - u, B_{22}, F](t)\|_{k+6} \\ & \le C(\|(\partial_x^2 u - u)(t)\|_{k+6,\omega^{-1}}|F(t)|_{[k/2]+3} + |(\partial_x^2 u - u)(t)|_{[k/2]+3}\|F(t)\|_{k+6,\omega^{-1}}) \\ & \le C(\|u(t)\|_{k+6} + \|\partial u(t)\|_{k+6})|u(t)|^2_{[k/2]+5},\end{aligned}$$

$$\begin{aligned}(3.16) \quad & \|[\partial_t u, B_{22}, \partial_t F](t)\|_{k+6} \\ & \le C(\|(\partial_t u(t)\|_{k+6,\omega^{-1}}|\partial_t F(t)|_{[k/2]+3} + |\partial_t u(t)|_{[k/2]+3}\|\partial_t F(t)\|_{k+6,\omega^{-1}}) \\ & \le C(\|u(t)\|_{k+6} + \|\partial u(t)\|_{k+6})(|u(t)|^2_{[k/2]+6} + |u(t)|^3_{[k/2]+4}),\end{aligned}$$

$$\begin{aligned}(3.17) \quad \|[F, B_{22}, F](t)\|_{k+6} & \le C\|F(t)\|_{k+6,\omega^{-1}}|F(t)|_{[k/2]+3} \\ & \le C(\|u(t)\|_{k+6} + \|\partial u(t)\|_{k+6})|u(t)|^3_{[k/2]+4}.\end{aligned}$$

The rest terms on the right hand side of (2.7) are estimated in the same way as above. Since the following inequality is valid:

$$\begin{aligned}\sup_{x\in \boldsymbol{R}} |\Gamma^\alpha u(t,x)| & \le \sup_{x\in \boldsymbol{R}} \left(\frac{1+t+|x|}{1+t}\right)^{1/2} |\Gamma^\alpha u(t,x)| \\ & = \frac{1}{(1+t)^{1/2}} \sup_{x\in \boldsymbol{R}} (1+t+|x|)^{1/2}|\Gamma^\alpha u(t,x)| \\ & \le \frac{1}{(1+t)^{1/2}} \sup_{x\in \boldsymbol{R}, t\in[0,T]} (1+t+|x|)^{1/2}|\Gamma^\alpha u(t,x)|\end{aligned}$$

for $t \in [0, T]$, we have

$$(3.18) \qquad |u(t)|_m \le \frac{1}{(1+t)^{1/2}}|u|_{m,T} \qquad \text{for } t \in [0,T].$$

Combining (3.15)–(3.18), we obtain

$$\begin{aligned}\|R(t)\|_{k+6} & \le \frac{C}{1+t}(\|u\|_{k+6,T} + \|\partial u\|_{k+6,T})(|u|^2_{[k/2]+6,T} + |u|^3_{[k/2]+4,T}) \\ & \le \frac{C}{1+t}(|||u|||^3_{k,T} + |||u|||^4_{k,T}) \qquad \text{for } t \in [0,T].\end{aligned}$$

Therefore we have

$$\int_0^T \|R(t)\|_{k+6}dt \leq C\log(1+T)(|||u|||_{k,T}^3 + |||u|||_{k,T}^4). \tag{3.19}$$

We next show the relation between the $\|\cdot\|_{k+6}$-norms of u and v to evaluate u through the estimate of v. We apply Propositions 2.3 and 2.4 to (2.2), (2.4) and (2.5) to obtain

$$\begin{aligned}(3.20)\qquad &\|v(t)\|_{k+6} + \|\partial v(t)\|_{k+6}\\ &\leq \|u(t)\|_{k+6} + \|\partial u(t)\|_{k+6}\\ &\quad + C(\|u(t)\|_{k+6} + \|\partial u(t)\|_{k+6})(|u(t)|_{[k/2]+2} + |u(t)|_{[k/2]+1}^2),\end{aligned}$$

$$\begin{aligned}(3.21)\qquad &\|u(t)\|_{k+6} + \|\partial u(t)\|_{k+6}\\ &\leq \|v(t)\|_{k+6} + \|\partial v(t)\|_{k+6}\\ &\quad + C(\|u(t)\|_{k+6} + \|\partial u(t)\|_{k+6})(|u(t)|_{[k/2]+2} + |u(t)|_{[k/2]+1}^2)\\ &\leq \|v(t)\|_{k+6} + \|\partial v(t)\|_{k+6} + C(|||u|||_{k,T}^2 + |||u|||_{k,T}^3),\end{aligned}$$

for $t \in [0, T]$. By (3.20), we easily see that

$$\|v(0)\|_{k+6} + \|\partial v(0)\|_{k+6} \leq C\varepsilon \tag{3.22}$$

for small ε.

The inequalities (3.10), (3.19), (3.21) and (3.22) yield the desired energy estimate. ■

Step 4. The proof of the Main Theorem concluded

Consequently, we have from Lemmas 3.2 and 3.3

$$|||u|||_{k,T} \leq C[\varepsilon + |||u|||_{k,T}^2 + |||u|||_{k,T}^3 + \log(2+T)(|||u|||_{k,T}^3 + |||u|||_{k,T}^4)] \tag{3.23}$$

provided that the solution u exists on the interval $[0, T]$ with $T > 0$ and that ε is sufficiently small, say, $\varepsilon < \varepsilon_1 \leq 1$. In what follows, we denote by C the constant appearing in (3.23).

Now we are in a position to conclude the proof of the Main Theorem.

Let A and B be positive numbers to be chosen appropriately later and let

$$T_\varepsilon^* = \sup\{T \in (0, T_\varepsilon); |||u|||_{k,T} \leq A\varepsilon\}.$$

We suppose that

$$\varepsilon^2 \log(2 + T_\varepsilon^*) \leq B. \tag{3.24}$$

Then we have, noting (3.1) and (3.23), that

$$\begin{aligned}(3.25)\qquad |||u|||_{k,T} &\le C[\varepsilon + A^2\varepsilon^2 + A^3\varepsilon^3 + \log(2+T)(A^3\varepsilon^3 + A^4\varepsilon^4)] \\ &\le C[1 + A^2(1+A\varepsilon)\varepsilon + A^3B(1+A\varepsilon)]\varepsilon,\end{aligned}$$

for all $T \in [0, T_\varepsilon^*)$. Now we choose two positive constants A and B such that

$$A > 2C, \qquad B < \frac{A-2C}{2A^3C}.$$

Then (3.25) implies that

$$|||u|||_{k,T} < \frac{1}{2} A\varepsilon$$

for all $T \in [0, T_\varepsilon^*)$ provided that

$$\varepsilon < \min\left\{\varepsilon_1, \frac{A - 2A^3BC - 2C}{2A^2(A + A^2BC + C)}\right\} =: \varepsilon_2.$$

But then, it follows from the definition of T_ε^* that $T_\varepsilon^* = T_\varepsilon$. This fact combined with Proposition 3.1 enables us to extend the solution $u(t)$ beyond $t = T_\varepsilon$, which is a contradiction to the definition of T_ε. Therefore (3.24) does not hold, so we have

$$T_\varepsilon \ge T_\varepsilon^* > \exp(B\varepsilon^{-2}) - 2 \ge \frac{1}{2}\exp(B\varepsilon^{-2})$$

provided that

$$\varepsilon \le \min\left\{\varepsilon_2, \left(\frac{B}{\log 4}\right)^{1/2}\right\} =: \varepsilon_0.$$

This completes the proof of the Main Theorem. ■

Concluding remark. After submitting this paper for publication, the authors knew two papers [13] and [5], which are concerned with the problem in this paper. In [13] Lindblad and Sogge have studied the global existence or almost global existence of solution to the nonlinear Klein-Gordon equation for small initial data, when the nonlinear function F is dependent only on u but F is not necessarily smooth. In [5] Delort has given the lower bound of the life span, that is,

$$T_\varepsilon > c\varepsilon^{-4}|\log \varepsilon|^{-6}$$

for (1.1)–(1.2) with $n = 1$, when the equation (1.1) is quadratically semilinear and

F satisfies the unit condition introduced by Kosecki [12]. But the initial data considered in [5] are less regular than in this paper.

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