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Free Boundary Problems for the Heat Equation and Application to a Bingham Flow

By

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Introduction

We concern ourselves with uniqueness and existence of solutions $(s(t), u(x,t), T)$ to the free boundary problem for the heat equation:

$$
(0.1)\quad
\begin{aligned}
&u_{xx} - u_t = q(x,t) \qquad \text{in } D(T) = \{(x,t); 0 < x < s(t), 0 < t < T\},\\
&u(0,t) = \varphi(t), \qquad 0 < t < T,\\
&u(s(t),t) = 0, \qquad 0 < t < T,\\
&u_x(s(t),t) = \lambda(s(t),t)s'(t) + \mu(s(t),t), \qquad 0 < t < T,
\end{aligned}
$$

and

$$(0.2)\qquad s(0) = b, \quad u(x,0) = h(x) \qquad \text{for } x \in (0,b)$$

satisfying

$$(0.3)\qquad s \in C([0,T]) \cap C^1((0,T)), \qquad u, u_x \in C(\bar{D}(T) \backslash [0,b] \times \{0\}),$$

where b is a positive constant and $q(x,t), \varphi(t), \lambda(x,t), \mu(x,t)$ are given functions sufficiently smooth on the closure of the quarter domain $Q = \{(x,t); x > 0, t > 0\}$, and we further assume that there exist positive constants M, λ_0 and μ_0 such that

$$(0.4)\qquad \lambda(x,t) \le -\lambda_0 \text{ on } \bar{Q},$$

$$(0.5)\qquad |q(x,t)| \le M, \qquad |\mu(x,t)| \le \mu_0 \text{ on } \bar{Q},$$

and that

(0.6) $\quad h$ is a piecewise continuous function belonging to $L^\infty(0,b)$.

There is an extensive literature on this problem. It has been shown in Fasano-Primicerio [3, II] that *under the assumption* $|h(x)| \le const.|x-b|^\alpha (\alpha > 0)$ *for* $x \in (0,b)$, *there exists a unique solution of* (0.1)–(0.2) *satisfying* (0.3). Concerning global existence of solutions satisfying (0.3) they proved in [3,I] that the following three cases occur: (A) *There exists a solution up to an arbitrarily large time.* (B) *There exists an extinction time* T_0 *such that* $s(t) > 0$ *for* $t < T_0$ *and* $\liminf_{t\to T_0} s(t) = 0$. (C) *There exists a blow-up time* T_1 *such that* $\liminf_{t\to T_1} s(t) > 0$ *and* $\liminf_{t\to T_1} s'(t) = -\infty$. Furthermore, in the special case that $q = \mu = 0$ and $\lambda = -1$, they established more precise result on existence and uniqueness in [5]: (1) *There exists no solution if* $\int_x^b d\xi \int_\xi^b (h(\eta)+1)d\eta \le 0$ *in a left neighborhood of* $x = b$. (2) *There exists at most one solution if* $h(x)+1 \ge 0$, $h(x)+1 \not\equiv 0$ *in a left neighborhood of* $x = b$. (3) *There exists a solution if* $h(x)+1 > 0$ *in a left neighborhood of* $x = b$.

The main purpose of the present paper is to extend the facts (1), (2) and (3) to the general case. That is,

Theorem 1. (cf. (1)) *Suppose that conditions* (0.4)–(0.6) *hold. There then exists no solution of equations* (0.1)–(0.2) *satisfying* (0.3) *if it holds that*

$$\int_x^b d\xi \int_\xi^b (h(\eta) - \lambda(\eta, 0))d\eta \le 0 \ \text{ in } (\beta, b) \qquad \text{for some } \beta \in (0, b). \tag{0.7}$$

Theorem 2. (cf. (2) and (3)) *There exists a unique solution, local in time, of equations* (0.1)–(0.2) *satisfying* (0.3) *under conditions* (0.4)–(0.6),

$$\begin{aligned} &q(x,t) + \lambda_t(x,t) \le \min\{\lambda_{xx}(x,t), \mu_x(x,t)\} && \text{in } (\beta, B)\times(0,\ell) \\ &\mu(x,t) + x\{\mu_x(x,t) - q(x,t) - \lambda_t(x,t)\} \le 0 && \text{in } (\beta, B)\times(0,\ell) \end{aligned} \tag{0.8}$$

and

$$h(x) - \lambda(x,0) \ge 0, \qquad \int_x^b (h(\xi) - \lambda(\xi, 0))d\xi > 0 \ \text{ in } (\beta, b) \tag{0.9}$$

for some $0 \le \beta < b < B$ *and* $\ell > 0$.

Remark 1. By the same argument as in [10] we can immediately prove that the function $s(t)$ is also sufficiently smooth in the interval $(0, T)$ (see also [7]).

Remark 2. Instead of boundary condition $u(0,t) = \varphi(t)$ we impose condition

$$u_x(0,t) = g(u(0,t), t) \qquad \text{on } (0, T). \tag{0.10}$$

Under some condition on g we can establish the same results as in Theorems 1 and 2.

The proof of Theorems 1 and 2 are similarly achieved as in [5], although there is a slight improvement in the course of the proof. Theorem 1 will be proved in Section 1, where we also give a condition for T to be a blow-up time (see Lemma 1.2). For the proof of Theorem 2 we prepare in Section 2 an auxiliary problem: *To find a solution of the equations* (0.1) *subject to the initial condition*:

$$(0.2_n) \qquad s_n(0) = b_n,\ u(x,0) = \tilde{h}(x) \qquad \text{for } x \in (0, b_n)\left(b_n = b + \frac{1}{n}\right),$$

where $\tilde{h}(x) = h(x)$ *for* $x < b$ *and* $= 0$ *for* $x \geq b$. We can construct a solution $(s_n(t), u_n(x,t), T_n)$ of equations (0.1)–(0.2$_n$) and set

$$(0.11) \qquad c_n(x,t) = \int_{s_n(t)}^{x} d\xi \int_{s_n(t)}^{\xi} \{u_n(\eta,t) - \lambda(\eta,t)\} d\eta.$$

It will be shown that the first condition of (0.8) and (0.9) imply the monotonicity of the sequences $\{s_n\}$, $\{c_n\}$ and $\{c_{n,x}\}$, and Lemma 1.2 implies $T_n \geq T^*$ for some T^* not depending on n. Let $s_n \to s$ and $c_n \to c$ as $n \to \infty$.

In Section 3 we shall prove, owing to the second condition of (0.8), that the triple $(s(t), c(x,t), T^*)$ is a unique solution of (1.2)–(1.3), satisfying $s \in C^1(0, T^*)$,

$$(0.12) \qquad s \in C([0,T]) \quad \text{and} \quad c, c_x \in C(\bar{D}(T)),$$

for $T = T^*$, that the triple $(s(t), u(x,t), T^*)$ with $u(x,t) = c_t(x,t) - R(x,t;s(t))$ is a desired solution (see Lemma 1.1) and the following corollaries:

Corollary 1. *Suppose that* (0.4)–(0.6), (0.8), (0.9) *and*

$$(0.13) \qquad \begin{aligned} &q(x,t) + \lambda_t(x,t) \leq \lambda_{xx}(x,t) \ \text{in } Q, \\ &h(x) - \lambda(x,0) \geq 0 \ \text{in } (0,b), \qquad \varphi(t) - \lambda(0,t) \geq 0 \ \text{in } (0,\infty) \end{aligned}$$

hold. Then, blow-up never occurs.

Corollary 2. *In addition to the hypothesis in Corollary* 1 *we assume that*

$$(0.14) \qquad \varphi(t) > 0 \ \text{in } (0,\infty)$$

hold. Then, there exists a unique solution of (0.1)–(0.2) *satisfying* (0.3) *up to an arbitrarily large time.*

Section 4 is devoted to the application of Theorem 2 and Corollaries to a one dimensional Bingham flow.

§ 1. Proof of Theorem 1 and key lemmas

Let $(s(t), u(x,t), T)$ be a solution of equations (0.1)–(0.2) satisfying (0.3). We then consider the related function as in (0.11):

$$(1.1) \qquad c(x,t) = \int_{s(t)}^{x} d\xi \int_{s(t)}^{\xi} \{u(\eta,t) - \lambda(\eta,t)\} d\eta.$$

It easily follows that $u(x,t) = c_t(x,t) - R(x,t;s(t))$ and the triple $(s(t), c(x,t), T)$ satisfies the equations

$$(1.2) \qquad \begin{aligned} &c_t - c_{xx} = \lambda(x,t) + R(x,t;s(t)) \text{ in } D(T), \\ &c(0,t) = c_0(0) + \int_0^t \{\varphi(\tau) + R(0,\tau;s(\tau))\} d\tau, \qquad 0 < t < T, \\ &c(s(t),t) = c_x(s(t),t) = 0, \qquad 0 < t < T, \end{aligned}$$

and the initial conditions

$$(1.3) \qquad s(0) = b, \qquad c(x,0) = c_0(x), \qquad x \in (0,b),$$

where

$$c_0(x) = \int_b^x d\xi \int_b^\xi \{h(\eta) - \lambda(\eta,0)\} d\eta = \int_x^b (\eta - x)\{h(\eta) - \lambda(\eta,0)\} d\eta,$$

$$R(x,t;s(t)) = (s(t)-x)\mu(s(t),t) - \int_{s(t)}^x d\xi \int_{s(t)}^\xi \{q(\eta,t) + \lambda_t(\eta,t)\} d\eta.$$

It is easy to see that

$$(1.4) \qquad c, c_x \in C^1(\bar{D}(T) \backslash [0,b] \times \{0\}).$$

Conversely, we can prove

Lemma 1.1. *Let* $(s(t), c(x,t), T)$ *be a solution of* (1.2) *satisfying* (0.12) *and* $s \in C^1(0,T)$. *Then, the regularity properties* (1.4) *is valid and the triple* $(s(t), u(x,t), T)$ *with* $u(x,t) = c_t(x,t) - R(x,t;s(t))$ *is a solution of* (0.1) *satisfying* (0.3) *and* (1.1).

Proof. We first represent $c(x,t)$, using Green's formula, in the integral form:

$$(1.5) \qquad \begin{aligned} c(x,t) = &\int_0^b c(\xi,0) G(x,t;\xi,0) d\xi + \int_0^t c(0,\tau) G_\xi(x,t;0,\tau) d\tau \\ &+ \iint_{D(t)} \{\lambda(\xi,\tau) + R(\xi,\tau;s(\tau))\} G(x,t;\xi,\tau) d\xi d\tau, \end{aligned}$$

where and in what follows G and N denote the Green and the Neumann functions on the half space $x > 0$, respectively. Differentiation with respect to x and integration by parts yield

$$(1.6)\qquad c_{xx}(x,t) = \int_0^b c_{\xi\xi}(\xi,0)G(x,t;\xi,0)d\xi$$
$$- \int_0^t \lambda(s(\tau),\tau)N_x(x,t;s(\tau),\tau)d\tau$$
$$+ \int_0^t \{\lambda(0,\tau) - \varphi(\tau)\}N_x(x,t;0,\tau)d\tau$$
$$- \int_0^t \{\lambda_\xi(s(\tau),\tau) - \mu(s(\tau),\tau)\}G(x,t;s(\tau),\tau)d\tau$$
$$+ \int_0^t \{\lambda_\xi(0,\tau) + R_\xi(0,\tau;s(\tau))\}G(x,t;0,\tau)d\tau$$
$$+ \iint_{D(t)} \{\lambda_{\xi\xi}(\xi,\tau) - q(\xi,\tau) - \lambda_\tau(\xi,\tau)\}G(x,t;\xi,\tau)d\xi d\tau.$$

We find, employing the usual argument (cf. Lemma 1 in [Ch. 8, 6]), that c_{xx} along with c_t is continuous on $\bar{D}(T)$ except for the set $[0,b] \times \{0\}$. Furthermore, the last equation of (1.2) implies (1.1) and $c_t = 0$ on $x = s(t)$. Hence, $u(s(t),t) = 0$, since $R(s(t),t;s(t)) = 0$. Therefore, it follows that c_{xt} is continuous on $\bar{D}(T)$ except for the set $[0,b] \times \{0\}$ and u satisfies (0.3) and (0.1) except for the last equation. Differentiations of the relations $u = c_t - R$ and (1.1) with respect to x and $x - t$, respectively, give

$$(1.7)\qquad u_x(x,t) = c_{xt}(x,t) + \mu(s(t),t) + \int_{s(t)}^x \{q(\eta,t) + \lambda_t(\eta,t)\}d\eta,$$
$$= \lambda(s(t),t)s'(t) + \mu(s(t),t) + \int_{s(t)}^x u_{xx}(\eta,t)d\eta,$$

which yields the last equation of (0.1). □

Proof of Theorem 1. Contrary to the assertion we suppose that there exists a solution $(s(t),u(x,t),T)$ of equations (0.1)–(0.2) satisfying (0.3) and consider the related function $c(x,t)$ defined by (1.1). It is easy to see that there exist constants $b_0 \in (\beta,b), B > b$ and $T_0 < T$ such that $b_0 < s(t) < B$ in $(0,T_0)$ and $\lambda(x,t) + R(x,t;s(t)) < 0$ in $D(T_0) \cap \{x > b_0\}$. The assumption (0.7) implies $c_t(\hat{x},0) = h(\hat{x}) + R(\hat{x},0;b) < 0$ for some $\hat{x} \in (b_0,b)$ and $c_0(x) \le 0$ for $x \in (\beta,b)$. Therefore, there exist $\hat{t} \in (0,T_0)$ such that $c(\hat{x},t) < 0$ and $s(t) > \hat{x}$ for $t \in (0,\hat{t})$. Applying the maximum principle to the domain $D(\hat{t}) \cap \{x > \hat{x}\}$, we conclude $c_x(s(t),t) > 0$, $t \in (0,\hat{t})$. This contradicts the last equation of (1.2). Thus, the proof is complete. □

Before ending the section we state the following lemma.

Lemma 1.2. (cf. Lemma 2.4 of [4]) *Let* $(s(t), u(x,t), T)$ *be a solution of* (0.1)–(0.2) *satisfying* (0.3), *and assume that* $q + \lambda_t \le \lambda_{xx}$ *in* $Q \cap \{t < T\}$ *and that* $\liminf_{t \to T} s(t) > 0$. *Then, blow-up does not occur at* $t = T$ *if there exist positive numbers* δ, d_0 *and* $\tau \in (0, T)$ *such that* $s(t) - d_0 > 0$, $\tau < t < T$, *and*

$$(1.8) \quad u(x,t) - \lambda(x,t) \ge \delta \text{ on } \Omega(d_0) = \{(x,t); s(t) - d_0 < x < s(t), \tau < t < T\}.$$

Proof. Suppose (1.8) to hold. Contrary to the assertion we assume that T is a blow-up time. It is easy to see that there exist positive constants H and $d \le d_0$ such that

$$(1.9) \quad \begin{aligned} &\lambda_0 > Hd \qquad \text{(see (0.4) for } \lambda_0) \\ &u(x,\tau) \ge H(x - \sigma), \qquad x \in (\sigma - d, \sigma) \; (\sigma = s(\tau)). \end{aligned}$$

Let us set, for $\varepsilon \in (\tau, T)$,

$$\alpha_\varepsilon = \inf\{s'(t); \tau < t < T - \varepsilon\}.$$

Evidently, $\alpha_\varepsilon \to -\infty$ as $\varepsilon \to 0$. So that without loss of generality we may assume that $\alpha_\varepsilon \le 0$. Choosing positive constants $\gamma < 1$ and η satisfying

$$(1.10) \qquad \gamma(\lambda_0 - \eta T) \ge Hd,$$

$$(1.11) \qquad \gamma\eta - (1 - \gamma)M \ge 0,$$

$$(1.12) \qquad u(x,t) - \gamma\lambda(x,t) - \gamma\eta T \ge 0 \qquad \text{on } x = s(t) - d, \; \tau < t < T,$$

we consider the domain $\Omega_\varepsilon = \Omega(d) \cap \{\tau < t < T - \varepsilon\}$ and the function w as follows:

$$w(x,t) = \gamma_\varepsilon \theta(x,t)(1 - \exp\{a_\varepsilon(x - s(t))\}) \qquad \text{with} \qquad \theta(x,t) = \lambda(x,t) - \eta(t - T),$$

where M is refered to (0.5), $\gamma_\varepsilon = \gamma(1 - e^{-a_\varepsilon d})^{-1}$ and $a_\varepsilon = C - \alpha_\varepsilon \ge C \ge 1$, C being a constant to be determined later. From (1.10) it results that $\theta(x,t) \le -\lambda_0 + \eta T < 0$. An elementary computation gives

$$w_{xx} - w_t \ge \gamma_\varepsilon(q + \eta) + \gamma_\varepsilon\left\{-q - \eta - a_\varepsilon\theta\left(a_\varepsilon + s' + \frac{2\theta_x}{\theta}\right)\right\}e^{a_\varepsilon(x-s)} \text{ in } \Omega_\varepsilon$$

$$w(x,\tau) = \gamma_\varepsilon\theta(x,\tau)(1 - e^{a_\varepsilon(x-\sigma)}), \qquad \sigma - d < x < \sigma,$$

$$w(s(t),t) = 0, \qquad \tau < t < T - \varepsilon,$$

$$w(s(t) - d, t) = \gamma\theta(s(t) - d, t), \qquad \tau < t < T - \varepsilon,$$

because $\theta_{xx} - \theta_t \ge q + \eta$. Take C, not depending on ε, so large that

$$(1.13) \qquad \gamma_\varepsilon < \frac{1+\gamma}{2} \qquad \text{and} \qquad w_{xx} - w_t \ge \gamma_\varepsilon(q + \eta) \text{ in } \Omega_\varepsilon.$$

By (1.11) we have $(u-w)_t - (u-w)_{xx} \geq 0$ in Ω_ε. After a simple calculation we have by (1.10)

$$w(x,\tau) \leq \gamma_\varepsilon(\lambda_0 - \eta T)(e^{a_\varepsilon(x-\sigma)} - 1) < H(x-\sigma), \qquad \sigma - d < x < \sigma.$$

Hence, (1.9) yield $w(x,\tau) \leq u(x,\tau)$, $\sigma - d < x < \sigma$. Clearly, $(u-w)(s(t),t) = 0$, and (1.12) guarantees $u - w \geq 0$ on $x = s(t) - d$, $\tau < t < T$.

Consequently, the maximum principle implies $u > w$ in Ω_ε and

$$u_x(s(t),t) \leq -\gamma_\varepsilon a_\varepsilon \theta(s(t),t) \leq -\gamma_\varepsilon a_\varepsilon \lambda(s(t),t), \qquad \tau < t < T - \varepsilon,$$

which leads to $s'(t) \geq -\gamma_\varepsilon a_\varepsilon - \mu/\lambda$. Taking infimum with respect to $t \in (\tau, T-\varepsilon)$, we have $C' - a_\varepsilon \geq -\gamma_\varepsilon a_\varepsilon (C' = C + \mu_0/\lambda_0)$. It then follows from (1.13) that $s'(t) \geq C'(\gamma+1)(\gamma-1)^{-1}$ and hence T is not a blow-up time. This is a contradiction. □

§ 2. Preliminaries to the proof of Theorem 2

Throughout the section we assume (0.4)–(0.6) and (0.8)–(0.9) to hold.

Lemma 2.1. *Let U be a solution of equations*

$$U_{xx} - U_t = q(x,t) \ \textit{in} \ (0,B) \times (0,\infty),$$

$$(2.1) \qquad U(x,0) = \min(-\lambda_1, \tilde{h}(x)), \qquad 0 < x < B,$$

$$U(0,t) = \min(-\lambda_1, \varphi(t)), \qquad U(B,t) = -\lambda_1, \qquad 0 < t < \infty,$$

where λ_1 is a constant such that $0 < \lambda_1 < \lambda_0$ and $\tilde{h}$ is the same as in (0.2_n). Then, there exist $b_0 \in (\beta, b)$ and $T_0 \in (0, \ell)$ such that $MT_0 \leq \lambda_1$ and

$$(2.2) \qquad \lambda(x,t) < U(x,t) < 0 \qquad \textit{on } N_b = [b_0, B] \times (0, T_0].$$

Proof. It easily follows from (0.9) that there exist $b_0 \in (\beta, b)$ and $T_0 > 0$ such that

$$U(b_0,t) - \lambda(b_0,t) > 0, \qquad 0 \leq t \leq T_0,$$

$$U(x,0) - \lambda(x,0) \geq 0, \qquad b_0 < x < B.$$

Keeping in mind that $U(B,t) - \lambda(B,t) > 0$ for $t \geq 0$ and that $q + \lambda_t \leq \lambda_{xx}$ in N_b, we obtain from the maximum principle that $U - \lambda > 0$ on N_b. The same device concludes that $U + M(T_0 - t) < 0$ on $[0,B] \times (0,T_0]$. This completes the proof. □

Let $(s_n(t), u_n(x,t), T_n)$, $T_n \leq T_0$ $(n \geq N)$ be a solution of (0.1)–(0.2_n) such that $b_0 < s_n(t) < B$ for $t \in (0, T_n)$, where N is a positive integer such that $b + N^{-1} < B$.

The unique existence of such solutions is guaranteed by Fasano-Primicerio's theorem which is described at the beginning of the paper, since $\tilde{h} = 0$ near $x = b_n$. The following lemma is then immediately established.

Lemma 2.2. *Let c_n be the related function mentioned by* (0.11). *Then, we obtain that*

$$(2.3) \qquad U(x,t) < u_n(x,t) \qquad in\ D_n(T_n) = \{(x,t); 0 < x < s_n(t), 0 < t < T_n\}$$

and that $(s_n(t), c_n(x,t), T_n)$ is a solution of the related problem (1.2)–(1.3) *with b and h replaced by b_n and $\tilde{h}$, respectively, and satisfies*

$$(2.4) \qquad c_n(x,t) > 0, \qquad c_{n,x}(x,t) < 0 \qquad in\ D_n(T_n) \cap \{x > b_0\}.$$

For $\alpha > 0$ such that $b_0 + \alpha < b$ we introduce a solution $(s_\alpha(t), u_\alpha(x,t), T_\alpha)$, $T_\alpha \leq T_0$, of equation (0.1) subject to initial condition

$$(0.2_\alpha) \qquad s_\alpha(0) = b_0 + \alpha (= b_\alpha), \qquad u(x,0) = h_0(x) \text{ for } x \in (0, b_\alpha),$$

satisfying $b_0 < s_\alpha(t) < B$ for $t \leq T_\alpha$, where $h_0(x) = h(x)$ for $x \leq b_0$ and $= 0$ for $x > b_0$.

Lemma 2.3. (cf. Lemma 3.1 of [5]) *We can choose α in $(0, b - b_0)$ independently from $n \geq N$ so that*

$$(2.5) \qquad s_\alpha(t) < s_n(t), \qquad t < \min(T_\alpha, T_n),$$

$$(2.6) \qquad s_{n+1}(t) < s_n(t), \qquad t < \min(T_n, T_{n+1}).$$

Proof. We start with the proof of (2.5). Temporarily, we suppose that there exists t^*, $0 < t^* < \min(T_\alpha, T_n)$ such that $s_\alpha(t) < s_n(t)$ for $t \in (0, t^*)$ and $s_\alpha(t^*) = s_n(t^*)$. Let $c_\alpha(x,t)$ be the function related to u_α in the same manner as in (0.11).

Recalling that condition $q + \lambda_t \leq \mu_x$ in N_b implies that for $(s_i(t), t) \in N_b$ $(i = 1, 2)$

$$(2.7) \qquad R_x(x,t; s_2(t)) - R_x(x,t; s_1(t)) \leq 0 \qquad \text{if } s_1(t) \leq s_2(t),$$

and setting

$$w(x,t) = c_{n,x}(x,t) - c_{\alpha,x}(x,t),$$

$$D_\alpha(t^*) = \{(x,t); 0 < x < s_\alpha(t), 0 < t < t^*\},$$

we get,

$$w_t - w_{xx} = R_x(x,t; s_n(t)) - R_x(x,t; s_\alpha(t)) \leq 0 \qquad \text{in } D_\alpha(t^*)$$

and

$$w(x,0) = -\int_x^{b_n} (\tilde{h}(\eta) - \lambda(\eta,0))d\eta + \int_x^{b_\alpha} (h_0(\eta) - \lambda(\eta,0))d\eta$$

$$< -\int_{b_0}^{b} (h(\eta) - \lambda(\eta,0))d\eta - \int_{b_0}^{b_\alpha} \lambda(\eta,0)d\eta, \qquad 0 < x < b_\alpha.$$

Consequently, condition (0.9) leads to the existence of small α not depending on n such that $w(x,0) \le 0$ for $x \in (0, b_\alpha)$. It is evident that $w(s_\alpha(t), t) < 0$, $0 < t \le t^*$, which follows from (2.4).

To derive $w(0,t) < 0$, $0 < t \le t^*$, we temporarily assume that there exists $t_0 \in (0, t^*]$ such that $w(0, t_0) \ge w(x,t)$ in $D_\alpha(t^*)$. The maximum principle guarantees that $w_x(0, t_0) < 0$. However, $w_x = c_{n,xx} - c_{\alpha,xx} = u_n - u_\alpha = 0$ at $(0, t_0)$. This concludes that $w(0,t) < 0$, $0 < t \le t^*$. Since $w(s_\alpha(t^*), t^*) = 0$, we obtain $w_x(s_\alpha(t^*), t^*) > 0$. On the other hand, it is easily seen that $w_x(x, t^*) = 0$ for $x = s_\alpha(t^*)$. This is absurd.

The proof of (2.6) is also carried out by the same argument as above. □

Let d be a positive constant such that $b_0 < s_\alpha(t) - d$ for $t \in (0, T_\alpha)$ and $\tau_n \in (0, T_n)$. Keeping in mind (2.2) and (2.3), we find a positive constant δ_n satisfying

(2.8) $$u_n(x,t) - \lambda(x,t) \ge \delta_n \qquad \text{in } \{(x,t); b_0 < x < s_n(t), \tau_n < t < T_n\}.$$

This permits us to apply Lemma 1.2 to the solution $(s_n(t), u_n(x,t), T_n)$. As a consequence, we have

(2.9) $$T_n \ge T^* = \min(T_\alpha, T_N) \qquad \text{for } n \ge N.$$

Therefore, τ_n and δ_n may be chosen independently of n.

Owing Lemma 2.3, we can state

Lemma 2.4. *It holds that*

(2.10) $$c_n(x,t) \ge c_{n+1}(x,t) \qquad \text{in } D_{n+1}(T^*),$$

(2.11) $$c_{n,x}(x,t) \le c_{n+1,x}(x,t) \qquad \text{in } D_{n+1}(T^*).$$

Proof. To prove (2.11) we set $w = c_{n+1,x} - c_{n,x}$. Then, the same argument as in Lemma 2.3 concludes that $w_t - w_{xx} \ge 0$ in $D_{n+1}(T^*)$, $w(x,0) \ge 0$, $0 < x < b_{n+1}$, $w(0,t) \ge 0$, $0 < t < T^*$, and $w(s_{n+1}(t), t) > 0$, $0 < t < T^*$. Accordingly, $w \ge 0$ in $D_{n+1}(T^*)$. This proves (2.11) and (2.10). □

§ 3. Proof of Theorem 2 and corollaries

Let us extend the function u_n defined in $D_n(T^*)$ and the related function c_n described by (0.11) over the domain $Q^* = (0,B) \times (0,T^*)$, defining $u_n = c_n = 0$ on the rest domain, where T^* is a positive constant appearing in (2.9).

Proof of Theorem 2. We first state the following lemmas.

Lemma 3.1. *It holds that*

$$u_n \to u \in C(\bar{D}(T^*)\backslash\Gamma) \text{ weakly}^* \text{ in } L^\infty(Q^*), \tag{3.1}$$

where $\Gamma = \{(s(t), t); 0 < t < T^*\}$). *Moreover,*

$$u_x \in L^2(Q^* \cap \{x > b_0\}). \tag{3.2}$$

Proof. We can describe u_n by the same manner as in (1.5), letting ζ be a nonnegative function in $C^\infty(\bar{Q}^*)$ such that $\zeta = 1$ for $x < s(t) - \varepsilon$ and $\zeta = 0$ for $x > s(t)$, in the integral form:

$$\begin{aligned}(\zeta u_n)(x,t) = &\int_0^b \zeta(\xi,0)h(\xi)G(x,t;\xi,0)d\xi \\ &+ \int_0^t \varphi(\tau)G_\xi(x,t;0,\tau)d\tau \\ &- \iint_{D_n(t)} (\zeta q - (\zeta_{\xi\xi} + \zeta_\tau)u_n)(\xi,\tau)G(x,t;\xi,\tau)d\xi d\tau \\ &+ \iint_{D_n(t)} 2(\zeta_\xi u_n)(\xi,\tau)G_\xi(x,t;\xi,\tau)d\xi d\tau,\end{aligned} \tag{3.3}$$

where ε denotes a small positive constant. It easily follows from the maximum principle that $\{u_n\}$ is bounded in $L^\infty(Q^*)$, we may choose a subsequence, again denoted by $\{u_n\}$, so that u_n converges weakly* in $L^\infty(Q^*)$ to a function u. As is easily seen from (3.2), it holds that u_n converges pointwise to u, which is bounded and continuous on $\bar{D}(T^*)\backslash\Gamma(\Gamma = \{(s(t), t); 0 < t < T^*\})$. This proves (3.1).

To deal with (3.2) we introduce an infinitely differentiable function $\chi(x)$ so that $\chi(0) = 0$ and $\chi(x) = 1$ for $x > b_0$. Set $v = \chi u_n$. Multiplying the equation

$$v_{xx} - v_t = \chi q + 2\chi' u_{n,x} + \chi'' u_n (= r)$$

by v and integrating with respect to τ over $(0, t)$ after integrating with respect to ξ over $(0, s_n(\tau))$, we have

$$\frac{1}{2}\int_0^B (v(\xi,t)^2 - v(\xi,0)^2)d\xi + \iint_{D_n(t)} (v_\xi^2 + rv)d\xi d\tau = 0.$$

A simple calculation gives

$$\begin{aligned}v_\xi^2 + rv &= (\chi u_{n,\xi})^2 + 4\chi u_{n,\xi}\chi' u_n + (\chi' u_n)^2 + \chi^2 q u_n + \chi\chi'' u_n^2 \\ &\geq \frac{1}{2}(\chi u_{n,\xi})^2 - 7(\chi' u_n)^2 + \chi^2 q u_n + \chi\chi'' u_n^2\end{aligned}$$

and hence

$$\int_0^{s(t)} \chi^2 u_n(\xi,t)^2 d\xi + \iint_{D_n(t)} (\chi u_{n,\xi})^2 d\xi d\tau$$
$$\leq \int_0^b \chi^2 h(\xi)^2 d\xi + 2\iint_{D_n(t)} \{7(\chi' u_n)^2 - \chi^2 q u_n - \chi\chi'' u_n^2\} d\xi d\tau.$$

This implies (3.2), since u_n and s_n are uniformly bounded. □

Lemma 3.2. *The sequences of functions* $\{c_n\}$, $\{c_{n,x}\}$ *and* $\{s_n\}$ *monotonically and uniformly converge to* c, c_x *and* s *on* $\bar{Q}^*$, $\bar{Q}^*$ *and* $[0, T^*]$, *respectively and the triple* $(s(t), c(x,t), T^*)$ *becomes a solution of* (1.2)–(1.3) *satisfying* (0.12). *Moreover, it holds that* $(s(t), c(x,t)) = (\tilde{s}(t), \tilde{c}(x,t))$, $t \in (0, \min(T^*, \tilde{T}))$, *for any solution* $(\tilde{s}(t), \tilde{c}(x,t), \tilde{T})$ *of* (1.2)–(1.3) *satisfying* (0.12) *and* $\tilde{s} \in C^1(0, \tilde{T})$.

Proof. Rewrite $c_n(x,t)$ in the integral form like (1.5). From Lemmas 2.3 and 2.4 it follows that c_n, $c_{n,x}$ and s_n, monotonically converge to c, c_x and s, respectively. Letting $n \to \infty$, we have (1.5). Consequently, s is upper semicontinuous and c is continuous on $\bar{Q}^*$. The Dini theorem leads to uniform convergence of $c_n \to c$. It is easy to see by (0.9) that $c(x,0) > 0$ for $x < b$. On the other hand we derive from (2.8) with δ_n replaced by δ independent of n that $c_n(x,t) \geq (\delta/2)(x - s_n(t))^2$. Therefore, $c(x,t) > 0$ for $x < s(t)$ and $c(x,t) = 0$ for $x > s(t)$. Accordingly, s is lower semicontinuous and hence continuous on $[0, T^*]$. The Dini theorem leads to the uniform convergence of $s_n \to s$.

We now consider the sequence $\{c_{n,x}\}$. By definition we have

$$c_{n,x}(x,t) = \int_{s_n(t)}^{x} \{u_n(\eta,t) - \lambda(\eta,t)\} d\eta, \qquad x < s_n(t). \tag{3.4}$$

Letting n to infinity in (3.4), we conclude that c_x is continuous on Q^* and $c_x = 0$ on Γ, which concludes the first part of the lemma.

We now prove the second part. To do so let us assume that $\tilde{T} \leq T^*$ and that $b_0 < \tilde{s}(t) < B$ for $t \in (0, \tilde{T})$. By the same argument as in Lemmas 2.3 and 2.4 we can prove that $\tilde{s}(t) < s_n(t)$ and $c_{n,x} - \tilde{c}_x \leq 0$, $0 < x < \tilde{s}(t)$, $0 < t < \tilde{T}$. Hence, $\tilde{s}(t) \leq s(t)$ and $v_x \leq 0$, where $v = c - \tilde{c}$. Observing that the second condition of (0.8) implies

$$R(x,t;s_2(t)) - R(x,t;s_1(t)) \leq 0 \qquad \text{if } s_1(t) \leq s_2(t) \text{ and } (s_i(t),t) \in N_b \ (i = 1,2),$$

we have $v(0,t) \leq 0$, $0 < t < \tilde{T}$. So that $v(\tilde{s}(t),t) = 0$. This leads to $v = 0$ and $\tilde{s} = s$. □

From Lemmas 3.1 and 3.2 it easily follows that the functions u, c and s fulfill the relations (1.1) and $u(x,t) = c_t(x,t) + R(x,t;s(t))$. To complete the proof of

Theorem 2 we employ Lemma 1.1. To do so it remains to verify that $s \in C^1(0, T^*)$. It follows from (3.2) that $u_x(\cdot, t) \in L^2(b_0, B)$ for almost all $t \in (0, T^*)$. Hence, we can choose a monotone sequence $t_i \in (0, T^*)$ converging to zero as $i \to \infty$ so that

$$u(x, t_i) \in L^\infty(0, B) \cap C(0, B) \qquad \text{and} \qquad |u(x, t_i)| \le H_i(b_i - x)^{1/2},\ x < b_i = s(t_i),$$

for some positive constant H_i. By Fasano-Primicerio's existence theorem (see [3, II]) we have a solution $(s_i(t), u_i(x, t), T_i - t_i)$ of (0.1) subject to initial condition on $t = t_i$:

$$(0.2_i) \qquad s_i(t_i) = b_i,\ u_i(x, t_i) = u(x, t_i) \qquad \text{for } x \in (0, b_i).$$

It easily follows from the last part of Lemma 3.2 that the related function $c_i(x, t)$ is equal to $c(x, t)$ and hence $s_i(t) = s(t)$ and $s \in C^1(t_i, \min(T_i, T^*))$.

To prove $s \in C^1(0, T^*)$ we have only to verify $T_i \ge T^*$. Suppose that $T_i < T^*$. Recalling (2.3) and (2.9), we have $U \le u$ in $D(T^*)$, and hence $U(x, t) \le u_i(x, t)$ in $D(T_i)\backslash D(t_i)$. Therefore, (2.2) implies that $u_i - \lambda \ge \delta_i$ in $\{(x, t); b_0 < x < s_i(t), t_i < t < T_i\}$ for some $\delta_i > 0$. This enables to apply Lemma 1.2 to solution $(s_i(t), u_i(x, t), T_i - t_i)$ to get $T_i \ge T^*$.

It easily follows from Lemma 1.1 that u satisfies (0.1)–(0.3). The uniqueness of u is guaranteed by the last part of Lemma 3.2. This completes the proof of Theorem 2. □

Proof of Corollaries. By the maximum principle we have $u(x, t) - \lambda(x, t) > 0$ in $D(T)$ for any T such that $s(t) > 0$, $t < T$. The proof of Corollary 1 is thus established by using Lemma 1.2.

Contrary to the assertion in Corollary 2 we assume that there exists an extinction time T. It is easy to see that $\liminf_{t \to T} c(0, t) = 0$. If $\limsup_{t \to T} c(0, t) > 0$, we then find a sequence $\{t_j\}$ converging to T such that $c_t(0, t_j) = 0$ and $s(t_j) \to 0$ as $t_j \to T$. Inserting this into $c_t(0, t) = \varphi(t) + R(0, t; s(t))$ and letting $j \to \infty$, we have $c_t(0, t_j) \to \varphi(T) > 0$. This contradicts $c_t(0, t_j) = 0$. Therefore, we may assume that $\lim_{t \to T} c(0, t) = 0$, which implies $s(t) \to 0$ as $t \to T$. However, (1.2) leads to $c(0, t) \to c_0(0) + \int_0^T \varphi(t) dt > 0$ as $t \to T$, which proves Corollary 2. □

§ 4. Application to a Bingham flow

We now consider a Bingham fluid which flows in the z-direction through a domain in $\boldsymbol{R}^3$ bounded by two planes $x = 0$ and $x = L$, and is governed by the equations:

$$v_t = \sigma_x + f \qquad \text{in } Q = D \times (0, \infty),\ D = (0, L),$$
$$v(0,t) = v(L,t) = 0 \qquad \text{in } (0, \infty),$$
$$\sigma = v_x + k\frac{v_x}{|v_x|} \qquad \text{where } v_x \neq 0 \text{ and } |\sigma| \leq k \text{ otherwise},$$

where v is the velocity, σ the share stress, k the yield limit and f the external force. Under the condition that

$$F(x,y,t) \equiv \frac{1}{y-x}\left(\int_x^y f(\xi,t)d\xi - 2k\right) < \min\{f(x,t), f(y,t)\} \tag{4.1}$$
$$\text{in the set } \Delta = \{(x,y,t); 0 \leq x < y \leq L, t \geq 0\},$$

the Bingham flow can be reduced to Problem: To find two triples $(s_i(t), v_i(x,t), T)$ with $0 < s_1(t) < s_2(t) < L$ satisfying

$$\begin{aligned} &v_{i,t} - v_{i,xx} = f(x,t),\ (s_i(t)-x)v_{i,x}(x,t) > 0 \qquad \text{in } D_i(T),\\ &v_1(0,t) = v_2(L,t) = 0, \qquad 0 < t < T,\\ &v_{1,t}(s_1(t),t) = v_{2,t}(s_2(t),t) = F(s_1(t), s_2(t), t), \qquad 0 < t < T\\ &v_{i,x}(s_i(t),t) = 0, \qquad 0 < t < T,\\ &s_i(0) = b_i,\ v_i(x,0) = v_0(x), \qquad x \in (0,b_1) \cup (b_2, L) \end{aligned} \tag{4.2}$$

and

(4.3)
$$s_i \in C([0,T]) \cap C^1(0,T), \qquad v_i, v_{i,x} \in C(\bar{D}_i(T)), \qquad v_{i,t} \in C(\bar{D}_i(T) \backslash (\bar{D} \times \{0\}))$$

where and in what follows

$$D_1(T)(\text{resp. } D_2(T)) = \{(x,t); 0 < x < s_1(t)(\text{resp. } s_2(t) < x < L), 0 < t < T\}$$

and we assume

$$v_0 \in C^2(\bar{D}) \text{ with } v_0 = 0 \text{ on } \partial D, \quad v_0'(b_i) = 0, \quad v_0(b_i) = V_0 > 0 \quad (i = 1,2). \tag{4.4}$$

The condition (4.1) corresponds to the physical request that the external force must exceed the force acting on the plastic part: $s_1(t) < x < s_2(t)$, $0 < t < T$ (see Additional remark mentioned at the end of the section).

The share stress σ which is governed by the constitutive law which is stated at the beginning of the section is then given by $\sigma = v_{1,x} + k$ in $D_1(T), = v_{2,x} - k$

in $D_2(T)$ and

$$\sigma(x,t)=(x-s_1(t))F(s_1(t),s_2(t),t)-\int_{s_1(t)}^{x}f(\xi,t)d\xi+k \qquad \text{otherwise.}$$

Moreover, after an elementary computation we have

$$\sigma(x,t)>-k \iff F(s_1(t),s_2(t),t)>F(s_1(t),x,t), \quad s_1(t)<x<s_2(t),$$

$$\sigma(x,t)<k \iff F(s_1(t),s_2(t),t)>F(x,s_2(t),t), \quad s_1(t)<x<s_2(t)$$

and

$$(4.1) \iff F_x(x,y,t)<0 \text{ and } F_y(x,y,t)>0 \text{ in } \Delta.$$

Hence, it follows that $|\sigma(x,t)|<k$, $s_1(t)<x<s_2(t)$.

It is easy to see that the function $u_i(x,t)=-v_{i,t}(x,t)$ satisfies the equations of type (0.1)–(0.2):

$$(4.5) \qquad u_{i,xx}-u_{i,t}=q_i(x,t) \qquad \text{in } D_i(T),$$

$$u_1(0,t)=\varphi_1(t),\ u_2(L,t)=\varphi_2(t) \qquad 0<t<T,$$

$$u_i(s_i(t),t)=-F_i(s_1(t),s_2(t),t), \qquad 0<s_1(t)<s_2(t)<L,\ 0<t<T,$$

$$u_{i,x}(s_i(t),t)=(F_i(s_1(t),s_2(t),t)+\lambda_i(s_i(t),t))s_i'(t)+\mu_i(s_i(t),t), \qquad 0<t<T,$$

$$s_i(0)=b_i,\ u_1(x,0)\ (\text{resp. } u_2(x,0))=h(x), \qquad x\in(0,b_1)\ (\text{resp.} \in (b_2,L)),$$

where

$$q_i=f_t, \quad \varphi_i=0, \quad F_i=F, \quad \lambda_i=-f, \quad \mu_i=0, \quad h(x)=-f(x,0)-v_0''(x).$$

Conversely, letting $(s_i(t),u_i(x,t),T)$ be a solution of (4.5) satisfying

$$(4.6) \qquad s_i\in C([0,T])\cup C^1(0,T), \qquad u_i,u_{i,x}\in C(\bar{D}_i(T)\backslash(\bar{D}\times\{0\})),$$

we can establish that the functions

$$(4.7) \qquad v_i(x,t)=V_0+\int_0^t F(s_1(\tau),s_2(\tau),\tau)d\tau-\int_{s_i(t)}^{x}d\xi\int_{s_i(t)}^{\xi}(u_i(\eta,t)+f(\eta,t))d\eta$$

solve equations (4.2)–(4.3) except for the inequality $(s_i(t)-x)v_{i,x}(x,t)>0$ in $D_i(T)$.

The problem (4.2)–(4.3) has been discussed in detail in Comparini [1], provided that the function f depends only on t (see also Sekimoto [8, 9]). To deal with this problem we consider here the problem (4.5)–(4.6), assuming that F_1, F_2 are smooth functions defined on the set Δ, that $0<b_1<b_2<L$, that q_i, φ_i, λ_i, μ_i are given functions sufficiently smooth on $\bar{Q}$, that there exist positive

constants M and μ_0 such that

$$(4.8)\qquad \begin{aligned} &|q_i(x,t)| \leq M,\ |\mu_i(x,t)| \leq \mu_0 \qquad \text{on } \bar{Q}, \\ &\lambda_1(x,t) + F_1(x,y,t) < 0,\ \lambda_2(y,t) + F_2(x,y,t) < 0 \qquad \text{on } \Delta \end{aligned}$$

and that

(4.9) h is a piecewise continuous function belonging to $L^{\infty}(D)$.

If F_1 and F_2 identically vanish, this problem is nothing but the one considered in Theorems 1 and 2. Employing the similar argument as in [3], we can construct a unique solution of (4.5)–(4.6), if the initial data h possesses the properties

$$|h(x) + F_i(b_1, b_2, 0)| \leq \text{const.}|x - b_i|^{\alpha} (\alpha > 0), \qquad i = 1, 2.$$

It is easy to see that Theorems 1 and 2 are able to be generalized as follows:

Theorem 1′. *Suppose that conditions* (4.8) *and* (4.9) *hold. There then exists no solution of equations* (4.5)–(4.6) *if there exists* $i \in \{1,2\}$ *such that*

$$\int_x^b d\xi \int_\xi^b (h(\eta) - \lambda_i(\eta, 0)) d\eta \leq 0 \qquad \text{near } x = b_i.$$

Theorem 2′. *Suppose that conditions* (4.8) *and* (4.9) *hold. There then exists a unique time-local solution of equations* (4.5)–(4.6) *if the following conditions hold:*

$$(4.10)\qquad \begin{aligned} &q_1(x,t) + \lambda_{1,t}(x,t) \leq \min\{\lambda_{1,xx}(x,t), \mu_{1,x}(x,t)\} \\ &q_2(y,t) + \lambda_{2,t}(y,t) \leq \min\{\lambda_{2,yy}(y,t), \mu_{2,y}(y,t)\} \\ &F_{1,x}(x,y,t) + \mu_1(x,t) + x\{\mu_{1,x}(x,t) - q_1(x,t) - \lambda_{1,t}(x,t)\} \leq 0 \\ &F_{2,y}(x,y,t) + \mu_2(y,t) + (y - L)\{\mu_{2,y}(y,t) - q_2(y,t) - \lambda_{2,t}(y,t)\} \geq 0 \end{aligned}$$

in a neighborhood of $(x,y,t) = (b_1, b_2, 0)$ *and*

$$(4.11)\quad h(x) - \lambda_i(x,0) \geq 0,\ \int_x^{b_i} (h(\xi) - \lambda_i(\xi, 0)) d\xi \neq 0 \qquad \textit{near } x = b_i\ (i = 1, 2).$$

Corollary 1′. *In addition to conditions* (4.10) *and* (4.11) *we assume that*

$$(0.13')\qquad \begin{aligned} &q_i(x,t) + \lambda_{i,t}(x,t) \leq \lambda_{i,xx}(x,t) \ \textit{in } Q\ (i = 1, 2), \\ &h(x) - \lambda_1(x,0) \geq 0 \ \textit{in } (0, b_1), \qquad h(x) - \lambda_2(x,0) \geq 0 \ \textit{in } (b_2, L), \\ &\varphi_1(t) - \lambda_1(0,t) \geq 0, \qquad \varphi_2(t) - \lambda_2(L,t) \geq 0 \ \textit{in } (0, \infty) \end{aligned}$$

hold. Then, blow-up never occurs.

The proof will be accomplished by introducing the related two functions

$$c_i(x,t) = \int_{s_i(t)}^{x} d\xi \int_{s_i(t)}^{\xi} \{u_i(\eta,t) - \lambda_i(\eta,t)\} d\eta \qquad \text{in } D_i(T),$$

which, together with $(s_i(t), T)$, satisfy $c_{i,t}(x,t) = u_i(x,t) + R_i(x,t;s_1,s_2)$ and the equations like (1.2)–(1.3), where

$$R_i(x,t;s_1,s_2) = F_i(s_1(t),s_2(t),t) + (s_i(t)-x)\mu_i(s_i(t),t)$$

$$- \int_{s_i(t)}^{x} d\xi \int_{s_i(t)}^{\xi} \{q_i(\eta,t) + \lambda_{i,t}(\eta,t)\} d\eta.$$

It goes without saying that the extinction time T is nothing but the time when the flow ceases ($v(x,t) > 0$ for $t < T$ and $\liminf_{t\to T} v(x,t) = 0$ for all x).

In Corollary 1′ it may occurs that $s_2(t) - s_1(t) \to 0$ as t tends to some T_1. However, we can state the following result.

Proposition 4.1. *Suppose that* (4.1), (4.4) *and*

(4.12) $\quad f(x,t) \geq 0, f_{xx}(x,t) \leq 0$ *in* Q, $\quad v_0''(x) \leq 0$ *in* $(0,b_1) \cup (b_2,L)$,

(4.13) $\quad v_0'(x) \neq 0$ *in* $(\beta_1,b_1) \cup (b_2,\beta_2)$ (*for some* $\beta_1 \in [0,b_1), \beta_2 \in (b_2,L]$)

hold. Then, there exists a unique solution of (4.2)–(4.3) *until the flow ceases.*

Proof. The hypothesis (4.12) and (4.13) implies conditions (4.10), (4.11) and (0.13′). Corollary 1′ gives a solution $(s_i(t), u_i(x,t), T)$ of (4.5)–(4.6). By the transformation (4.7) we see that $(s_i(t), v_i(x,t), T)$ becomes a unique solution of (4.2)–(4.3), since $u_i + f > 0$ in $D_i(T)$.

To complete the proof of the proposition we have only to show that there is no time T_1 such that $s_2(t) - s_1(t) \to 0$ as $t \to T_1$, in the other words, the boundedness of $v_{i,t}$, because $F(s_1(t), s_2(t), t) \to -\infty$ as $s_2(t) - s_1(t) \to 0$. To do so we extend v_i onto $D \times (0,T)$ by setting $v(x,t) = v_i(s_i(t),t)$ on $s_1(t) < x < s_2(t)$ and introduce a function $g(x)$ on $D = (0,L)$ for a fixed $t_0 \in (0,T)$, setting $\beta_i = s_i(t_0)$,

$$g(x) = \begin{cases} f(x,t_0) - v_{1,t}(x,t_0), & 0 < x < \beta_1, \\ f(x,t_0) - F(\beta_1,\beta_2,t_0), & \beta_1 \leq x \leq \beta_2, \\ f(x,t_0) - v_{2,t}(x,t_0), & \beta_2 < x < L. \end{cases}$$

It is easy to see that $g(x) > 0$ and $\int_D g(x)dx > 2k$. We then have

$$v(x, t_0) = \begin{cases} \int_0^x d\xi \int_\xi^{\beta_1} g(\eta)d\eta, & 0 < x < \beta_1, \\ \int_0^{\beta_1} \eta g(\eta)d\eta, & \beta_1 \le x \le \beta_2, \\ \int_x^L d\xi \int_{\beta_2}^{\xi} g(\eta)d\eta, & \beta_2 < x < L, \end{cases}$$

which is a stationary solution of (4.2) with f replaced by g.

We now approximate the constitutive law by $\sigma = S_\varepsilon(v_x)$, $\varepsilon > 0$, and construct a solution $v_{g,\varepsilon}$ of the elliptic equation $-S_\varepsilon(v_x)_x = g$ satisfying $v = 0$ on ∂D, where $S_\varepsilon(\eta) - \eta$ is a smooth and increasing odd function such that $S_\varepsilon(\eta) - \eta = k$ for $\eta > \varepsilon$ and $S_\varepsilon''(\eta) \le 0$ for $\eta > 0$. By v_ε we denote a solution of the nonlinear parabolic equation $v_{\varepsilon,t} - S_\varepsilon(v_{\varepsilon,x})_x = f$ in $Q_0 = D \times (t_0, T)$ such that $v_\varepsilon = v_{g,\varepsilon}$ on B and $v_\varepsilon = 0$ on A.

We are going to prove that $v_{\varepsilon,t}$ is uniformly bounded in Q_0. Doing so, we introduce two functions:

$$\Psi(t) = \int_0^t \max_{x \in D} f_t(x, \tau)d\tau \quad \text{and} \quad \psi(t) = \int_0^t \min_{x \in D} f_t(x, \tau)d\tau.$$

Set $Z = \Psi - v_{\varepsilon,t}$. We immediately obtain

$$Z_t - \{S_\varepsilon'(v_{\varepsilon,x})Z_x\}_x = \Psi_t - f_t \ge 0 \quad \text{in } Q_0,$$

$$Z(x, t_0) = \Psi(t_0) + g(x) - f(x, t_0) \quad \text{in } D,$$

$$Z(0, t) = Z(L, t) = \Psi(t) \quad \text{in } (t_0, T).$$

Applying the maximum principle to Z, we conclude that Z is uniformly bounded from below with respect to ε. Analogously, it follows that $z = \psi - v_{\varepsilon,t}$ is uniformly bounded from above with respect to ε.

The usual argument guarantees that there exists a subsequence of v_ε, again denoted by v_ε, such that $v_{\varepsilon,t}$ converges to v_t weakly* in $L^\infty(Q_0)$ (cf. Chap. VI of [2]). □

Concerning the global existence we can establish, using the same technique as in Corollary 2,

Proposition 4.2. *Suppose that the same hypothesis as in Proposition* (4.1) *is valid. If it holds that*

$$\int_0^L f(x, t)dx > 2k \quad \textit{for all } t, \tag{4.14}$$

then, there exists a unique solution of (4.2)–(4.3) *up to an arbitrarily large time. Conversely, if it holds that*

$$(4.15) \qquad \int_0^L f(x,t)dx < 2k \text{ for all } t, \qquad \int_0^L \int_0^{T_1} f(x,t)dxdt - 2kT_1 < -LV_0,$$

then, there exists an extinction time $T < T_1$.

Proof. Let us suppose that there exists an extinction time T. For brevity we set

$$(4.16) \qquad V(t) = V_0 + \int_0^t F(s_1(\tau), s_2(\tau), \tau)d\tau, \qquad 0 \le t < T.$$

Since $u_i + f > 0$ $(i = 1, 2)$ in $D_i(T)$, it then follows from (4.7) that

$$0 \le \int_{s_i(t)}^x d\xi \int_{s_i(t)}^{\xi} (u_i(\eta, t) + f(\eta, t))d\eta \le \int_{s_i(t)}^{o_i} d\xi \int_{s_i(t)}^{\xi} (u_i(\eta, t) + f(\eta, t))d\eta = V(t),$$

where $o_1 = 0$, $o_2 = L$. Evidently, we have $\liminf_{t \to T} V(t) = 0$. If $\limsup_{t \to T} V(t) > 0$, we can find a sequence $\{t_j\}$ converging to T such that $V_t(t_j) = 0$ and $V(t_j) \to 0$ as $t_j \to T$. Since $u_i + f > 0$ in $D_i(T)$, we have $s_1(t_j) \to 0$ and $s_2(t_j) \to L$ as $t_j \to T$. Hence, (4.16) guarantees that $0 = F(s_1(t_j), s_2(t_j), t_j) \to F(0, L, T)$ as $j \to \infty$. This contradicts (4.14) and (4.15). So that it is allowed to assume that $\lim_{t \to T} V(t) = V(T) = 0$, which results $s_1(t) \to 0$, $s_2(t) \to L$ as $t \to T$ and $V_t(T) \le 0$.

We first suppose (4.14) to hold. Then, $V_t(T) = F(0, L, T) > 0$. This concludes the first part of the proposition. Defining as $s_1(t) = 0$ and $s_2(t) = L$ for $t \ge T$ we get a continuous function $V(t)$ for all $t > 0$. Condition (4.15) implies $V(T_1) < 0$, which prove $T < T_1$. □

Additional remark. It is conjectured that Proposition 4.1 will remain valid even if condition (4.1) is fulfilled at only one point $(x, y, t) = (b_1, b_2, 0)$. In fact, this is true for the symmetric case that $f(x, t) = f(L - x, t)$ and $v_0(x) = v_0(L - x)$ for all x. The detail will appear elsewhere.

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nuna adreso:

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