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On Polynomial Liénard Systems Which Have Invariant Algebraic Curves

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§ 1. Introduction

In this paper, we consider the polynomial Liénard system

$$(\mathrm{L}) \qquad \begin{cases} \dot{x} = y, \\ \dot{y} = -f(x)y - g(x), \end{cases} \qquad \text{where } (\dot{\ }) := (d/dt).$$

Our purpose is to give an equivalent condition under which the system (L) has an invariant algebraic curve. We say that a complex algebraic curve $\Phi(x, y) = 0$ is invariant if it is a union of some complex solutions. Recently, Odani [4] gave a condition under which the system (L) has no invariant algebraic curves for $\deg f \geq \deg g$. However, his result cannot be applied to some interesting systems such as Examples 1, 2 in Section 3. So we want to generalize his result so as to be applicable to such systems.

Let M and N be the degree of the polynomials f and g with complex coefficients, respectively. Also we suppose that the conditions (i) $f \neq 0$, (ii) $M + 1 \geq N$. Our results are stated as follows.

Theorem. *Under the conditions* (i)–(ii), *the Liénard system* (L) *has an invariant algebraic curve if and only if there is an invariant curve* $y = P(x)$ *satisfying*

$$(*) \qquad g(x) = -[f(x) + P'(x)]P(x),$$

where $P(x)$ *or* $P(x) + \int f(x)dx$ *is a polynomial of degree at most one.*

Corollary. *If* f *and* g *are real polynomials satisfying the conditions* (i)–(ii), *then the Liénard system* (L) *has no algebraic closed orbits.*

This is an answer of the conjecture in [4]. The condition of our theorem is useful for seeking concretely the form of an invariant algebraic curve so as to be shown by two examples in Section 3.

§ 2. Proof of theorem

To prove the theorem, we assume that the system (L) has an irreducible algebraic curve $\Phi(x,y)=0$ as an invariant curve. Then, by the lemma of [4], we obtain the identity

$$(1) \qquad y\frac{\partial\Phi(x,y)}{\partial x}+[-f(x)y-g(x)]\frac{\partial\Phi(x,y)}{\partial y}=h(x,y)\Phi(x,y).$$

By the same argument as [4], we obtain

$$h(x,y)=h_0(x), \qquad \Phi(x,y)=\sum_{j=0}^{k}\Phi_j(x)y^j, \qquad \Phi_k\in \boldsymbol{C}-\{0\}.$$

So we assume without loss of generality that Φ is normalized in the sense that $\Phi_k(x)=1$. By changing the variable y into $w-\varphi(x)$ for the identity (1), it is transformed into the following identity

$$(2) \qquad [w-\varphi(x)]\frac{\partial\tilde{\Phi}(x,w)}{\partial x}+[-\tilde{f}(x)w-\tilde{g}(x)]\frac{\partial\tilde{\Phi}(x,w)}{\partial w}=h_0(x)\tilde{\Phi}(x,w),$$

where

$$\tilde{f}(x)=\tilde{F}'(x), \quad \tilde{F}(x)=F(x)-\varphi(x), \quad F(x)=\int f(x)dx,$$

$$\tilde{g}(x)=g(x)-[f(x)-\varphi'(x)]\varphi(x), \quad \tilde{\Phi}(x,w)=\Phi(x,w-\varphi(x)).$$

Consider a finite power series in the form

$$(3) \qquad \varphi(x)=\sum_{j=1-M}^{1}a_jx^j, \qquad a_1\neq 0,\ a_j\in\boldsymbol{C}.$$

We call such a series a finite Laurent series of degree one. Note that we can choose $\varphi(x)$ so that $\tilde{g}(x)=R_0$, where R_j is a unknown finite Laurent series of degree at most j. The idea is due to [5]. We set

$$(4) \qquad \tilde{\Phi}(x,w)=\sum_{j=0}^{k}\tilde{\Phi}_j(x)w^j, \qquad \tilde{\Phi}_k(x)=1.$$

From now on we abbreviate the parentheses of $\tilde{\Phi}_k(x), \varphi(x)$, etc.

By comparing the coefficient of w^j in (2), we obtain for $0\leq j\leq k$

$$(5) \qquad -\tilde{\Phi}'_{j-1}+\varphi\tilde{\Phi}_j{}'+[h_0+j\tilde{f}]\tilde{\Phi}_j+(j+1)\tilde{g}\tilde{\Phi}_{j+1}=0.$$

Let an integer m be the maximal degree of the polynomial $\tilde{\Phi}_j$ and an integer n

the maximal suffix attaining it, that is,

$$m = \max\{\deg \tilde{\Phi}_j \colon 0 \le j \le k\} \qquad \text{and} \qquad n = \max\{j \colon \deg \tilde{\Phi}_j = m\}.$$

By the same argument as [4], we obtain

$$\deg \tilde{\Phi}_{k-j} = j(M+1) \qquad \text{for } 0 \le j \le k-n. \tag{6}$$

In particular, by putting $j = k - n$ in (6), we get

$$m = \deg \tilde{\Phi}_n = r(M+1), \tag{7}$$

where $r = k - n$. By applying (7) to (5), we get

$$h_0(x) + n\tilde{f}(x) = -ma_1 + R_{-1}. \tag{8}$$

Then for $0 \le j \le r$ the polynomial $\tilde{\Phi}_{k-j}$ is written in the form

$$\tilde{\Phi}_{k-j} = {}_rC_j \tilde{F}^j - j\,{}_rC_j a_1 b_M^{j-1}(M+1)x^{(j-1)(M+1)+1} + R_{(j-1)(M+1)}, \tag{9}$$

where ${}_rC_j$ denote the binominal coefficient and b_j the coefficient of x^j of $\tilde{F}$. By means of (8), we can prove (9) by induction.

By putting $j = r$ in (9), we get

$$\tilde{\Phi}_n = \tilde{F}^r - ra_1 b_M^{r-1}(M+1)x^{m-M} + R_{m-M-1}. \tag{10}$$

So we obtain

$$\tilde{\Phi}_n' = r\tilde{f}\tilde{F}^{r-1} - ra_1 b_M^{r-1}(m-M)(M+1)x^{m-M-1} + R_{m-M-2}. \tag{11}$$

Moreover, by putting $j = r - 1$ in (9), we get

$$\tilde{\Phi}_{n+1} = r\tilde{F}^{r-1} + R_{m-2M-1}. \tag{12}$$

By putting $j = n$ in (5), we get

$$\tilde{\Phi}_{n-1}' = \varphi\tilde{\Phi}_n' + [h_0 + n\tilde{f}]\tilde{\Phi}_n + (n+1)\tilde{g}\tilde{\Phi}_{n+1}.$$

By substituting (10), (11) and (12) to it, we obtain

$$\tilde{\Phi}_{n-1}' = r\varphi\tilde{f}\tilde{F}^{r-1} - ma_1\tilde{F}^r + ra_1^2 b_M^{r-1}M(M+1)x^{m-M} + R_{m-M-1}. \tag{13}$$

On the other hand, by putting $j = n - 1$ to (5), we get

$$\tilde{\Phi}_{n-1}' = R_{m-M-1}. \tag{14}$$

By (13) and (14), we obtain

$$r[\varphi\tilde{f} - (M+1)a_1\tilde{F} + M(M+1)a_1^2 x]\tilde{F}^{r-1} = R_{m-M-1}. \tag{15}$$

By the definition of $\tilde{f}$, $\tilde{F}$ and $\tilde{g}$, we get

$$r[g-(M+1)a_1F+(M+1)^2a_1^2x]F^{r-1}=R_{m-M-1}.$$

By comparing the terms of degree $> m-M-1$, we get

$$r[g-(M+1)a_1F+(M+1)^2a_1^2x]=c, \qquad \text{where } c\in \boldsymbol{C}. \tag{16}$$

With the above preliminaries, we shall prove this theorem by dividing into the following two categories.

First, we consider the case of $r=0$. Then, by the definition of m, we obtain

$$\tilde{\Phi}_j=R_0 \qquad \text{for every } 0\le j\le k.$$

Thus we obtain

$$\begin{aligned}\Phi(x,y)&=\tilde{\Phi}(x,y+\varphi(x))\\&=\sum_{j=0}^{k}\tilde{\Phi}_j(x)[y+\varphi(x)]^j\\&=\sum_{j=0}^{k}c_j(y+a_1x+a_0)^j, \qquad \text{where } c_k=1,\ c_j\in \boldsymbol{C}.\end{aligned}$$

Since $\Phi(x,y)=0$ is an irreducible algebraic curve, we have

$$\Phi(x,y)=y+P(x), \quad P(x)=-a_1x-a_0.$$

Next, we consider the case of $r\neq 0$. We set

$$P(x)=-F(x)+(M+1)a_1x+c/r(M+1)a_1.$$

Then we obtain the equality $(*)$. Thus, the system (L) has the algebraic curve $y=P(x)$ as an invariant curve.

On the other hand, we see easily that the converse holds by the lemma of [4]. Therefore the proof of our theorem is now completed. □

As the special case of our theorem, we shall consider non-existence of algebraic closed orbits. So we shall give a proof of the corollary.

Proof of Corollary. We assume that the system (L) has an algebraic closed orbit. Since the closed orbit surrounds an equilibrium point, we can change the variables so that the origin becomes an equilibrium point surrounded by the closed orbit. By using the same method as our theorem, we need to consider two cases. In the first case, the system (L) has no invariant algebraic curves but a straight line. This is in contradiction to the assumption. In the second case, the system (L) has the algebraic curve $y=P(x)$ as an invariant curve. Since $P(0)=0$, it gets through the origin. Thus it must get across the closed orbit. This is in contradiction to the assumption. Therefore we have that the system (L) has no algebraic closed orbits. □

§ 3. Applications

In this section, we shall apply our theorem to two concrete examples. One is Bogdanov-Takens system, which is an important example in Birfucation Theory; see [2]. The other is FitzHugh-Nagumo system, which is an important model for nerve membrane; see [1], [3]. By using the condition of our theorem, we obtain the following results easily.

Example 1. Consider Bogdanov-Takens system

$$\text{(BT)} \qquad \begin{cases} \dot{x}=y \\ \dot{y}=\mu_1+\mu_2 y+x^2+xy. \end{cases}$$

Then (a) in the case of $\mu_1=-(\mu_2+1)^2$, it has the straight line $y=-x+\mu_2+1$ as an invariant curve; (b) in the case of $\mu_2=-2$, it has the algebraic curve $y=(1/2)x^2+(1/2)\mu_1$ as an invariant curve; (c) otherwise it has no invariant algebraic curves.

Example 2. Consider FitzHugh-Nagumo system

$$\text{(FHN)} \qquad \begin{cases} \dot{x}=y-(1/3)x^3+x+\mu \\ \dot{y}=-\rho(x+by-a), \end{cases} \qquad \text{where } \rho\neq 0.$$

Then (a) in the case of $2\rho b^2+6b-9=b\mu+a=0$, it has the algebraic curve $y=-(\rho b/3)x$ as an invariant curve; (b) otherwise it has no invariant algebraic curves.

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