On the Summability of the Formal Solutions of a Class of
Inhomogeneous Linear Difference Equations

By

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Abstract. We discuss different, equivalent, ways of summing the formal solution \( \hat{f} \) of
a particular inhomogeneous linear difference equation possessing a 'level 1'\(^+\). We
prove that \( \hat{f} \) is Borel-summable in the directions of the right half-plane. Moreover,
the formal Borel transform \( \varphi \) of \( \hat{f} \) is weakly accelerable in the direction of
the negative real axis by means of an acceleration operator \( \mathcal{A}_\theta \) corresponding to
the change of variable from \( z \) to \( \psi_\theta(z) := z(\log z + i\theta) \) for any \( \theta \in \mathbb{R} \). The resulting weak
accelerate \( \varphi_\theta \) is quasi-analytic on the negative real axis for all but a countable
number of values of \( \theta \). Its Laplace transform, in the variable \( \psi_\theta(z) \), is an analytic
solution of the equation. This solution can also be represented as a Laplace integral
of \( \varphi \), over a suitable, in general non-rectilinear, path.

For singular values of \( \theta \), \( \varphi_\theta \) has quasi-analytic singularities at the points \( e^{(2n+1)i\theta} \),
\( n \in \mathbb{Z} \). The nature of these singularities is described by a quasi-analytic version of
the 'bridge equation'.

These results form an illustration of the theory of weak acceleration operators and
'cohesive' functions of J. Ecalle.

Introduction

This paper is concerned with the inhomogeneous linear difference equation

\[
(0.1) \quad y(z) - a(z)y(z + 1) = b(z)
\]

where \( a, b \in z^{-1}C\{z^{-1}\} \) and \( \lim_{z \to \infty} za(z) \neq 0 \).

The corresponding homogeneous equation has a formal solution \( \hat{y}_0 \) of the form

\[
(0.2) \quad \hat{y}_0(z) = \exp\,(z \log z + \lambda z)z^{\sigma - 1/2}z^0(z)
\]

where \( \lambda, \sigma \in C \) and \( f^0 \) is a formal power series in \( z^{-1} \), which is Gevrey of order
1\(^0\). The equation (0.1) possesses a unique formal solution \( \hat{f} = \sum_{n=1}^{\infty} \phi_n z^{-n} \) with
the property that

\[1\] The series \( \sum_{n=0}^{\infty} a_n z^{-n} \) is Gevrey of order \( k \) if there exist positive constants \( A \) and \( C \) such that
\( |a_n| \leq CA^n(n!)^{1/k} \) for all \( n \in \mathbb{N} \).
\(|\phi_n| \leq CA^n \left( \frac{n}{\log n} \right)^n\), where \(C > 0, A > 0\)

(cf. [6], [13, proposition 1.11]). This property is slightly stronger than the
Gevrey condition of order 1, but weaker than the Gevrey condition of order \(k\)
for any \(k > 1\). It is closely related to the occurrence of a dominant factor of the
type \(\exp(z \log z)\) in the formal solution (0.2). It can be shown that, in general,
the above estimates cannot be improved. Accordingly, we shall say that equa-
tion (0.1) has the levels 1 and \(1^+\). Equation (0.1) is the simplest nontrivial
example of a difference equation in which level \(1^+\) occurs. A linear difference
equation always possesses the level 1 and may possess any number of additional
levels < 1.

Formal solutions of linear (and nonlinear) differential equations may be
summed, in a canonical way, to actual, analytic solutions by means of a
summation process introduced by J. Ecalle (cf. [7, 8, 9, 1, 2, 17, 16]). This
process is a generalization of the Borel summation in the case of an equation
of level 1. It involves so-called acceleration operators, which 'accelerate' from
one level to the next. The question arises naturally whether similar techniques
can be applied to formal solutions of difference equations. The most im-
portant difference with the situation in the theory of differential equations
seems to lie in the occurrence of the level \(1^+\). In [6], J. Ecalle has made a
rather detailed study of equation (0.1), in order to explain the consequences of
the presence of this level. In the present paper we further elaborate this
example. We study the relation between Ecalle's results and the existence
(proved in [12] and [15] for a larger class of equations) of particular analytic
solutions of (0.1), and show that these solutions can be interpreted as sums of
the formal solution \(\hat{f}\).

The paper is arranged as follows. In §1 we define an infinite number of
analytic solutions of (0.1), each of which is characterized by the fact that it is
represented asymptotically by the formal series \(\hat{f}\) as \(z \to \infty\) in a particular
region of the complex plane. We discuss the relation between these solu-
tions. This analysis yields an infinite number of analytic invariants of
the equation and we prove the completeness of this set of invariants.

In §2 we study the properties of the formal Borel transform \(\varphi\) (of level 1)
of the formal solution \(\hat{f}\). It is an entire function of the complex variable \(s\),
with exponential growth as \(s \to \infty\) in a right half plane. As \(s \to \infty\) parallel to
the negative real axis it may grow either superexponentially or exponentially,
depending on the value of \(\text{Im } s\). We show that each of the solutions defined in
§1 can be represented by a Laplace integral of \(\varphi\) over a suitable path.

In §3 we introduce so-called 'weak accelerates' \(\varphi_\theta\) of \(\varphi\) corresponding to a
change of variable from \(z\) to the 'faster' variable \(z(\log z + i\theta)\), for different real
values of \( \theta \). For every \( \theta \in \mathbb{R}, \varphi_\theta \) is an analytic function of a complex variable \( t \) in those regions of the Riemann surface of \( \log t \) where \( \Re t > -1 \). Moreover, it is quasi-analytic (more precisely ‘cohesive’) on the half lines \( \arg t = (2n + 1)\pi, n \in \mathbb{Z} \), for all but a countable number of ‘singular’ values of \( \theta \). This is in agreement with a result of J. Ecalle, to the effect that ‘weak accelerates are cohesive’ (cf. [5]). Each of the solutions of (0.1) defined in §1 can be represented as the Laplace transform, in the variable \( z(\log z + i\theta) \), of \( \varphi_\theta \), for an appropriate value of \( \theta \), the path of integration being either the positive or the negative real axis. The formal solution \( \hat{f} \) is, so to speak, accelero-summable in the direction \( k\pi \) for all \( k \in \mathbb{Z} \).

Finally, in §4, we derive a quasi-analytic version of the ‘bridge equation’ for each singular value of \( \theta \). This equation describes the nature of the quasi-analytic singularity of \( \varphi_\theta \) at the point \( e^{\pi i} \). Each equation involves exactly one of the invariants of level \( 1^+ \) found in §1.

1. Analytic classification of Equation (0.1)

We write

\[
(1.1) \quad a(z) = e^{-\lambda-1}(z + \sigma)^{-1}\{1 + c(z)\}
\]

where \( c \in z^{-2}\mathbb{C}\{z^{-1}\} \) and we choose \( \Im \lambda \in [0, 2\pi) \). The numbers \( \lambda \) and \( \sigma \) are the ‘formal invariants’ of (0.1), i.e. they are invariant with respect to a change of the dependent variable of the type \( y \mapsto Fy + G \), with \( F, G \in \mathbb{C}[\llbracket z^{-1} \rrbracket] \). The homogeneous equation

\[
(1.2) \quad y(z) = a(z)y(z + 1)
\]

has two solutions \( y_+^0 \) and \( y_-^0 \) defined by

\[
(1.3) \quad y_+^0(z) = \Gamma(z + \sigma)e^{(\lambda+1)z}\prod_{n=0}^{\infty}\{1 + c(z + n)\}
\]

and

\[
y_-^0(z) = 2\pi i \Gamma(1 - z - \sigma)^{-1} e^{(\lambda+1-\sigma)z - i\pi \sigma}\prod_{n=1}^{\infty}\{1 + c(z - n)\}^{-1}
\]

\( y_+^0 \) and \( y_-^0 \) are analytic in the regions \( D_+ \) and \( D_- \), respectively, where

\[
D_{\pm} = \{z \in \mathbb{C} : \pm \Re z > R \text{ or } |\Im z| > R\}
\]

\( R \) is a positive number such that both \( a \) and \( b \) are analytic in \( D_+ \cup D_- \). \( y_+^0 \) and \( y_-^0 \) are represented asymptotically by \( \hat{y}_0^0 \) (cf. (0.2)) as \( z \to \infty \), \( |\arg z| < \pi - \varepsilon \) and as \( z \to \infty \), \( -2\pi + \varepsilon < \arg z < -\varepsilon \), respectively, for any \( \varepsilon > 0 \). (Note: in the
evaluation of multiform functions, we will always use the branch corresponding
to arg $z \in (-\pi, 0)$ in $D_-$ and to arg $z \in (-\pi, \pi)$ in $D_+$. Obviously, the quotient

$$p_0^0(z) := y_+^0(z)y_-^0(z)^{-1} = \left\{1 - e^{-2niz(z+\sigma)}\right\}^{-1} \prod_{n=-\infty}^{\infty} \{1 + c(z+n)\}$$

is a periodic function of period 1, which tends to 1 as $\operatorname{Im} z \to -\infty$. Moreover,
there exist complex numbers $P_n^0$, $n \in \mathbb{Z}$, such that

$$\log \left\{\{1 - e^{-2niz(z+\sigma)}\}p_0^0(z)\right\} = \begin{cases} 
\sum_{n=1}^{\infty} P_n^0 e^{2niz} & \text{if } \operatorname{Im} z > R \\
-\sum_{n=-\infty}^{-1} P_n^0 e^{2niz} & \text{if } \operatorname{Im} z < -R 
\end{cases}$$

It will be shown that the numbers $P_n^0$ are analytic invariants of the equation (0.1). We will call them analytic invariants of level 1.

Equation (0.1) possesses a solution $f$ defined by

$$f(z) = b(z) + \sum_{n=1}^{\infty} a(z)…a(z+n-1)b(z+n)$$

or, equivalently,

$$f(z) = y_+^0(z) \int_{C(z)} y_+^0(\zeta)^{-1} \left\{1 - e^{2niz(z-\zeta)}\right\}^{-1} b(\zeta) d\zeta$$

where $C(z)$ is the contour consisting of the half line from $z - 1/2 + i\delta + \infty$ to
$z - 1/2 + i\delta$, the segment $[z - 1/2 + i\delta, z - 1/2 - i\delta]$ and the half line from
$z - 1/2 - i\delta$ to $z - 1/2 - i\delta + \infty$ with $\delta > 0$ (cf. fig. 1). $f$ is analytic in $D_+$ and
represented asymptotically by $f$ as $z \to \infty$ in any direction but that of the negative real axis.

Let $\operatorname{Re} z < -R$ and let $p, q \in D_+ \cap D_-$ such that $\operatorname{Re} p = \operatorname{Re} q = \operatorname{Re} z - 1/2$
and $\operatorname{Im} p = -\operatorname{Im} q > |\operatorname{Im} z|$. By $U_-(z)$ we denote the contour consisting of the
half line from $p + \infty$ to $p$, the directed segment $[p, q]$, and the half line from $q$
to $q + \infty$ (cf. fig. 2).

For all $n \in \mathbb{Z}$, the function $f_n$ defined by

$$f_n(z) = y_+^0(z) \int_{U_-(z)} \frac{e^{2niz(z-\zeta)} b(\zeta)}{y_+^0(\zeta) \left\{1 - e^{2niz(z-\zeta)}\right\}} d\zeta$$

is a solution of equation (0.1), which can be continued analytically to $D_-$. For
all $z \in D_- \cap D_+$ we have

$$f_n(z) = f(z) + y_-^0(z) \int_{U_-} \frac{e^{2niz(z-\zeta)} b(\zeta)}{y_-^0(\zeta) \left\{1 - e^{2niz(z-\zeta)}\right\}} d\zeta$$
where $U_\_ = U_\_(-R - \delta), \delta > 0$ (cf. fig. 3). Hence it follows that $f_n$ is represented asymptotically by $\hat{f}$ as $z \to \infty$ in such a manner that either of the following conditions is satisfied

$$\text{Im } z \to \infty \quad \text{and} \quad \text{Re } \{z(\log z + \lambda + 2n\pi i)\} \to -\infty$$

or

$$\text{Im } z \to -\infty \quad \text{and} \quad \text{Re } \{z(\log z + \lambda + 2(n - 1)\pi i)\} \to -\infty.$$ 

Hence one easily deduces the following proposition, which is a particular case of [15, proposition 2.3.1].

**Proposition 1.1.** Let $n \in \mathbb{Z}$. $f_n$ is represented asymptotically by $\hat{f}$ as $z \to \infty$ in $D_\_$ in such a manner that

$$\text{Re } \{z(\log z + i\theta)\} < C$$
for some $C \in \mathbb{R}$ and a real number $\theta$ such that

$$\text{Im} \lambda + 2(n - 1)\pi < \theta < \text{Im} \lambda + 2n\pi$$

**Remark 1.2.** If $C$ is sufficiently small, the relation $\text{Re} \{z(\log z + i\theta)\} = C$ defines $\text{Re} \ z$ as a function $x(y)$ of $y := \text{Im} \ z$ with the property that $x(y) = O(|y|/\log|y|)$ as $y \to \pm \infty$. If $\pi/2 \leq \theta \leq 3\pi/2$ this curve lies entirely in the left half plane. If $\theta < \pi/2$, the function $x(y)$ decreases monotonically from $\infty$ to $-\infty$, whereas for $\theta > 3\pi/2$ it increases monotonically from $-\infty$ to $\infty$. (cf. fig. 4)

For more details cf. [12, 15].

Let

$$P_n := \int_{U_-} y_-^0(\zeta)^{-1}e^{-2n\pi i\zeta}b(\zeta)d\zeta$$

and

$$p_n := (f_n - f_n)(y_-^0)^{-1}. \tag{1.9}$$

From (1.8) we deduce that

$$f_n(z) - f_{n+1}(z) = P_ne^{2nmiz}y_-^0(z) \tag{1.10}$$

and
(1.11) \[ p_n(z) = \begin{cases} - \sum_{m=n}^{\infty} P_m e^{2\imath m z} & \text{if } \text{Im } z > R \\ \sum_{m=-\infty}^{n-1} P_m e^{2\imath m z} & \text{if } \text{Im } z < -R. \end{cases} \]

**Proposition 1.3.** The numbers \( P_n^0(n \in \mathbb{Z}_+) \) and \( P_n(n \in \mathbb{Z}) \) form a complete and free system of analytic invariants of the equation (0.1).

**Proof.** Suppose that
\[ y(z) - \tilde{a}(z)y(z + 1) = \tilde{b}(z) \]
is another equation of the type (0.1) having the same formal invariants, i.e. the same values of \( \lambda \) and \( \sigma \). (1.12) is analytically equivalent to (0.1) if there exist invertible elements \( F, G \in C\{z^{-1}\} \) such that any solution \( \tilde{y} \) of (1.12) is carried into a solution \( y \) of (0.1) by means of the substitution
\[ y(z) = F(z)\tilde{y}(z) + G(z). \]
This implies that \( F \) and \( G \) satisfy the following equations
\[ \tilde{a}(z)F(z) = a(z)F(z + 1) \]
and
\[ G(z) = a(z)G(z + 1) + b(z) - \tilde{b}(z)F(z). \]
Consequently, the functions \( \tilde{y}_0^0 \) and \( y_0^0 \) defined by
\[ \tilde{y}_0^0 := F^{-1}y_0^0 \]
are solutions of the homogeneous equation associated with (1.12), similar to \( y_0^0 \) and \( y_0^0 \), respectively, and we have
\[ \tilde{y}_0^0(y_0^-)^{-1} = y_0^0(y_0^-)^{-1} = p^0. \]
Furthermore, the functions \( \tilde{f} \) and \( \tilde{f}_n \) defined by
\[ \tilde{f} = \tilde{y}_0^0(y_0^-)^{-1}(f - G) \]
and
\[ \tilde{f}_n = \tilde{y}_0^0(y_0^-)^{-1}(f_n - G), \quad n \in \mathbb{Z} \]
are solutions of (1.12) similar to \( f \) and \( f_n \), respectively, and we have
\[ \{\tilde{f}_n(z) - \tilde{f}_{n+1}(z)\}y_0^0(z)^{-1} = \{f_n(z) - f_{n+1}(z)\}y_0^0(z)^{-1} = P_n e^{2\imath n z}, \quad n \in \mathbb{Z}. \]
From (1.13) and (1.14) we conclude that the numbers $P_n^0$ and $P_n$ ($n \in \mathbb{Z}$) are analytic invariants of the equation (0.1). The completeness of this set of invariants can be proved by means of the following, well-known type of argument. Suppose that (0.1) and (1.12) have the same set of invariants. Let $y^0_\pm$, $\tilde{f}$ and $\tilde{f}_n$ be defined analogously to $y^0_\pm$, $f$ and $f_n$, respectively. Then the identities (1.13) and (1.14) hold. Now, let $F$ and $G$ be defined by

$$F := y^0_+ (\tilde{y}_+^0)^{-1}, \quad G = f - \tilde{f} F.$$  

From (1.9) and (1.13) it follows that

$$F = y^0_- (\tilde{y}_-^0)^{-1} \quad \text{and} \quad f - f_n = (\tilde{f} - \tilde{f}_n) F,$$

hence

$$G = f_n - \tilde{f}_n F \quad \text{for any } n \in \mathbb{Z}.$$  

Apparently, both $F$ and $G$ are analytic in a reduced neighbourhood of $\infty$. The formal equivalence of (0.1) and (1.12) implies that $F$ and $G$ admit formal power series expansions in $z^{-1}$ as $z \to \infty$ in any direction. Consequently, both $F$ and $G$ are analytic at $\infty$.

Now suppose we are given two formal invariants $\lambda$ and $\sigma$, and numbers $P_n^0$, $n \in \mathbb{Z}$, and $P_n$, $n \in \mathbb{Z}$, with the property that the infinite series in the righthand sides of (1.4) and (1.11) with $n = 0$ converge. Then these series define periodic functions to be denoted by $\psi$ and $p_0$, respectively, analytic in the union of half-planes $\text{Im } z > R \cup \text{Im } z < -R$ for some $R > 0$. Let $R' > R$ and let $\eta_+$ and $\eta_-$ be defined by

$$\eta_{\pm}(z) = \int_{i\infty}^{iR'} \frac{\psi(\zeta)}{2\pi i (\zeta - z)} d\zeta + \int_{-i\infty}^{-iR'} \frac{\psi(\zeta)}{2\pi i (\zeta - z)} d\zeta, \quad \pm \text{Re } z > 0$$

$\eta_+$ and $\eta_-$ can be continued analytically to U-shaped regions of the type $D_+$ and $D_-$, respectively, and admit the same asymptotic expansion $\tilde{\eta} \in z^{-1} C[z^{-1}]$ (which is Gevrey of order 1, cf. [18]). Moreover, for $|\text{Im } z| > R'$,

$$\eta_+(z) - \eta_-(z) = \psi(z).$$

From the fact that $\psi$ is periodic, with period 1, we deduce that

$$\eta_+(z) - \eta_+(z + 1) = \eta_-(z) - \eta_-(z + 1)$$

and, consequently, the function $\gamma$ defined by

$$\gamma(z) := \eta_+(z) - \eta_+(z + 1)$$

is analytic in a reduced neighbourhood of $\infty$, and represented asymptotically by $\tilde{\eta}(z) - \tilde{\eta}(z + 1)$ as $z \to \infty$ in any direction. It follows that $\gamma \in z^{-2} C[z^{-1}]$ and
the function $a$ defined by

$$a(z) := (z + a)^{-1} e^{\gamma(z) - \lambda - 1}$$

belongs to $z^{-1}C[z^{-1}]$. It is easily seen that $a$ can be written in the form (1.1). Furthermore, let

$$y_0^0(z) := 2\pi i \Gamma(1 - z - a)^{-1} e^{\gamma(z) + (\lambda + 1 - i\sigma)z - i\sigma}$$

and let $f_+$ and $f_-$ be defined by

$$f_\pm(z) = \int_{-iR'}^{iR'} \frac{p_0(\xi) y_0^0(\xi)}{2\pi i (\xi - z)} d\xi + \int_{-iR'}^{-iR'} \frac{p_0(\xi) y_0^0(\xi)}{2\pi i (\xi - z)} d\xi, \quad \pm z > 0.$$ 

Obviously, $f_+$ is holomorphic in a U-shaped region of the type $D_+$ and $f_-$ can be continued analytically to a left half-plane by rotation of the paths of integration. $f_+$ and $f_-$ admit the same asymptotic expansion $\hat{f} \in z^{-1}C[z^{-1}]$ as $z \to \infty$, $|\arg z| < \pi$ and $-3\pi/2 < \arg z < -\pi/2$, respectively. Moreover,

(1.15) 

$$f_+ - f_- = p_0 y_0^0$$

and

(1.16) 

$$y_0^0(z) = a(z) y_0^0(z + 1).$$

Now, let

$$b(z) := f_+(z) - a(z) f_+(z + 1).$$

From (1.15) and (1.16) we deduce that $b(z) = f_-(z) - a(z) f_-(z + 1)$, and thus $b$ is analytic in a reduced neighbourhood of $\infty$. Furthermore, $b$ admits an asymptotic expansion $b \in z^{-1}C[z^{-1}]$, defined by: $b(z) = \hat{f}(z) - a(z) \hat{f}(z + 1)$ as $z \to \infty$ in any direction. Consequently, $b \in z^{-1}C[z^{-1}]$. In is easily verified that the analytic invariants of the difference equation $y(z) - a(z) y(z + 1) = b(z)$ are the numbers $P_n^0 (n \in \mathbb{Z}_*)$ and $P_n (n \in \mathbb{Z})$. 

**Remark 1.4.** Proposition 1.3 is a special case of a more general result stated in [14].

2. Properties of the formal Borel transform of $\hat{f}$

Let $\hat{\Theta}: z^{-1}C[z^{-1}] \to C[s]$ denote the formal Borel transformation, i.e. the linear mapping defined by the substitution

$$z^{-n} \mapsto \frac{s^{n-1}}{(n-1)!}.$$

Application of \( \hat{B} \) to the formal solution \( \hat{f} \) of (0.1) yields the convergent series
\[
\sum_{n=1}^{\infty} \phi_n \frac{x^n}{(n-1)!}.
\]

In view of (0.3) this series defines an entire function to be denoted by \( \varphi \). It satisfies the following convolution equation
\[
\varphi = \alpha \ast e^{-s} \varphi + \beta
\]
where \( \alpha \) and \( \beta \) denote the formal Borel transforms of \( a \) and \( b \), respectively. Thus \( \alpha \) and \( \beta \) are entire functions with at most exponential growth of order 1 as \( s \to \infty \) in any direction of the complex plane. Moreover, \( \alpha(0) = \exp(-\lambda - 1) \).

For all \( s \in \mathbb{C} \) such that \( \text{Re} \, s > 0 \), \( \varphi \) coincides with the ordinary Borel transform \( \mathcal{B}f \) of the function \( f \) defined in the previous section, represented by the integral
\[
\mathcal{B}f(s) = \frac{1}{2\pi i} \int_{U_+} f(z) e^{sz} dz,
\]
where \( U_+ \) is a contour in \( D_+ \) consisting of the half lines \( l_1: \text{Im} \, z = -R_1, \text{Re} \, z \leq R_1 \) and \( l_2: \text{Im} \, z = R_1, \text{Re} \, z \leq R_1 \) and the segment from \( R_1(1 - i) \) to \( R_1(1 + i) \) with \( R_1 > R \), described in the positive sense (cf. fig. 5). (Note that \( f(z) = O(z^{-1}) \) as \( z \to \infty \), uniformly in \( D_+ \), and hence this Borel transform exists.)

For each \( n \in \mathbb{Z} \) let \( s_n \in \mathbb{C} \) such that \( \text{Re} \, s_n \leq 0 \) and
\[
|\text{Im} \, (s_n + \lambda) + 2(n-1)\pi| < \frac{\pi}{2}
\]
\( \Gamma_n \) will denote the path from \( O \) to \( \infty \) consisting of the segment \([0, s_n]\) and the half line from \( s_n \) to \( \infty \) parallel to the negative real axis.
Proposition 2.1. \( \varphi \) grows at most exponentially of order 1 as \( s \to \infty \), \( \Re s > c \) for some \( c \in \mathbb{R} \), and as \( s \to \infty \) on \( \Gamma_n \) for any \( n \in \mathbb{Z} \). The solutions \( f \) and \( f_n \) \( (n \in \mathbb{Z}) \) defined in §1 admit the following integral representations

\[
(2.4) \quad f(z) = \int_0^\infty \varphi(s) e^{-sz} ds, \quad \Re z > R
\]

\[
(2.5) \quad f_n(z) = \int_{\Gamma_n} \varphi(s) e^{-sz} ds, \quad \Re z < -R, \ n \in \mathbb{Z}
\]

Proof. From (2.2) it is easily seen that \( \varphi \) grows at most exponentially as \( s \to \infty \), \( \Re s > c \) for any \( c > 0 \). (2.4) can be proved by inserting (2.2) into the right-hand side of (2.4) and changing the order of integration.

Let \( a_1 := (i-1)R_1 \), \( a_2 := (-i-1)R_1 \), with \( R_1 > R \). Let \( n \in \mathbb{Z} \). From (1.9) it follows that

\[
(2.6) \quad \int_{a_1}^{a_1+\infty} f(z) e^{sz} dz = \int_{a_1}^{a_1+\infty} f_n(z) e^{sz} dz + \int_{a_1}^{a_1-\infty} p_n(z) y^0(z) e^{sz} dz, \quad j = 1, 2.
\]

We put

\[
(2.7) \quad I^-_j(s) := \int_{a_1}^{a_1-\infty} p_n(z) y^0(z) e^{sz} dz, \quad j = 1, 2.
\]

Starting from (2.2) and using (2.6) one obtains, by continuous deformation of the paths of integration, the following representation of \( \varphi \) in the half plane \( \Im s > 0 \).

\[
\varphi(s) = \frac{1}{2\pi i} \left[ I^-_1(s) - I^-_2(s) + \int_{a_1}^{a_2} f_n(z) e^{sz} dz + \int_{a_1}^{a_1-\infty} f(z) e^{sz} dz + \int_{-a_1+\infty}^{-a_1+i\infty} f(z) e^{sz} dz - \int_{a_1+i\infty}^{a_1+\infty} f(z) e^{sz} dz \right].
\]

In view of the asymptotic properties of \( f \) and \( y^0 \) it easily follows that \( \varphi \) grows at most exponentially of order 1 as \( \Im s \to \infty \) in the strip \( c_1 < \Re s < c_2 \) for any pair of real numbers \( c_1 < c_2 \). In a similar manner one proves that \( \varphi \) grows at most exponentially of order 1 as \( \Im s \to -\infty \) in such a strip. Now, let

\[
(2.8) \quad I^+_j(s) := \int_{a_1}^{a_1+\infty} f(z) e^{sz} dz, \quad j = 1, 2.
\]

In the half plane \( \Re s < 0 \) \( \varphi \) may be represented as follows

\[
(2.9) \quad \varphi(s) = \frac{1}{2\pi i} \left[ I^-_1(s) - I^-_2(s) - I^+_1(s) + I^+_2(s) + \int_{a_1}^{a_2} f_n(z) e^{sz} dz \right]
\]
It is readily verified that the last three integrals in (2.9) have exponential growth of order 1 as $s \to \infty$ in this half plane. Now suppose that $\text{Im} \ s = \text{Im} \ s_n$. Due to (2.3), there exist real numbers $\theta_1$ and $\theta_2$ such that
\begin{equation}
\frac{3\pi}{2} < \theta_1 < \text{Im} \ (s_n + \lambda) + 2n\pi \quad \text{and} \quad \text{Im} \ (s_n + \lambda) + 2(n - 1)\pi < \theta_2 < \frac{\pi}{2}.
\end{equation}

Let $C_j$ denote the path from $a_j$ to $\infty$ defined by
\begin{equation}
C_j := \{z: -2\pi < \text{arg} \ z < 0, \ \text{Re} \ \{z(\log z + i\theta_j)\} = c_j, \ \text{Im} \ a_j \text{ Im} \ (z - a_j) > 0\},
\end{equation}
where $c_j := \text{Re} \ \{a_j(\log a_j + i\theta_j)\}$, $j = 1, 2$. For all $z$ with the property that $\text{Re} \ \{z(\log z + i\theta_j)\} \leq c_j$ we have
\begin{equation}
\text{Re} \ \{z(\log z + s + \lambda)\} \leq c_j + \{\theta_j - \text{Im} \ (s_n + \lambda)\} \text{ Im} \ z + \text{Re} \ z \text{ Re} \ (s + \lambda)
\end{equation}
Using the asymptotic properties of $y_0^+$, together with (1.11), and (2.10)–(2.12), we conclude that
\begin{equation}
I_j^-(s) = \int_{C_j} \rho_n(z)y_0^+(z)e^{\rho z}dz
\end{equation}
for all $s$ such that $\text{Im} \ s = \text{Im} \ s_n$. From the fact that, for all $z \in C_j$,
\[\text{Re} \ z \log |z| = c_j + (\text{arg} \ z + \theta_j) \text{ Im} \ z\]
it follows that $\cos (\text{arg} \ z) \to 0$ as $z \to \infty$ on $C_j$ ($j = 1, 2$), i.e. $\text{Im} \ z \to \infty$ if $j = 1$ and $\text{Im} \ z \to -\infty$ if $j = 2$. In view of (2.10) this implies that $\text{Re} \ z > 0$ if $z \in C_j$ and $|z|$ is sufficiently large. Consequently, there exists a real number $c \leq \text{Re} \ a_1$ such that
\[\text{Re} \ z > c \quad \text{for all} \quad z \in C_1 \cup C_2.
\]

With (2.13) it is now easily verified that the functions $I_j^-$ have at most exponential growth as $\text{Re} \ s \to -\infty$, $\text{Im} \ s = \text{Im} \ s_n$ ($j = 1, 2$) and thus the same is true of $\varphi$. (Alternatively, the growth properties of $\varphi$ can be derived from (2.1) by means of techniques similar to those used in [3]).

It remains to be proved that (2.5) holds. We use the representation (2.9) and insert it into the right-hand side of (2.5). By a change in the order of integration we obtain the following identities
\begin{equation}
\int_{I_n} I_j^+(s)e^{-\rho z}ds = \int_{a_j}^{a_j+\infty} \frac{f(\xi)}{\xi - z}d\xi, \quad j = 1, 2
\end{equation}
and
\begin{equation}
\int_{I_n} ds \int_{a_1}^{a_2} d\xi f_n(\xi)e^{s(\xi - z)} = \int_{a_1}^{a_2} \frac{f_n(\xi)}{\xi - z}d\xi
\end{equation}
provided \( \text{Re} \, z < \text{Re} \, a_1 \). Furthermore, we have

\[
(2.16) \quad \int_0^{a_j} I_1(s) e^{-sz} \, ds = \int_{a_j}^{d_j-\infty} p_n(\zeta) y_0^0(\zeta) \frac{e^{\xi_n(\zeta-z)}}{\zeta-z} \, d\zeta.
\]

Using (2.13) and changing the order of integration we find

\[
(2.17) \quad \int_{a_j}^{d_j-\infty} I_j(s) e^{-sz} \, ds = - \int_{C_j} p_n(\zeta) y_0^0(\zeta) \frac{e^{\xi_n(\zeta-z)}}{\zeta-z} \, d\zeta, \quad j = 1, 2
\]

for all \( z \) in the half plane \( \text{Re} \, z < c \). From the asymptotic properties of \( y_0^0 \) and \( p_n \), in combination with (2.10)–(2.12), we infer that

\[
(2.18) \quad \int_{C_j} p_n(\zeta) y_0^0(\zeta) \frac{e^{\xi_n\zeta}}{\zeta-z} \, d\zeta = \int_{a_j}^{d_j-\infty} p_n(\zeta) y_0^0(\zeta) \frac{e^{\xi_n\zeta}}{\zeta-z} \, d\zeta
\]

for all \( z \) in the strip \( \mathcal{S} := \{ z \in \mathbb{D} : \text{Re} \, z < c \text{ and } |\text{Im} \, z| < |\text{Im} \, a_1| \} \). From (2.16)–(2.18) it follows that, for all \( z \in \mathcal{S} \),

\[
\int_{a_j}^{d_j} I_j(s) e^{-sz} \, ds = - \int_{a_j}^{d_j-\infty} p_n(\zeta) y_0^0(\zeta) \frac{e^{\xi_n\zeta}}{\zeta-z} \, d\zeta, \quad j = 1, 2.
\]

Combining this with (2.9), (2.14) and (2.15), using (1.9) and noting that, for all \( z \in \mathcal{S} \),

\[
\int_{a_j}^{d_j} \frac{f(\zeta)}{\zeta-z} \, d\zeta = 0, \quad j = 1, 2
\]

we conclude that, for all \( z \in \mathcal{S} \),

\[
\int_{a_j}^{d_j} \varphi(s) e^{-sz} \, ds = \frac{1}{2\pi i} \left[ \int_{a_1}^{d_1} \frac{f_n(\zeta)}{\zeta-z} \, d\zeta - \int_{a_2}^{a_1} f_n(\zeta) \, d\zeta - \int_{a_3}^{a_2} f_n(\zeta) \, d\zeta \right]
\]

\[
= f_n(z).
\]

This completes the proof of proposition 2.1.  

\[ \Box \]

3. Week accelerates of \( \varphi \)

Let \( \theta \in \mathbb{R} \) and \( \psi_\theta(z) := z(\log z + i\theta) \). The formal Borel transform \( \varphi \) of \( \hat{f} \), defined in the previous section can be 'accelerated' from level 1 to level \( 1^+ \) by means of the acceleration operator \( \mathcal{A}_\theta \). This is an integral operator with kernel \( C_\theta(t, s) \) defined by
\begin{equation}
C_\theta(t, s) = \begin{cases}
\frac{1}{2\pi i} \int_{U_+} \exp\left\{t\psi_\theta(z) - sz\right\} d\psi_\theta(z) & \text{if } |\arg t| < \frac{\pi}{2}, s \in \mathbb{C} \\
\frac{1}{2\pi i} \int_{U_-} \exp\left\{t\psi_\theta(z) - sz\right\} d\psi_\theta(z) & \text{if } \frac{\pi}{2} < |\arg t| < \frac{3\pi}{2}, s \in \mathbb{C}
\end{cases}
\end{equation}

where $U_+$ and $U_-$ are contours of the type used in (2.2) and (1.8), respectively. $C_\theta$ is an analytic function of $t$ on the Riemann surface of $\log t$ and is an entire function of $s$.

For a fixed $t$, let $s_\theta(t)$ be a point in the half plane $|\arg(s/t)| < \pi/2$ with the property that

$$\left| \frac{\text{Im } s_\theta(t)}{t} + \arg t - \theta \right| < \frac{\pi}{2}$$

and let $\Gamma_\theta(t)$ denote the path from $O$ to $\infty$ consisting of the segment $(0, s_\theta(t))$ and the half line $l_\theta(t) : \arg(s - s_\theta(t)) = \arg t$. By means of the saddle point method it can be shown that

$$C_\theta(t, s) \sim \left\{ \frac{t}{2\pi} \exp\left( \frac{s}{t} - 1 - i\theta \right) \right\}^{1/2} \cdot s \exp\left\{ -t \exp\left( \frac{s}{t} - 1 - i\theta \right) \right\}$$

as $s \to \infty$ on $l_\theta(t)$.

**Definition 3.1.** Let $\theta \in \mathbb{R}$. For all $t$ on the Riemann surface of $\log t$ with the property that $\text{Re } t > -1$, the 'weak accelerate' $\varphi_\theta$ of $\varphi$ is defined by

$$\varphi_\theta(t) = \int_{\Gamma_\theta(t)} C_\theta(t, s) \varphi(s) ds.$$

We define the **singular values** of $\theta$ to be the numbers $\text{Im } \lambda + 2n\pi$, $n \in \mathbb{Z}$. It will be proved that, for non-singular values of $\theta$, $\varphi_\theta$ can be continued quasi-analytically to the half lines $\arg t = (2m + 1)\pi$, i.e. it is infinitely many times differentiable and completely determined by the sequence of derivatives at any point of these half lines. For each $n \in \mathbb{Z}$ there exists a real number $\theta$ such that the solution $f_n$ of (0.1) defined in §1 can be represented as a Laplace transform, in the variable $\psi_\theta(z)$, of $\varphi_\theta$. Here, the axis of integration is the half line $\arg t = \pi$.

Let $\theta \in \mathbb{R}$. We begin by deriving some useful integral representations of $\varphi_\theta$ in terms of solutions of the difference equation, valid in different regions of the Riemann surface of $\log t$. From the fact that

$$\psi_\theta(ze^{2ni}) = e^{2ni}\psi_{\theta+2n}(z)$$
it follows that
\begin{equation}
\varphi_\theta(te^{2\pi i}) = \varphi_{\theta-2\pi}(t).
\end{equation}

Hence we may restrict ourselves to $-\pi/2 \leq \arg t < 3\pi/2$.

**Lemma 3.2.** For $|\arg t| < \pi/2$, $\varphi_\theta$ can be represented by
\begin{equation}
\varphi_\theta(t) = \frac{1}{2\pi i} \int_{U_+} f(z)e^{t\psi_\theta(z)}d\psi_\theta(z),
\end{equation}
where $U_+$ denotes the contour used in (2.2) (cf. fig. 5). (Note that $\varphi_\theta$ is an ordinary Borel transform of $f$ with respect to the variable $\psi_\theta(z)$.)

**Proof.** It suffices to prove the lemma for $\arg t = 0$. The assertion then follows from the uniqueness of the analytic continuation. For $\arg t = 0$ we have
\[-\frac{\pi}{2} < \theta - \frac{1}{t} \Im s_\theta(t) < \frac{\pi}{2}.
\]
Let $\theta_+$ and $\theta_-$ be real numbers satisfying the inequalities
\begin{equation}
-\frac{\pi}{2} < \theta_+ < \theta - \frac{1}{t} \Im s_\theta(t) < \theta_- < \frac{\pi}{2}.
\end{equation}

For any $K > 1$, $C_+(K)$ and $C_-(K)$ will denote the paths from $K$ to $\infty$ defined by
\begin{equation}
C_\pm(K) := \{z: -\pi < \arg z < \pi, \Re \{z(\log z + i\theta_\pm)\} = c \text{ and } \pm \Im z > 0\}
\end{equation}
where $c := K \log K$ and $C(K)$ will denote the contour consisting of $C_+(K)$ and $C_-(K)$ and described in the direction of increasing imaginary part. For all $z \in C_\pm(K)$ we have
\[\Re z \log |z| = K \log K + (\theta_\pm + \arg z) \Im z\]

With (3.4) this implies that
\begin{equation}
\rho(K) := \inf_{z \in C(K)} \Re z \to \infty \quad \text{as } K \to \infty.
\end{equation}

Suppose that $K$ is so large that $\rho(K) > R_1$. For all $s \in L(t)$ and all $z$ in the region enclosed by $C(K)$ and $U_+$ the following inequality holds
\begin{equation}
\Re \{t\psi_\theta(z) - sz\} \leq tc + \{t\theta_\pm - t\theta + \Im s_\theta(t)\} \Im z - R_1 \Re s \quad \text{if } \pm \Im z \geq 0.
\end{equation}

With (3.1) and (3.4) it follows that
\begin{equation}
\int_{L(t)} C_\theta(t,s)\varphi(s)ds = \frac{1}{2\pi i} \int_{L(t)} d\varphi(s) \int_{C(K)} d\psi_\theta(z) \exp \{t\psi_\theta(z) - sz\}\]

\( \varphi \) grows at most exponentially of order 1 as \( s \to \infty \) on \( \Gamma_\theta(t) \). With the aid of (3.6) and (3.7) we conclude that the order of integration in the right-hand side of (3.8) may be changed if \( K \) is sufficiently large. Thus we obtain

\[
\int_{l_\theta(t)} C_\theta(t, s) \varphi(s) ds = \frac{1}{2\pi i} \int_{C(K)} d\psi_\theta(z) \exp \{ t\psi_\theta(z) \} \int_{l_\theta(t)} d\varphi(s) \exp (-sz) \tag{3.9}
\]

From now on let us assume that \( \theta \geq 0 \) and \( \text{Im } s_\theta(t) = t\theta \). The proof for the case that \( \theta < 0 \) is analogous. Let \( \tilde{\theta} > \theta \) and denote by \( \tilde{C}(K) \) the contour consisting of \( C_+(K) \) and

\[
\tilde{C}_-(K) := \{ z : |\text{arg } z| < \pi, \text{Re } \{ z(\log z + i\tilde{\theta}) \} = c \text{ and } \text{Im } z \leq 0 \}. \]

It is easily verified that

\[
\int_{0}^{s_\theta(t)} C_\theta(t, s) \varphi(s) ds = \int_{0}^{s_\theta(t)} d\varphi(s) \frac{1}{2\pi i} \int_{\tilde{C}(K)} d\psi_\theta(z) \exp \{ t\psi_\theta(z) - sz \} = \frac{1}{2\pi i} \int_{\tilde{C}(K)} d\psi_\theta(z) \exp \{ t\psi_\theta(z) \} \int_{0}^{s_\theta(t)} d\varphi(s) \exp (-sz) \tag{3.10}
\]

By rotation of the path \( l_\theta(t) \), it can be shown that

\[
\int_{l_\theta(t)} d\varphi(s) e^{-sz} = e^{-s_\theta(t)^2} O(1)
\]

as \( z \to \infty \) in the region enclosed by \( C_-(K) \) and \( \tilde{C}_-(K) \). Hence it follows that

\[
\int_{C_-(K)} d\psi_\theta(z) e^{t\psi_\theta(z)} \int_{l_\theta(t)} d\varphi(s) e^{-sz} = \int_{\tilde{C}_-(K)} d\psi_\theta(z) e^{t\psi_\theta(z)} \int_{l_\theta(t)} d\varphi(s) e^{-sz} \tag{3.11}
\]

Combining (3.10) and (3.11) and using proposition 2.1 we find

\[
\varphi_\theta(t) = \int_{\tilde{C}(K)} d\psi_\theta(z) f(z) \exp \{ t\psi_\theta(z) \}.
\]

The result follows by deformation of the contour \( C(K) \).

**Lemma 3.3.** Suppose that \( \theta \neq \text{Im } \lambda \) (mod 2\( \pi \)).

Let \( n \) be the smallest integer with the property that \( \theta < \text{Im } \lambda + 2n\pi \). Let \( \theta_1 \) and \( \theta_2 \) be real numbers such that

\[
\text{Im } \lambda + 2(n - 1)\pi < \theta_2 < \theta < \theta_1 < \text{Im } \lambda + 2n\pi \tag{3.12}
\]

and let \( C_j \) denote the path from \( a_j \) to \( \infty \) defined by (2.11), \( j = 1, 2 \). Let \( C(n) \) denote the contour consisting of \( C_1, C_2 \) and the segment \( [a_1, a_2] \), described in the
positive sense. For all \( t \in (0, e^{n \lambda}) \), \( \varphi_{\theta}(t) \) may be represented by the integral

\[
\varphi_{\theta}(t) = \frac{1}{2\pi i} \int_{C(n)} f_n(z) e^{i\psi_{\theta}(z)} d\psi_{\theta}(z).
\]

**Proof.** Let \( \theta \in \mathbb{R} \) and \( n \in \mathbb{Z} \). For all \( t \) in the vertical strip \( T := \{ t : \pi/2 < \arg t < 5\pi/2, -1 < \Re t < 0 \} \) we define

\[
I_j^+(t) := \int_{a_j}^{a_j + \infty} f(z) e^{i\psi_{\theta}(z)} d\psi_{\theta}(z), \quad j = 1, 2
\]
\[
I_j^-(t) := \int_{a_j}^{a_j - \infty} p_n(z) y_{-}^0(z) e^{i\psi_{\theta}(z)} d\psi_{\theta}(z), \quad j = 1, 2
\]

With the aid of (1.9) one easily verifies, by continuous deformation of the path of integration in (3.3), that for all \( t \in T \) the following identity holds

\[
\varphi_{\theta}(t) = \frac{1}{2\pi i} \left[ I_1^-(t) - I_2^-(t) + I_2^+(t) - I_1^+(t) + \int_{a_1}^{a_2} f_n(z) e^{i\psi_{\theta}(z)} d\psi_{\theta}(z) \right]
\]

Now suppose that the conditions of the lemma are satisfied. Let \( j \in \{1, 2\} \). For all \( z \) with the property that \( -2\pi < \arg z < 0 \) and \( \Re \{ z(\log z + i\theta_j) \} \geq c_j \) and all \( t \in (0, e^n) \) we have

\[
\Re t \psi_{\theta}(z) \leq tc_j - t(\theta - \theta_j) \Im z.
\]

With (3.12) and (3.14) it follows that, for all \( t \in (0, e^n) \),

\[
I_j^+(t) = \int_{C_j} f(z) e^{i\psi_{\theta}(z)} d\psi_{\theta}(z).
\]

For all \( z \) with the property that \( -2\pi < \arg z < 0 \) and \( \Re \{ z(\log z + i\theta_j) \} \leq c_j \) and all \( t \in (0, e^n) \), we have

\[
\Re \{ t\psi_{\theta}(z) + z(\log z + \lambda) \} \leq (1 + t)c_j + \{(1 + t)\theta_j - t\theta - \Im \lambda\} \Im z + \Re \lambda \Re z.
\]

In view of (3.12) and the asymptotic properties of \( y_{-}^0 \) and \( p_n \), this implies that, for all \( t \in (0, e^n) \), the path of integration in (3.15) may be changed to \( C_j \) and thus

\[
I_j^-(t) = \int_{C_j} p_n(z) y_{-}^0(z) e^{i\psi_{\theta}(z)} d\psi_{\theta}(z).
\]

From (3.17) and (3.18) we conclude, with the aid of (1.9), that, for all \( t \in (0, e^n) \),

\[
I_j^+(t) - I_j^-(t) = \int_{C_j} f_n(z) e^{i\psi_{\theta}(z)} d\psi_{\theta}(z), \quad j = 1, 2
\]

and the proof of the lemma is completed by inserting this into (3.16).
Proposition 3.4. Suppose that \( \theta \neq \text{Im} \lambda \mod 2\pi \). Then the function defined by the right-hand side of (3.13) is quasi-analytic on the half line \( \arg t = \pi \) (more precisely, it belongs to the Denjoy class \( 1D \), cf. [4, 9, 5, 10]) and analytic in the half plane: \( \pi/2 < \arg t < 3\pi/2 \) and \( \text{Re} \ t < -1 \). In this half plane it may be represented as follows

\[
(3.19) \quad \varphi_\theta(t) = \frac{1}{2\pi i} \int_{U_-} f_n(z) e^{i\psi_\theta(z)} d\psi_\theta(z)
\]

where \( U_- \) denotes the contour used in (1.8).

Proof. Due to (3.12), \( f_n(z) \) is represented asymptotically by \( \tilde{f}(z) \) as \( z \to \infty \) on \( C(n) \) (cf. proposition 1.1). Hence there exists a positive constant \( K \) such that, for all \( z \in C(n) \),

\[
(3.20) \quad |f_n(z)| \leq \frac{K}{|z|}.
\]

Furthermore we have, for all \( z \in C_j \),

\[
\text{Re} \ \psi_\theta(z) = c_j + (\theta_j - \theta) \text{Im} \ z, \quad j = 1, 2.
\]

In view of (3.12), this implies the existence of a positive number \( \varepsilon \) such that

\[
(3.21) \quad \text{Re} \ \psi_\theta(z) \geq c_j + \varepsilon |\text{Im} \ z|
\]

for all \( z \in C_j, j = 1, 2 \). From (3.20) and (3.21) it follows that the integral on the right-hand side of (3.13) exists and is infinitely many times differentiable on the half line \( \arg t = \pi \). Furthermore, it is easily seen that it coincides with the function defined by (3.19) when \( \text{Re} \ t < -1 \) and that this function is analytic in the half plane: \( \pi/2 < \arg t < 3\pi/2 \) and \( \text{Re} \ t < -1 \).

In order to prove the quasi-analyticity of \( \varphi_\theta \) on the half line \( \arg t = \pi \), we estimate the growth of \( |\varphi_\theta^{(m)}| \) as \( m \to \infty \). We have

\[
\varphi_\theta^{(m)}(t) = \frac{1}{2\pi i} \int_{C(n)} f_n(z) \psi_\theta(z)^m e^{i\psi_\theta(z)} d\psi_\theta(z), \quad m \in \mathbb{N}.
\]

Let

\[
I_{j,m}(t) := \int_{C_j} f_n(z) \psi_\theta(z)^m e^{i\psi_\theta(z)} d\psi_\theta(z) \quad j = 1, 2, m \in \mathbb{N}.
\]

Put \( |\text{Im} \ z| = x \) and let \( \{z_j(x) : x \in (\text{Im} \ a_1, \infty)\} \) be a parametrization of \( C_j \) \( (j = 1, 2) \). Thus we have

\[
\text{Re} \ z_j(x) \log |z_j(x)| = \text{Im} \ z_j(x)(\arg z_j(x) + \theta_j) + c_j, \quad j = 1, 2.
\]

It follows that

\[
|z_j(x)| = x(1 + o(1)), \quad x \to \infty.
\]
Consequently, there exists a positive constant $c$ such that

$$|\psi_\theta(z_j(x))| \leq cx \log x.$$  

(3.22)

From (3.20), (3.21) and (3.22) we deduce that

$$|I_{j,m}(t)| \leq K_1 c^m \int_{\text{Im } a_1}^{\infty} (x \log x)^m e^{(t+tx)} dx, \quad j = 1, 2, m \in \mathbb{N},$$  

(3.23)

where $K_1$ is a positive constant, independent of $m$. If $m$ is sufficiently large, the integrand on the right-hand side of (3.23) has a unique saddle point $x_m$ on the path of integration, with the property that

$$x_m = \frac{m}{\epsilon |t|} \left(1 + O\left(\frac{1}{\log m}\right)\right), \quad m \to \infty$$

where $O(1/\log m)$ is uniformly bounded with respect to $t$ on any closed bounded interval of the half line $\arg t = \pi$. Applying the saddle point method to the right-hand side of (3.23) we obtain the following estimate, which is uniformly valid on every closed bounded interval of the half line $\arg t = \pi$,

$$|I_{j,m}(t)| \leq K_2 A^m (m \log m)^m \quad j = 1, 2, m \in \mathbb{N}$$

where $K_2$ and $A$ are positive constants. Hence it can be concluded that $\varphi_\theta$ is quasi-analytic on the half line $\arg t = \pi$. More precisely, it is what is named by Ecalle a 'cohesive' function, belonging to the Denjoy class $^1D$.

The following result is easily derived from (3.3) and (3.13).

**Proposition 3.5.** The solutions $f$ and $f_n$ ($n \in \mathbb{Z}$) of (0.1) admit the following Laplace integral representations

$$f(z) = \int_0^{\infty} \varphi_\theta(t) e^{-t\psi_\theta(z)} dt, \quad \text{Re } \psi_\theta(z) > k_1$$

where $\theta$ is any real number and $k_1 > 0$, and

$$f_n(z) = \int_0^{\infty e^{\pi i}} \varphi_\theta(t) e^{-t\psi_\theta(z)} dt, \quad \text{Re } \psi_\theta(z) < k_2$$

where $\theta \in (\text{Im } \lambda + 2(n - 1)\pi, \text{Im } \lambda + 2n\pi)$ and $k_2 < 0$.

4. A quasi-analytic version of the 'bridge equation'

If $\theta \in \mathbb{R}$ is a singular value for (0.1), i.e. if $\theta = \text{Im } \lambda \mod 2\pi$, then $\varphi_\theta$ has quasi-analytic singularities at the points $t = e^{(2n+1)i\pi}, n \in \mathbb{Z}$. These singularities turn out to be of a rather simple type, $\varphi_\theta$ being the sum of a quasi-analytic
function on the half line $\arg t = (2n + 1)\pi$ and an analytic function in the sector $|\arg (t - e^{(2n+1)i})| < 3\pi/2$. In view of (3.2) we need only consider the singularity at $t = e^{ni}$. As before, let $a_1 = (i - 1)R_1$ and $a_2 = (-i - 1)R_1$, with $R_1 > R$. Furthermore, let $a_0 := -R_1$.

**Lemma 4.1.** Let $n \in \mathbb{Z}$ and $\theta = \text{Im } \lambda + 2n\pi$. Let $\theta_1$ and $\theta_2$ be real numbers such that

$$
\text{Im } \lambda + 2(n - 1)\pi < \theta_2 < \theta < \theta_1 < \text{Im } \lambda + 2(n + 1)\pi
$$

$C_1(n)$ will denote the path in $D_-$ consisting of the segment $[a_0, a_1]$ and the curve $C_1$, and $C_2(n)$ will denote the path in $D_-$ consisting of the segment $[a_0, a_2]$ and the curve $C_2$, where $C_j$ is defined by (2.11) for $j = 1, 2$.

For all $t \in (0, e^{ni})$, $\varphi_\theta(t)$ may be represented as follows

$$
\varphi_\theta(t) = \frac{1}{2\pi i} \left[ \int_{C_1(n)} f_n(z)e^{i\psi_\theta(z)}d\psi_\theta(z) - \int_{C_1(n)} f_{n+1}(z)e^{i\psi_\theta(z)}d\psi_\theta(z) \right]
$$

$$
- \int_{a_0}^{a_1} \int_{a_0}^{a_1} y_0(z)e^{2niz + i\psi_\theta(z)}d\psi_\theta(z)
$$

The first two terms on the right-hand side of (4.2) are quasi-analytic functions on the half line $\arg t = \pi$. Both belong to the Denjoy class $1D$. The function defined by the last term can be continued analytically to the sector $|\arg (t - e^{ni})| < 3\pi/2$.

**Proof.** We will use the same notations as in the proof of lemma 3.3. Starting from the representation (3.16), which is valid for any $\theta \in \mathbb{R}$, one easily verifies that (3.17) still holds and so does (3.18) in the case that $j = 2$. Due to the last inequality in (4.1) we now have

$$
\int_{a_1}^{a_1} p_{n+1}(z)y_0(z)e^{i\psi_\theta(z)}d\psi_\theta(z) = \int_{C_1} p_{n+1}(z)y_0(z)e^{i\psi_\theta(z)}d\psi_\theta(z)
$$

According to (1.11),

$$
p_n(z) = p_{n+1}(z) - P_n e^{2niz} \quad \text{if } \text{Im } z > R
$$

From (4.3) and (4.4) we derive the identity

$$
I_1(t) = \int_{C_1} p_{n+1}(z)y_0(z)e^{i\psi_\theta(z)}d\psi_\theta(z) - \int_{a_1}^{a_1} y_0(z)e^{2niz + i\psi_\theta(z)}d\psi_\theta(z)
$$

(4.2) now follows easily from (3.16) with the aid of (1.10).

The quasi-analytic nature of the first two terms on the right-hand side of (4.2) can be proved by an argument similar to the one used in the proof of proposition 3.4. The third integral obviously defines a function that is analytic in the half plane $|\arg (t - e^{ni})| < \pi/2$. It can be continued analytically to the
sector mentioned in lemma 4.1 by continuous deformation of the path of integration, in such a manner that this path stays entirely within the region $D_-$.

From lemma 4.1 it follows that the singular point $e^{n\lambda}$ can be 'bypassed quasi-analytically', or rather 'cohesively', both in the positive and in the negative sense (for the details cf. [11, §3.12])\(^2\). This gives rise to two distinct quasi-analytic branches of $\varphi_\theta$, to be denoted by $\varphi_\theta^+$ and $\varphi_\theta^-$, respectively. The difference $\varphi_\theta^+ - \varphi_\theta^-$ equals the difference of the two corresponding branches of the function defined by the last term in (4.2). Let

$$\omega = e^{n\lambda}$$

and let $\Delta^\theta_{\omega} \varphi_\theta$ be defined by

$$\Delta^\theta_{\omega} \varphi_\theta(t) = \varphi_\theta^+(\omega + t) - \varphi_\theta^-(\omega + t)$$

for all $t$ on the half line $\arg t = \pi$. Then we have the following result.

**Proposition 4.2.** Let $\theta \in \mathbb{R}$.

\begin{equation}
\Delta^\theta_{\omega} \varphi_\theta = \begin{cases} 
P_n \eta_\theta & \text{if } \theta = \text{Im } \lambda + 2n\pi \text{ for some } n \in \mathbb{Z} \\
0 & \text{otherwise}
\end{cases}
\end{equation}

where

\begin{equation}
\eta_\theta(t) = \frac{1}{2\pi i} \int_{U_-} y_\theta^0(z)e^{2n\lambda z + (t-1)\psi_\theta(z)}d\psi_\theta(z)
\end{equation}

$U_-$ is the contour used in (1.8).

This equation may be regarded as a quasi-analytic version of the 'bridge equation' (cf. [6]). For each singular value of $\theta$ it yields exactly one analytic invariant of level $1^+$.

**References**


\(^2\) In the present example the process of bypassing the singular point can be described as follows. For some $t$ on the half line $\arg t = \pi$ such that $|t| < 1$, $\varphi_\theta(t)$ is transformed into $\varphi_\theta^+(t)$, where $\theta'$ is either slightly smaller or slightly larger than $\theta$ and non-singular. Next, $\varphi_\theta$ is continued quasi-analytically to some $t'$ on the same half line, such that $|t'| > 1$ and, finally, $\varphi_\theta(t')$ is transformed back into $\varphi_\theta(t')$. 


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