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Some Existence Results for Sublinear Elliptic Problems in $\boldsymbol{R}^N$

By

Marino Badiale and Fernando Dobarro

(Università di Padova, Italia and Instituto Argentino de Matemáticas, Argentina)

Abstract. In this paper we prove existence of (at least) a solution for the sublinear elliptic problems $(P_\pm)$ (see below), under different sets of hypotheses on the potential q. In particular in all our results we allow q to change sign, which is relevant for many applications.

1. Introduction

This paper is concerned with the existence of nontrivial solutions to the sublinear elliptic problems

$$(P_+)\qquad \begin{cases} -\Delta u + qu = u^\alpha & \text{in } \boldsymbol{R}^N \\ u \geq 0\,, \quad u \in H^1_{loc}(\boldsymbol{R}^N) \end{cases}$$

$$(P_-)\qquad \begin{cases} -\Delta u + qu = -u^\alpha & \text{in } \boldsymbol{R}^N \\ u \geq 0\,, \quad u \in H^1_{loc}(\boldsymbol{R}^N) \end{cases}$$

where $\alpha \in]0, 1[$ and $q\colon \boldsymbol{R}^N \to \boldsymbol{R}$ is a function which may change sign.

Problems $(P_\pm)$ arise in different situations. A first example is the parabolic degenerate equation

$$v_t - \Delta v^m = \rho(x)v^p \qquad (x, t) \in \boldsymbol{R}^N \times \boldsymbol{R}_+ \tag{1.1}$$

in the special case $p = m$ with $m > 1$. In this case one can look for solutions of (1.1) by separation of variables, i.e. by setting

$$v(x, t) = u^{1/m}(x)w(t)\,.$$

Then one gets easily the equations

$$\Delta u(x) + \rho(x)u(x) = Cu^\alpha(x) \tag{1.2}$$

$$w_t = Cw^{1/\alpha} \tag{1.3}$$

where C is a constant and $\alpha = 1/m < 1$. Problem (1.1) is then reduced to (1.2) and then, by a slight change, to $(P_\pm)$. This idea of separation of variables was mentioned in [BK], [AP]. There it is introduced for the study of the porous media equation

$$\rho(x)u_t = \Delta u^m$$

where ρ is a non negative function. One can find this idea of separation of variables also in [BNP]. There it is applied to the nonlinear diffusion equation (1.1), with $\rho(x) = \lambda$, a nonnegative constant. In the papers [AP], [BNP], the corresponding equations $(P_\pm)$ (in bounded domains) are also connected with the large time behavior of solutions of (1.1). Equation (1.1) appears in several situations: in the study of population dynamics, of reaction-diffusion processes, of filtration in porous media with absorption (see the papers quoted above and also [BP], [BPT], [BKP], [E], [N], [K], [A], [GM], [GN], [Sc]). We remark that in several cases it is important, for applications, to suppose that q changes sign; in our existence results this is allowed (see below).

In a totally different context positive solutions of $(P_\pm)$ are related with problems on scalar curvature of warped products of semiriemannian manifolds. Consider two semiriemannian manifolds $M = (M_m, g)$ and $N = (N_n, h)$ of dimension m, n and metrics g, h respectively. Consider a function $f\colon M \to \boldsymbol{R}_+$. Then the warped product $M \times_f N = ((M \times_f N)_{m+n}, g + f^2 h)$ has been defined by Bishop and O'Neil ([BN], [O], [B]). If M, N are Riemannian manifold and one writes $f = u^{2/(n+1)}$ then one gets the following equation:

$$(1.4) \qquad -\frac{4}{n+1}\Delta_g u + R(x)u(x) + H(y)u^{(n-3)/(n+1)}(x) = \tilde{R}(x, y)u(x)\,.$$

Equation (1.4) was first obtained in [DL]. In (1.4) x denotes the coordinates on M, y on N, Δ_g is the Laplace-Beltrami operator on M and R, H, $\tilde{R}$ are the scalar curvatures of M, N, $M \times_f N$ respectively. If one supposes H constant, then $\tilde{R}$ is independent of y and one easily gets the equation

$$(1.5) \qquad -\Delta_g u + qu = -Hu^\alpha\,, \qquad x \in M\,,$$

where $q = R - \tilde{R}$. Equations (1.4), (1.5) were studied in [DL], [CDM]. In this last paper the authors get necessary and sufficient conditions for the existence of a nonnegative solution of (1.5), assuming M compact. In connection with this geometric problem, our results are a generalization of the previous results (in particular of those obtained in [CDM]) to the case of a non compact M. We work in a very particular case of a non compact M, that is in the case $M = \boldsymbol{R}^N$. Then the Laplace-Beltrami operator Δ_g reduces to the usual Laplace operator and the scalar curvature $R(x)$ of M is zero. We remark that, as in [DL] and [CDM], in our results a crucial role is played by the sign of $\lambda_1(q)$ (see section 2 for the definition of $\lambda_1(q)$).

Our paper is organized as follows: Section 1 is the introduction, in Section 2 we give the notations and some preliminary remarks, in Section 3 we study problem (P_-) and in Section 4 we study problem (P_+). In Section 5 we give some other existence results for $(P_\pm)$ under the assumptions that the potential q is unbounded at infinity.

Our main existence results are stated in Theorem 3.2, Theorem 4.1, Theorem 5.1 and Proposition 5.4. Throughout this paper we consider problems on all $\boldsymbol{R}^N$. We note however that our arguments apply to problems like $(P_\pm)$ on more general unbounded sets (exterior domains, for example).

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2. Notations and preliminaries

We introduce in this section our basic hypotheses and definitions. As first thing we remark that we search solutions of $(P_\pm)$ in the usual weak sense; that is, a solution of (P_+) is a function $u \in H^1_{loc}(\boldsymbol{R}^N)$ such that $u \geq 0$ and

$$\int_{\boldsymbol{R}^N} \nabla u \nabla \phi + \int_{\boldsymbol{R}^N} q u \phi = \int_{\boldsymbol{R}^N} u^\alpha \phi , \qquad \forall \phi \in C_0^\infty(\boldsymbol{R}^N) .$$

The meaning of solution for (P_-) is obvious. We remark that one can get more regularity for the solutions of $(P_\pm)$, by usual arguments; see the following sections for precise statements. We remark that we don't get, in general, informations on the asymptotic behavior (as $|x| \to \infty$) of our solutions: we know only that they are in $H^1_{loc}(\boldsymbol{R}^N)$. However for problem (P_-) we get more precise information: our solutions are in fact in $H^1(\boldsymbol{R}^N)$. We get solutions vanishing at infinity also for problem (P_+) if some particular hypotheses on q is assumed (see Section 5).

Throughout this paper we assume

$$q \in L^\infty_{loc}(\boldsymbol{R}^N)$$

and we define

$$\lambda_1(q) = \inf \left\{ \int_{\boldsymbol{R}^N} |\nabla \phi|^2 + \int_{\boldsymbol{R}^N} q \phi^2 \,\middle|\, \phi \in C_0^\infty(\boldsymbol{R}^N), \int_{\boldsymbol{R}^N} \phi^2 = 1 \right\} .$$

In [CDM] the authors give the analogous definition of $\lambda_1(q)$ for the problem on a compact manifold. They prove that $\lambda_1(q) > 0$ (respectively $\lambda_1(q) < 0$) is a necessary and sufficient condition for the existence of non negative solutions of (1.5) when $H < 0$ (respectively $H > 0$). In this paper we shall use hypotheses of this kind, but we are not able to prove existence of solutions for $(P_\pm)$ assuming only $\lambda_1(q) > 0$ or $\lambda_1(q) < 0$: we need also some hypotheses on the behavior of $q(x)$ as $|x| \to \infty$. As we work in an unbounded domain, this seems to us very reasonable.

We introduce now some notations. If $p \geq 1$ $L^p(\boldsymbol{R}^N)$ is the usual space of functions such that $|u|^p$ is integrable; $|u|_p = \left(\int_{\boldsymbol{R}^N} |u|^p\right)^{1/p}$ is the usual norm of $L^p(\boldsymbol{R}^N)$. If $|\nabla u| \in L^2(\boldsymbol{R}^N)$ we define $|u|_{1,2} = \left(\int_{\boldsymbol{R}^N} |\nabla u|^2\right)^{1/2}$. $D^{1,2}$ is the closure of $C_0^\infty(\boldsymbol{R}^N)$ with respect to the norm $|\cdot|_{1,2}$. If $u \in \boldsymbol{R}$ we define, as usual, $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and if $\phi\colon \boldsymbol{R}^N \to \boldsymbol{R}$ then $\phi^+(x) = (\phi(x))^+$, $\phi^-(x) = (\phi(x))^-$. We shall write $\to$ for strong convergence and $\rightharpoonup$ for weak convergence. For $R > 0$ we define $B_R = \{x \in \boldsymbol{R}^N \mid |x| < R\}$ and $B = B_1$. We define as usual $|\cdot|_\infty$ the norm of the space $L^\infty(\boldsymbol{R}^N)$ and $|\cdot|_{\infty,R}$ the norm of $L^\infty(B_R)$. $2^* = 2N/(N-2)$ is the exponent of Sobolev embedding. We shall denote $o(1)$ any vanishing real sequence and by C (or C_i, $i = 1, 2, \ldots$) any positive constant.

3. Problem (P_-)

We shall study problem (P_-) by a variational technique. As first thing we shall introduce the space E in which we set the problem. Then we shall define in E a variational problem whose solutions are solutions of (P_-).

Consider the norm on $C_0^\infty(\boldsymbol{R}^N)$ given by $\|\phi\| = |\phi|_{\alpha+1} + |\phi|_{1,2}$. We define E to be the closure of $C_0^\infty(\boldsymbol{R}^N)$ with respect to this norm. We remark that $|u|_2^2 \leq C|u|_{1,2}^{2^*} + |u|_{\alpha+1}^{\alpha+1}$, so that $E \subset H^1(\boldsymbol{R}^N)$ and the inclusion is continuous.

We define

$$M = \left\{ u \in E \,\middle|\, \int_{\boldsymbol{R}^N} |\nabla u|^2 dx + \int_{\boldsymbol{R}^N} q|u|^2 dx = -1 \right\}$$

and

$$m = \inf\left\{ \int_{\boldsymbol{R}^N} |u|^{\alpha+1} dx \,\middle|\, u \in M \right\}.$$

We shall assume the following hypotheses on q:

(q_1) $q \in L^\infty(\boldsymbol{R}^N)$,
(q_2) $\lambda_1(q) < 0$,
(q_3) $q^- \in L^{N/2}(\boldsymbol{R}^N)$.

We shall study the following variational problem.

$$(vp_1) \qquad \text{find } u \in M \text{ such that } \int_{\boldsymbol{R}^N} |u|^{\alpha+1} dx = m.$$

Remark that $\lambda_1(q) < 0$ implies $M \neq \emptyset$, so that (vp_1) is not trivial.

We shall prove the following theorem.

Theorem 3.1. *Assume* (q_1), (q_2), (q_3). *Then* $\exists u \in M$, $u \geq 0$, *such that* $\int_{\boldsymbol{R}^N} |u|^{\alpha+1} dx = m$.

By usual arguments one sees that a nonnegative solution of (vp_1) gives a weak solution of the equation $-\Delta u + qu = \lambda u^\alpha$, where $\lambda < 0$. So by a rescaling we get a solution to (P_-). From Theorem 3.1 we then deduce

Theorem 3.2. *Assume* (q_1), (q_2), (q_3). *Then there exists a nontrivial weak solution* u *of* (P_-). *Also* $u \in E$ *(so* $u \in H^1(\boldsymbol{R}^N)$*) and* $u \geq 0$.

See Remark 3.5 below for a brief comment on hypotheses (q_2), (q_3).

Proof of Theorem 3.1. Theorem 3.1 will be proved by some lemmas. The hypotheses of the lemmas will be those of Theorem 3.1. We remark that if $\{u_k\}_k$ is a minimizing sequence for (vp_1), then also $\{|u_k|\}_k$ it is, so we can always suppose $u_k \geq 0$, for a minimizing sequence.

Lemma 3.1. *It is* $m > 0$.

Proof of Lemma 3.1. Of course $m \geq 0$, so we must prove $m \neq 0$. Suppose by contradiction that $\{u_k\}_k$ is a minimizing sequence for (vp_1) such that $\int_{\boldsymbol{R}^N} |u_k|^{\alpha+1} dx \to 0$. Define $a_k = \left(\int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx \right)^{1/2}$. Of course we have, up to a subsequence, that one of the following two cases must occur:

$$\alpha_1) \quad a_k \to \lambda \geq 0\,, \qquad \alpha_2) \quad a_k \to \infty\,.$$

In the case $\alpha_1)$ $\{|u_k|_{1,2}\}_k$ is bounded; also $\{|u_k|_{\alpha+1}\}_k$ is bounded, so $\{u_k\}_k$ is bounded in E and then in $H^1(\boldsymbol{R}^N)$. So we get, up to a subsequence, $u_k \rightharpoonup u_0$ in $H^1(\boldsymbol{R}^N)$ and $u_k \to u_0$ in $L^{\alpha+1}(B_R)$ and in $L^2(B_R)$, for each $R > 0$. As $\int_{\boldsymbol{R}^N} |u_k|^{\alpha+1} dx \to 0$ we have $u_0 = 0$.

Fix now $\varepsilon > 0$. By (q_3) there is R_ε such that $\int_{|x| > R_\varepsilon} (q^-)^{N/2} < (\varepsilon)^{N/2}$. We have:

$$\begin{aligned}
(3.1) \qquad -1 &= \int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx + \int_{\boldsymbol{R}^N} q|u_k|^2 dx \geq -\int_{\boldsymbol{R}^N} q^- |u_k|^2 dx \\
&= -\int_{B_{R_\varepsilon}} q^- |u_k|^2 dx - \int_{|x| > R_\varepsilon} q^- |u_k|^2 dx \\
&\geq \sigma_k - \left(\int_{|x| > R_\varepsilon} (q^-)^{N/2} dx \right)^{2/N} \left(\int_{|x| > R_\varepsilon} |u_k|^{2^*} dx \right)^{2/2^*} \geq \sigma_k - \varepsilon C
\end{aligned}$$

where $\sigma_k \to 0$ as $k \to \infty$ and C is a constant independent from ε and k. To get (3.1) we have used Hölder inequality, Sobolev embedding and the hypothesis that $\{a_k\}_k$ is bounded. It is then obvious that for large k's and small ε's we get a contradiction. This proves that $\alpha_1)$ cannot hold.

Suppose now that $\alpha_2)$ holds. Define $v_k = u_k/a_k$, so that $|v_k|_{1,2} = 1$; of course

$$\int_{\boldsymbol{R}^N} |v_k|^{\alpha+1} dx \to 0 \qquad \text{and} \qquad v_k \rightharpoonup 0 \qquad \text{in } H^1(\boldsymbol{R}^N),$$

as before. Also we have

$$1 + \int_{\boldsymbol{R}^N} q|v_k|^2 dx = o(1) \qquad \text{that is} \qquad \int_{\boldsymbol{R}^N} q|v_k|^2 dx \to -1 . \tag{3.2}$$

We can get a contradiction from (3.2) with the same argument as above. Indeed we have, up to a subsequence, $v_k \to 0$ in $L^{\alpha+1}(B_R)$ and in $L^2(B_R)$, for all $R > 0$. Then for any $\varepsilon > 0$ we fix R_ε as above and we get

$$\begin{aligned} \int_{\boldsymbol{R}^N} q v_k^2 &\geq -\int_{\boldsymbol{R}^N} q^- v_k^2 = -\int_{|x|<R_\varepsilon} q^- v_k^2 - \int_{|x|\geq R_\varepsilon} q^- v_k^2 \\ &\geq -\int_{|x|<R_\varepsilon} q^- v_k^2 - \left(\int_{|x|\geq R_\varepsilon} (q^-)^{N/2}\right)^{2/N} \left(\int_{|x|\geq R_\varepsilon} |u_k|^{2^*}\right)^{2/2^*} \\ &\geq -\int_{|x|<R_\varepsilon} q^- v_k^2 - \varepsilon C , \end{aligned}$$

where C is independent from k and ε. It is $\int_{|x|<R_\varepsilon} q^- v_k^2 \to 0$ as $k \to \infty$, so we get $\liminf_{k\to\infty} \int_{\boldsymbol{R}^N} q v_k^2 \geq -\varepsilon C$, $\forall \varepsilon > 0$. This implies $\liminf_{k\to\infty} \int_{\boldsymbol{R}^N} q v_k^2 \geq 0$, a contradiction with (3.2).

In any case from $m = 0$ we get a contradiction, so it must be $m > 0$ and Lemma 3.1 is proved. □

Lemma 3.2. *Let $\{u_k\}_k$ be a minimizing sequence for (vp_1). Then there are c_1, $c_2 > 0$ such that for all k it is $c_1 \leq \int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx \leq c_2$.*

Proof of Lemma 3.2. By contradiction: if the thesis were not true we should have, up to a subsequence,

$$\int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx \to \infty \qquad \text{or} \qquad \int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx \to 0 .$$

In the first case we argue exactly as in Lemma 3.1: we define a_k, v_k as before and we get $v_k \rightharpoonup 0$ in $H^1(\boldsymbol{R}^N)$ and $\int_{\boldsymbol{R}^N} q v_k^2 \to 0$. From this we get a contradiction exactly as before. In the case $\int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx \to 0$ we get easily that $u_k \rightharpoonup 0$ in $H^1(\boldsymbol{R}^N)$ and then $u_k \to 0$ in $L^2(B_R)$ for all $R > 0$. Repeating the argument used above for v_k we get

$$\liminf_k \int_{\boldsymbol{R}^N} q|u_k|^2 dx \geq 0 . \tag{3.3}$$

On the other hand $u \in M$ so we have

$$-1 = \int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx + \int_{\boldsymbol{R}^N} q|u_k|^2 dx \geq \int_{\boldsymbol{R}^N} q|u_k|^2 dx . \tag{3.4}$$

Of course (3.3) and (3.4) give a contradiction. Lemma 3.2 is then proved. □

Lemmas 3.1, 3.2 allow us to consider a minimizing sequence $\{u_k\}_k$ for (vp_1) such that

$$\int_{\boldsymbol{R}^N} |u_k|^{\alpha+1} dx \to m > 0 ; \qquad \int_{\boldsymbol{R}^N} |\nabla u|^2 dx \to \lambda > 0 .$$

We now apply the Concentration-Compactness Principle of P. L. Lions (CCP). We recall that CCP has been one of the main tools used in the last years for the study of variational problems on unbounded domains. See [L1], [L2], [L3], [L4], [EL], for an idea of CCP and its applications. The principle is a general tool which must be adapted to the different situations. We state it in the form more convenient for our purposes. The adaptation is an easy task which we leave to the reader.

Concentration-Compactness Principle

Let $\rho_k = |\nabla u_k|^2 + |u_k|^{\alpha+1} \in L^1(\boldsymbol{R}^N)$. We have $\int_{\boldsymbol{R}^N} \rho_k \to \beta \in]0, +\infty[$ (up to a subsequence). Then one of the following possibilities must hold, up to a subsequence:

(i) There is a sequence $\{y_k\}_k \subset \boldsymbol{R}^N$ such that

$$\forall \varepsilon > 0,\ \exists R_\varepsilon,\ k_\varepsilon > 0 \text{ such that } \int_{y_k + B_{R_\varepsilon}} \rho_k > \beta - \varepsilon , \qquad \forall k \geq k_\varepsilon \tag{3.5}$$

(ii)
$$\lim_k \left(\sup_{y \in \boldsymbol{R}^N} \int_{y + B_R} \rho_k \right) = 0 , \qquad \forall R > 0 .$$

(iii) There are the following sequences: $\{u_{1,k}\}_k$, $\{u_{2,k}\}_k \subset E$, $\{y_k\}_k \subset \boldsymbol{R}^N$, $\{R_k\}_k \subset \boldsymbol{R}$ and there are $R > 0$, $\delta \in]0, \beta[$ such that:

cc_1) $\|u_k - (u_{1,k} + u_{2,k})\| \to 0$,

cc_2) $\operatorname{supp} u_{1,k} \subset y_k + B_{2R}$, $\operatorname{supp} u_{2,k} \subset \boldsymbol{R}^N \backslash (y_k + B_{R_k})$,

cc_3) $u_{1,k}|_{y_k+B_R} = u_k|_{y_k+B_R}$, $u_{2,k}|_{\boldsymbol{R}^N\backslash(y_k+B_{2R_k})} = u_k|_{\boldsymbol{R}^N\backslash(y_k+B_{2R_k})}$,

cc_4) $\displaystyle\int_{\boldsymbol{R}^N} |\nabla u_{1,k})|^2 dx + \int_{\boldsymbol{R}^N} |u_{1,k}|^{\alpha+1} \to \beta - \delta$,

$$\int_{\boldsymbol{R}^N} |\nabla u_{2,k})|^2 dx + \int_{\boldsymbol{R}^N} |u_{2,k}|^{\alpha+1} \to \delta,$$

cc_5) $R_k \to +\infty$.

In the following lemma we prove the cases ii), iii) of CCP cannot hold.

Lemma 3.3. *Let* $\{u_k\}_k$ *be a minimizing sequence for* (vp_1) *such that* $\int_{\boldsymbol{R}^N} |u_k|^{\alpha+1} dx \to m > 0$ *and* $\int_{\boldsymbol{R}^N} |\nabla u|^2 dx \to \lambda > 0$. *Then cases* ii), iii) *of CCP cannot hold.*

Proof of Lemma 3.3. Case ii) gives us immediately a contradiction, with the same argument of previous lemmas: indeed it gives $\int_{B_R} q|u_k|^2 dx \to 0$ as $k \to \infty$, for all $R > 0$, from which one derives $\liminf_k \int_{\boldsymbol{R}^N} q|u_k|^2 dx \geq 0$. As before, this gives a contradiction with the hypothesis $u \in M$. In case iii) we distinguish two cases:

j_1) $\{|y_k|\}_k$ is bounded; j_2) $|y_k| \to \infty$ up to subsequences.

In case j_1) we easily get

$$\int_{\boldsymbol{R}^N} q^- |u_{2,k}|^2 dx \to 0. \tag{3.6}$$

To prove (3.6) choose $\varepsilon > 0$ and fix $R_\varepsilon > 0$ such that $\int_{|x|>R_\varepsilon} (q^-)^{N/2} dx < \varepsilon$; as $\{|y_k|\}_k$ is bounded and $R_k \to \infty$ it is $\boldsymbol{R}^N \backslash (y_k + B_{R_k}) \subset \boldsymbol{R}^N \backslash B_{R_\varepsilon}$ for large k's, so that

$$\begin{aligned}
\int_{\boldsymbol{R}^N} q^- |u_{2,k}|^2 dx &= \int_{\boldsymbol{R}^N\backslash(y_k+B_{R_k})} q^- |u_{2,k}|^2 dx \\
&\leq \left(\int_{\boldsymbol{R}^N\backslash(y_k+B_{R_k})} (q^-)^{N/2} dx \right)^{2/N} \left(\int_{\boldsymbol{R}^N\backslash(y_k+B_{R_k})} |u_{2,k}|^{2^*} dx \right)^{2/2^*} \\
&\leq \varepsilon C,
\end{aligned} \tag{3.7}$$

where C is a constant independent from ε and k. From (3.7) one easily deduces (3.6).

At this point we have again to distinguish two cases; indeed the argument is slight different if $\int_{\boldsymbol{R}^N} |\nabla u_{2,k}|^2 dx \to \eta > 0$ or if $\int_{\boldsymbol{R}^N} |\nabla u_{2,k}|^2 dx \to 0$, up to a subsequence. In the first case we get

$$\begin{aligned}-1 &= \int_{\boldsymbol{R}^N} |\nabla u_{1,k}|^2 dx + \int_{\boldsymbol{R}^N} q|u_{1,k}|^2 dx + \int_{\boldsymbol{R}^N} |\nabla u_{2,k}|^2 dx + \int_{\boldsymbol{R}^N} q|u_{2,k}|^2 dx + o(1)\\ &\geq \int_{\boldsymbol{R}^N} |\nabla u_{1,k}|^2 dx + \int_{\boldsymbol{R}^N} q|u_{1,k}|^2 dx + \eta + o(1),\end{aligned}$$

so we get

$$\int_{\boldsymbol{R}^N} |\nabla u_{1,k}|^2 dx + \int_{\boldsymbol{R}^N} q|u_{1,k}|^2 dx \leq -1 - \eta - \varepsilon_k ,$$

where $\varepsilon_k \to 0$.

Let us define

$$\lambda_k = \frac{1}{\left|\int_{\boldsymbol{R}^N} |\nabla u_{1,k}|^2 dx + \int_{\boldsymbol{R}^N} q|u_{1,k}|^2 dx\right|^{1/2}} \leq \frac{1}{(1 + \eta + \varepsilon_k)^{1/2}} < 1 .$$

The inequalities hold for large k's. Let $v_k = \lambda_k u_{1,k}$. We have

$$\int_{\boldsymbol{R}^N} |\nabla v_k|^2 dx + \int_{\boldsymbol{R}^N} q|v_k|^2 dx = -1 \tag{3.8}$$

and

$$\begin{aligned}m + o(1) &= \int_{\boldsymbol{R}^N} |u_k|^{\alpha+1} dx = \int_{\boldsymbol{R}^N} |u_{1,k}|^{\alpha+1} dx + \int_{\boldsymbol{R}^N} |u_{2,k}|^{\alpha+1} dx + o(1)\\ &\geq \left(\frac{1}{\lambda_k}\right)^{\alpha+1} \int_{\boldsymbol{R}^N} |v_k|^{\alpha+1} + o(1) \geq (1 + \eta + \varepsilon_k)^{(\alpha+1)/2} \int_{\boldsymbol{R}^N} |v_k|^{\alpha+1} dx + o(1)\\ &\geq (1 + \eta + \varepsilon_k)^{(\alpha+1)/2} m + o(1) .\end{aligned}$$

Passing to the limit we then get $m \geq (1 + \eta)^{(\alpha+1)/2} m$, a contradiction.

In the case $\int_{\boldsymbol{R}^N} |\nabla u_{2,k}|^2 dx \to 0$ we must have $\int_{\boldsymbol{R}^N} |u_{2,k}|^{\alpha+1} \to \delta > 0$. (3.6) holds true also in this case, so we can write

$$\begin{aligned}-1 &= \int_{\boldsymbol{R}^N} |\nabla u_{1,k}|^2 dx + \int_{\boldsymbol{R}^N} q|u_{1,k}|^2 dx + \int_{\boldsymbol{R}^N} |\nabla u_{2,k}|^2 dx + \int_{\boldsymbol{R}^N} q|u_{2,k}|^2 dx + o(1) \\ &\geq \int_{\boldsymbol{R}^N} |\nabla u_{1,k}|^2 dx + \int_{\boldsymbol{R}^N} q|u_{1,k}|^2 dx + o(1)\end{aligned}$$

and

$$m = \int_{\boldsymbol{R}^N} |u_{1,k}|^{\alpha+1} dx + \int_{\boldsymbol{R}^N} |u_{2,k}|^{\alpha+1} dx + o(1) = \int_{\boldsymbol{R}^N} |u_{1,k}|^{\alpha+1} dx + \delta + o(1).$$

Defining λ_k and v_k exactly as above we get $\int_{\boldsymbol{R}^N} |\nabla v_k|^2 dx + \int_{\boldsymbol{R}^N} q|v_k|^2 dx = -1$. So we have

$$m = \left(\frac{1}{\lambda_k}\right)^{\alpha+1} \int_{\boldsymbol{R}^N} |v_k|^{\alpha+1} dx + \delta + o(1) \geq (1 + o(1))m + \delta + o(1)$$

and then $m \geq m + \delta$, a contradiction. We have ruled out the case j_1). It is then easy to treat the case j_2). It is enough to substitute $u_{1,k}$ and $u_{2,k}$ in the argument and to remark that if j_2) holds then $\int_{\boldsymbol{R}^N} q^- |u_{1,k}|^2 dx \to 0$. This derives from the fact that in case j_2) for any fixed $R' > 0$ there is $\bar{k}$ such that $\operatorname{supp} u_{1,k} \subset \boldsymbol{R}^N \backslash B_{R'}$. The argument can then be repeated to get a contradiction from j_2). In this way we have ruled out case iii) of CCP and lemma 3.4 is proved. □

We conclude the proof of Theorem 3.1 in the following lemma.

Lemma 3.4. *Problem* (vp_1) *has a nonnegative solution* u_0.

Proof of Lemma 3.4. Let $\{u_k\}_k$ be a minimizing sequence for (vp_1), as above. As we have seen, we can suppose $u_k \geq 0$, $\int_{\boldsymbol{R}^N} |u_k|^{\alpha+1} dx \to m > 0$ and $\int_{\boldsymbol{R}^N} |\nabla u|^2 dx \to \lambda > 0$. By Lemma 3.3 case i) of CCP holds. If $\{y_k\}$ is the sequence given in the statement of case i), then $\{|y_k|\}_k$ is bounded. Indeed, if $|y_k| \to \infty$ (up to a subsequence) we easily get

$$\lim_k \int_{B_R} |\nabla u_k|^2 = \lim_k \int_{B_R} q|u_k|^2 = 0$$

for all $R > 0$. From this and the hypothesis $u \in M$ we get a contradiction, as we have done several times. So $\{|y_k|\}_k$ is bounded. Now let $u_0 \in E$ such that

$$u_k \rightharpoonup u_0 \quad \text{in } H^1(\boldsymbol{R}^N).$$

As we are in case i) of CCP and $\{|y_k|\}_k$ is bounded, it is easy to verify that

$$(3.9)\quad \forall \varepsilon > 0 \quad \exists R_\varepsilon > 0 \quad \text{such that } \forall k \quad \int_{|x| \geq R_\varepsilon} |\nabla u_k|^2 + \int_{|x| \geq R_\varepsilon} |u_k|^{\alpha+1} < \varepsilon .$$

From (3.9) and compactness of Sobolev embeddings it is easy to deduce

$$\int_{\boldsymbol{R}^N} q|u_k|^2 \to \int_{\boldsymbol{R}^N} q|u_0|^2, \qquad \int_{\boldsymbol{R}^N} |u_k|^{\alpha+1} \to \int_{\boldsymbol{R}^N} |u_0|^{\alpha+1}.$$

Of course it is $\int_{\boldsymbol{R}^N} |\nabla u_0|^2 \leq \liminf_k \int_{\boldsymbol{R}^N} |\nabla u_k|^2$, so that

$$\int_{\boldsymbol{R}^N} (|\nabla u_0|^2 + q|u_0|^2) \leq \liminf_k \int_{\boldsymbol{R}^N} (|\nabla u_k|^2 + q|u_k|^2) = -1 .$$

Define now $\lambda = \left| \int_{\boldsymbol{R}^N} (|\nabla u_0|^2 + q|u_0|^2 \right| \geq 1$ and $v = u_0/(\lambda)^{1/2}$. It is $\int_{\boldsymbol{R}^N} |\nabla u_0|^2 + q|u_0|^2 = -1$ and $\int_{\boldsymbol{R}^N} |u_0|^{\alpha+1} = m\lambda^{-(\alpha+1)/2} \leq m$, so that $\lambda = 1$ and u_0 is a solution to our minimization problem. In this way Lemma 3.4 is proved, so Theorems 3.1 and 3.2 are proved. □

Remark 3.5. We comment here briefly hypotheses (q_2) and (q_3). As first thing it is easy to see that (q_2) is necessary for the existence of a nontrivial solution $u \in E$ to (P_-). Indeed if $u \in E$ is such a solution then from $-\Delta u + qu = -u^\alpha$ one easily deduces $\int_{\boldsymbol{R}^N} |\nabla u|^2 + \int_{\boldsymbol{R}^N} q|u|^2 = -\int_{\boldsymbol{R}^N} |u|^{\alpha+1} < 0$ and this implies $\lambda_1(q) < 0$. We note that (q_2) means, roughly speaking, that "q is negative enough", while (q_3) means that "q is not very negative at infinity". So what we are assuming in Section 3 is, roughly speaking, that "q is negative enough on a compact set".

4. Problem (P_+)

In this section we study problem (P_+) and we solve it by the method of sub- and super- solutions. So we shall construct, under suitable hypotheses on q, a sub- and a super- solution for (P_+) and they will be "well ordered"; then we shall show how to get a solution (this is an almost standard task).

We list now the hypotheses we shall use in this section:

(q_4) $q \in C^{0,\beta}_{loc}(\boldsymbol{R}^N)$ $\exists \beta > 0$,
(q_5) $\lambda_1(q) > 0$,
(q_6) $\exists R,\ \delta > 0$ such that $q(x) \geq \delta$ $\forall x \in {}^N \backslash B_R$.

The main result of this section is the following Theorem 4.1.

Theorem 4.1. *Assume* (q_4), (q_5), (q_6). *Then there is a strictly positive classical solution* u *to* (P_+).

Note that we don't have information on the behavior at ∞ of the solution we find in Theorem 4.1. This is in contrast with Section 3, where the solutions we find are known to be in $H^1(\boldsymbol{R}^N)$. In the following Section 5, under more stringent assumptions on q, we shall find solutions to (P_+) which vanish at ∞.

We shall prove Theorem 4.1 by constructing a sub- and a super- solution. As first thing we give the definition sub- and super- solution.

Definition 4.1. $u \in H^1_{loc}(\boldsymbol{R}^N)$ is a supersolution for (P_+) if for any $\phi \in C_0^\infty({}^N)$, $\phi \geq 0$, it is

$$\int_{\boldsymbol{R}^N} \nabla u \nabla \phi dx + \int_{\boldsymbol{R}^N} q u \phi dx \geq \int_{\boldsymbol{R}^N} |u|^{\alpha-1} u \phi dx . \tag{4.1}$$

Of course, the definition of a subsolution is obtained just reversing the inequalities in (4.1). In the following Proposition 4.2 we prove the existence of an ordered couple of sub- and super- solutions of (P_+).

Proposition 4.2. *Assume* (q_4), (q_5), (q_6). *Then there is a subsolution* $\underline{u}$ *and a supersolution* $\bar{u}$ *to* (P_+) *such that* $0 \leq \underline{u} \leq \bar{u}$ *and* $\underline{u} \neq 0$.

Proof of Proposition 4.2. As first thing we consider a solution $u_1 \in H_0^1(B_1) \cap L^\infty(B_1)$ of the problem

$$\begin{cases} -\Delta u + qu = u^\alpha & \text{in } B_1 , \\ u(x) > 0 , & u \in H_0^1(B_1) . \end{cases} \tag{1}$$

Following e.g. [BO] we know that such a solution exists. Also we know $u_1 \in C^2(\Omega) \cap C^{1,\beta}(\bar{\Omega})$ and $\partial u/\partial \nu < 0$, where ν is the outward normal at ∂B_1. Define $\underline{u_1} \in H^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$ by $\underline{u_1} = u_1$ in B_1, $\underline{u_1} = 0$ in $\boldsymbol{R}^N \backslash B_1$. It is then easy to prove that $\underline{u_1}$ is a subsolution for (P_+). Indeed if $\phi \in C_0^\infty(\boldsymbol{R}^N)$, $\phi \geq 0$, then

$$\int_{\boldsymbol{R}^N} \nabla \underline{u_1} \nabla \phi dx + \int_{\boldsymbol{R}^N} q \underline{u_1} \phi dx - \int_{\boldsymbol{R}^N} (\underline{u_1})^\alpha \phi dx = \int_{B_1} \nabla \underline{u_1} \nabla \phi dx + \int_{B_1} q \underline{u_1} \phi dx$$

$$- \int_{B_1} (\underline{u_1})^\alpha \phi dx = \int_{B_1} [-\Delta \underline{u_1} + q \underline{u_1} - (\underline{u_1})^\alpha] \phi dx + \int_{\partial B_1} \frac{\partial u}{\partial \nu} \phi \leq 0$$

because in the last line the first term is $= 0$ and the second is ≤ 0. In this way we have proved that $\underline{u_1}$ is a subsolution. We remark that $\mu\underline{u_1}$ is also a subsolution, for any $\mu \in [0, 1]$. We must now produce a supersolutuion. We begin by fixing $\lambda \in]0, \lambda_1(q)[$ and a $v \in H^1(\boldsymbol{R}^N) \cap C^\infty(\boldsymbol{R}^N)$ such that $v(x) > 0$ $\forall x$. It is then easy to get a $u \in H^1(\boldsymbol{R}^N) \cap C^{2,\beta}(\boldsymbol{R}^N)$ such that

$$-\Delta u + qu - \lambda u = v\,, \tag{4.3}$$

in a classical sense. For this one can, for example, minimize the functional $I(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\int qu^2 - \frac{1}{2}\lambda \int u^2 - \int vu$, $u \in H^1(\boldsymbol{R}^N)$ and apply usual regularity techniques. Also, as $v > 0$ and $\lambda < \lambda_1(q)$ one can easily prove $u \geq 0$ and then, by classical maximum principle, $u(x) > 0$ $\forall x$.

Consider now the function

$$w = w_{\mu,\nu} = \mu u + \nu\,,$$

where μ, $\nu > 0$. We claim that for μ, ν large enough w is a classical supersolution of (P_+), that is

$$-\Delta w(x) + q(x)w(x) \geq w^\alpha(x) \qquad \forall x \in \boldsymbol{R}^N\,. \tag{4.4}$$

Of course a classical supersolution is also a supersolution in the sense of Definition 4.1. If we substitute $w = \mu u + \nu$ in (4.4) and we recall (4.3) our claim reduces to

$$\mu v(x) + \lambda\mu u(x) + q(x)\nu \geq (\mu u(x) + \nu)^\alpha \qquad \forall x \in \boldsymbol{R}^N\,. \tag{4.5}$$

So we want to prove (4.5), for μ, ν large enough. We begin by choosing

$$\bar{\nu} = \max\left\{\left(\frac{\alpha}{\lambda}\right)^{1/(1-\alpha)}, \left(\frac{1}{\delta}\right)^{1/(1-\alpha)}\right\}.$$

It is easy to prove that $\forall \nu \geq \bar{\nu}$ and $\forall t \geq 0$ it is

$$\nu^\alpha \geq (\nu + t)^\alpha - \lambda t\,.$$

Recalling (q_6) it is also

$$q(x)\nu \geq \nu^\alpha \qquad \forall x \in \boldsymbol{R}^N \backslash B_R\,, \qquad \forall \nu \geq \bar{\nu}\,.$$

So we get

$$q(x)\nu \geq \nu^\alpha \geq (\nu + \mu u(x))^\alpha - \lambda\mu u(x) \quad \text{then} \quad \lambda\mu u(x) + q(x)\nu \geq (\nu + \mu u(x))^\alpha \qquad \forall \nu \geq \bar{\nu}\,, \tag{4.6}$$

$\forall \mu \geq 0$, $\forall x \in \boldsymbol{R}^N \backslash B_R$. Take now $\delta_1 > 0$ such that $u(x) \geq \delta_1$ $\forall x \in B_R$. It is then possible to choose $\bar{\mu}$ such that for all $\mu \geq \bar{\mu}$

$$(\mu|u|_{\infty,R} + \bar{v})^\alpha + \bar{v}|q^-|_{\infty,R} \leq \mu\delta_1\lambda .$$

Then for all $x \in B_R$ and $\mu \geq \bar{\mu}$ we get

$$(\mu u(x) + \bar{v})^\alpha - \bar{v}q(x) \leq (\mu|u|_{\infty,R} + \bar{v})^\alpha + \bar{v}|q^-|_{\infty,R} \leq \mu\delta_1\lambda \leq \lambda\mu u(x)$$

so that

$$\lambda\mu u(x) + \bar{v}q(x) \geq (\mu u(x) + \bar{v})^\alpha \qquad \forall\mu \geq \bar{\mu}\,, \quad \forall x \in B_R\,. \tag{4.7}$$

From (4.6) and (4.7) we get (4.5), if $\mu \geq \bar{\mu}$, so our claim is proved.

We have then found a supersolution $\bar{u} = \bar{\mu}u + \bar{v}$. It is $\bar{u}(x) > \bar{v}$ for all $x \in \boldsymbol{R}^N$. As $\mu\underline{u_1}$ is a subsolution $\forall\mu \in [0, 1]$, we can choose μ small enough so that $\mu\underline{u_1} \leq \bar{u}$. We then define $\underline{u} = \mu\underline{u_1}$ and we have a subsolution $\underline{u}$ and a supersolution $\bar{u}$ such that $0 \leq \underline{u} \leq \bar{u}$ and $\underline{u} \neq 0$. □

To complete the proof of Theorem 4.1 we have to build solution from the sub- and supersolution we have found above. We use here well known arguments, so we shall skip the details (for an idea on the sub- and supersolution arguments see [Am], [S], [He] and, for a variational approach, [St]).

For each n we consider the problem

$$(P_n) \qquad \begin{cases} -\Delta u + qu = u^\alpha & \text{in } B_n\,, \\ u = 0 & \text{in } \partial B_n\,. \end{cases}$$

If we define $\bar{u}_n = \bar{u}_{|B_n}$, we get that $\bar{u}_n$ is a supersolution to (P_n), for all n. We proceed now as follows. Define $\underline{u_2} \in H_0^1(B_2)$ in this way: $\underline{u_2} = \underline{u}$ in B_1, $\underline{u_2} = 0$ in $B_2\backslash B_1$. With the same argument used above we get that $\underline{u_2}$ is a subsolution of (P_2) and $0 \leq \underline{u_2} \leq \bar{u}_2$. By usual arguments we then get a solution u_2 of (P_2) such that $\underline{u_2} \leq u_2 \leq \bar{u}_2$. We have $u_2 \in H_0^1(B_2) \cap L^\infty(B_2)$ and by usual regularity arguments $u_2 \in C^{1,\alpha}(B_2)$; also $u_2 \in C^2(B_2^+)$ where $B_2^+ = \{x \in B_2 | u_2(x) > 0\}$. By strong maximum principle we get $u_2(x) > 0$ $\forall x \in B_2$ and then $u_2 \in C^2(B_2)$. Now define $\underline{u_3} \in H_0^1(B_3)$ as follows: $\underline{u_3} = u_2$ in B_2, $\underline{u_3} = 0$ in $B_3\backslash B_2$. Then $\underline{u_3}$ is a subsolution of (P_3), $0 \leq \underline{u_3} \leq \bar{u}_3$ and we get a solution u_3 of (P_3) such that $u_3 \in C^2(B_3)$, $\underline{u_3} \leq u_3 \leq \bar{u}_3$ and $u_3 > 0$ in B_3. In this way we get a sequence $\{u_n\}_n$ of functions such that $u_n \in H_0^1(B_n) \cap C^2(B_n)$, u_n is a solution of (P_n), $0 < u_n \leq u_{n+1}$ and $u_n \leq \bar{u}$ in B_n. By a diagonal argument (as $\{u_n\}_n$ is bounded in $H_0^1(B_R)$ $\forall R$) we get that there is a subsequence (which we label again $\{u_n\}_n$) and a $u_0 \in H_{loc}^1(\boldsymbol{R}^N)$ such that $u_n \rightharpoonup u_0$ weakly in $H^1(B_R)$, for each fixed $R > 0$. This implies that u_0 is a weak solution of (P_+). It is also $u_n(x) \to u_0(x)$ $\forall x$; as $0 < u_n(x) \leq u_{n+1}(x)$ $\forall x \in B_n$, this implies $u_0(x) > 0$ $\forall x \in \boldsymbol{R}^N$. Then, again with usual regularity arguments, u_0 is a classical solution of (P_+) and Theorem 4.1 is proved. □

5. Some results for unbounded potentials

In this section we prove some other existence results for problem $(P_\pm)$ assuming that $q^+(x)$ is unbounded as $|x| \to \infty$. To be precise, we shall use the following hypotheses:

(q_7) $q \in C(\boldsymbol{R}^N)$; $\exists R > 0$ such that $\{x \in \boldsymbol{R}^N | q(x) \le 0\} \subset B_R$,

(q_8) Let R as in (q_7). Define p in the following way: $p(x) = 0$ if $x \in B_{R+1}$, $p(x) = 1/q(x)$ if $x \in \boldsymbol{R}^N \backslash B_{R+1}$. Then $p \in L^{(1+\alpha)/(1-\alpha)}(\boldsymbol{R}^N)$.

Note that (q_7) implies $q \in L^\infty_{\mathrm{loc}}(\boldsymbol{R}^N)$ and $q^- \in L^\infty(\boldsymbol{R}^N)$.

We shall use variational methods, so we must specify the space in which to work. For $\phi \in C_0^\infty(\boldsymbol{R}^N)$ define $|\phi|_q = \left(\int_{\boldsymbol{R}^N} |q|\phi^2 dx\right)^{1/2}$ and $\|\phi\| = |\phi|_q + |\phi|_{1,2}$. We define E to be the closure of $\in C_0^\infty(\boldsymbol{R}^N)$ with respect to the norm $\|\cdot\|$. We define then

$$M = \left\{ u \in E \,\middle|\, \int_{\boldsymbol{R}^N} |u|^{\alpha+1} dx = 1 \right\}$$

and

$$m = \inf \left\{ \int_{\boldsymbol{R}^N} |\nabla u|^2 dx + \int_{\boldsymbol{R}^N} q|u|^2 dx \,\middle|\, u \in M \right\}.$$

We consider the following variational problem:

(vp_2) find $u \in M$, $u \ge 0$, such that $\int_{\boldsymbol{R}^N} |\nabla u|^2 dx + \int_{\boldsymbol{R}^N} q|u|^2 dx = m$.

It is easy to see that a solution u of (vp_2) satisfies, in the usual weak sense,

$$-\Delta u + qu = mu^\alpha .$$

We can then define $v = |m|^{-1/(1-\alpha)}u$, so that v satisfies $v \ge 0$ and

$$-\Delta v + qv = sv^\alpha ,$$

where $s =$ sign of m. On the other hand it is not difficult to prove the following proposition:

Proposition 5.1. *Assume* (q_7), (q_8). *Then* $m < 0$ *if and only if* $\lambda_1(q) < 0$ *and* $m > 0$ *if and only if* $\lambda_1(q) > 0$.

This section will be devoted to prove that there is a solution to (vp_2), that is to prove the following Proposition 5.2.

Proposition 5.2. *Assume* (q_7), (q_8). *Then* $\exists u \in M$, $u \ge 0$, *such that* $\int_{\boldsymbol{R}^N} |\nabla u|^2 dx + \int_{\boldsymbol{R}^N} q|u|^2 dx = m$.

Combining Propositions 5.1, 5.2 we get Theorem 5.1, which is the main result of this section.

Theorem 5.1. *Assume* (q_7), (q_8). *If* $\lambda_1(q) > 0$ *there is a nontrivial solution to* (P_+). *If* $\lambda_1(q) < 0$ *there is a nontrivial solution to* (P_-). *These solutions belong to* E.

We have now to prove Proposition 5.2.

Proof of Proposition 5.2. Let $\{u_k\}_k$ be a minimizing sequence for (vp_2). Obviously we can suppose $u_k \geq 0$ $\forall k$. We begin by proving the following claim:

$$\left\{\int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx\right\}_k \text{ is bounded}. \tag{5.1}$$

The proof of (5.1) is by contradiction and the argument is very similar to those we used in Section 3. Define $a_k = \left(\int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx\right)^{1/2}$ and suppose by contradiction that $a_k \to \infty$ (up to a subsequence). Define $v_k = u_k/a_k$ so that $|v_k|_{1,2} = 1$. As $\int_{\boldsymbol{R}^N} |v_k|^{\alpha+1} \to 0$ and $v^2 \leq |v|^{\alpha+1} + |v|^{2^*}$ $\forall v \in \boldsymbol{R}$ we have that $\{v_k\}_k$ is bounded in $H^1(\boldsymbol{R}^N)$, so $v_k \rightharpoonup v_0$ in $H^1(\boldsymbol{R}^N)$. As $|v_k|_{\alpha+1} \to 0$ we have $v_0 = 0$. Arguing exactly as in Section 3 one then finds $\liminf_k \int_{\boldsymbol{R}^N} qv_k^2 \geq 0$. On the other hand $\int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx + \int_{\boldsymbol{R}^N} q|u_k|^2 dx \to m$ implies $1 + \int_{\boldsymbol{R}^N} q|v_k|^2 dx \to 0$ so we get a contradiction. We have proved that $\left\{\int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx\right\}_k$ is bounded.

We now repeat for $\{u_k\}_k$ the argument we used above for $\{v_k\}_k$: as $\int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx$ and $\int_{\boldsymbol{R}^N} |u_k|^{\alpha+1} dx$ are bounded, we deduce that $\{u_k\}_k$ is bounded in $H^1(\boldsymbol{R}^N)$ so we can assume

$$u_k \rightharpoonup u_0 \quad \text{in } H^1(\boldsymbol{R}^N).$$

We remark that also the sequence $\left\{\int_{\boldsymbol{R}^N} q|u_k|^2\right\}_k$ is bounded, because $\left\{\int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx\right\}_k$ and $\left\{\int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx + \int_{\boldsymbol{R}^N} q|u_k|^2\right\}_k$ are both bounded. From the hypotheses on q^- and boundedness of $\int_{\boldsymbol{R}^N} |\nabla u_k|^2 dx$ it is easy to derive that $\left\{\int_{\boldsymbol{R}^N} q^-|v_k|^2\right\}_k$ is bounded, so we can deduce that also $\left\{\int_{\boldsymbol{R}^N} q^+|v_k|^2\right\}_k$ is bounded, a fact we shall use below.

Obviously we have, by weak lower semicontinuity,

$$\int_{\boldsymbol{R}^N} |\nabla u_0|^2 + q|u_0|^2 \le \liminf_k \int_{\boldsymbol{R}^N} |\nabla u_k|^2 + q|u_k|^2 = m\,; \qquad \int_{\boldsymbol{R}^N} |u_0|^{\alpha+1} \le 1\,.$$

Let us prove that $\int_{\boldsymbol{R}^N} |u_0|^{\alpha+1} \ge 1$. Obviously $\int_{B_R} |u_k|^{\alpha+1} \to \int_{B_R} |u_0|^{\alpha+1}$ $\forall R > 0$. Fix $\varepsilon > 0$ and take $R_\varepsilon > R$ (R given in (q_8)) such that

$$\left(\int_{\{|x|>R_\varepsilon\}} p^{(1+\alpha)/(1-\alpha)}dx\right)^{(1-\alpha)/2} < \varepsilon\,.$$

Also fix k_ε such that $\forall k \ge k_\varepsilon$ it is

$$\int_{B_{R_\varepsilon}} |u_k|^{\alpha+1} \le \int_{B_{R_\varepsilon}} |u_0|^{\alpha+1} + \varepsilon\,.$$

We then write

$$\begin{aligned}
(5.2)\quad 1 = \int_{\boldsymbol{R}^N} |u_k|^{\alpha+1} &= \int_{B_{R_\varepsilon}} |u_k|^{\alpha+1} + \int_{\{|x|>R_\varepsilon\}} |u_k|^{\alpha+1}\\
&\le \int_{B_{R_\varepsilon}} |u_0|^{\alpha+1} + \varepsilon + \int_{\{|x|>R_\varepsilon\}\cap\{q(x)>0\}} |u_k|^{\alpha+1}\\
&\le \int_{\boldsymbol{R}^N} |u_0|^{\alpha+1} + \varepsilon + \int_{\{|x|>R_\varepsilon\}} q^{-(\alpha+1)/2}|q^{1/2}u_k|^{\alpha+1}\\
&\le \int_{\boldsymbol{R}^N} |u_0|^{\alpha+1} + \varepsilon + \left(\int_{\{|x|>R_\varepsilon\}} (q)^{-(\alpha+1)/(1-\alpha)}\right)^{(1-\alpha)/2}\left(\int_{\boldsymbol{R}^N} q^+|u_k|^2\right)^{(1+\alpha)/2}\\
&\le \int_{\boldsymbol{R}^N} |u_0|^{\alpha+1} + \varepsilon + C\left(\int_{\{|x|>R_\varepsilon\}} (p)^{(\alpha+1)/(1-\alpha)}\right)^{(1-\alpha)/2} \le \int_{\boldsymbol{R}^N} |u_0|^{\alpha+1} + C\varepsilon
\end{aligned}$$

where C denote positive constants (possibly different from a line to another) which do not depend on ε, k. As (5.2) holds $\forall\varepsilon > 0$ we get obviously

$$\int_{\boldsymbol{R}^N} |u_0|^{\alpha+1} \ge 1$$

and so

$$\int_{\boldsymbol{R}^N} |u_0|^{\alpha+1} = 1\,.$$

Then u_0 satisfies

$$\int_{\mathbf{R}^N} |\nabla u_0|^2 + q|u_0|^2 \leq m\,; \qquad \int_{\mathbf{R}^N} |u_0|^{\alpha+1} = 1\,,$$

so it is a solution of (vp_2). Proposition 5.2 is proved, so also Theorem 5.1 is proved. □

The solutions that we have found in Theorem 5.1 belong to the space E; in this sense we say they vanish at infinty. So in the case $\lambda_1(q) > 0$ we get solutions to (P_+) vanishing at infinity. We did not get such an information for the solutions we have found in Section 4. The difference derives of course from hypothesis (q_7), which says (roughly) that $q(x) \to +\infty$ as $|x| \to +\infty$. We can get some other results of the same kind using sub- and super- solution method. In order to do this we consider for every $\beta > 0$ the functions

$$v_\beta(x) = \left(\frac{1}{1+|x|^2}\right)^\beta,$$

$$Q_\beta(x) = (1+|x|^2)^{\beta(1-\alpha)}\left\{1 + 4\beta\left(\frac{1}{1+|x|^2}\right)^{1+\beta(1-\alpha)} \times\left[(\beta+1)\left(1-\frac{1}{1+|x|^2}\right) - \frac{N}{2}\right]\right\}.$$

Note that $Q_\beta = O((1+|x|^2)^{\beta(1-\alpha)})$ as $|x| \to \infty$. Note also that Q_β assumes negative values: indeed $Q_\beta(0) = 1 - 2\beta N < 0$ if $\beta > 1/(2N)$.

We consider now the hypothesis:
(q_9) $q \in C^{0,\gamma}_{loc}(\mathbf{R}^N)$, $q(x) \geq Q_\beta(x)$ $\forall x \in \mathbf{R}^N$.

We have the following proposition:

Proposition 5.3. *Assume* (q_9). *Then* v_β, *defined above, is a (classical) supersolution to* (P_+).

We omit the proof of Proposition 5.3, which is just a calculation. As $v_\beta(x) > 0$ $\forall x$, we can choose a subsolution $\underline{u}$ as in Section 4 in such a way that $\underline{u} \leq v_\beta$. Repeating the arguments of Section 4 we get

Proposition 5.4. *Assume* (q_5), (q_9). *Then there is a classical solution* u *to* (P_+) *such that* $\underline{u} \leq u \leq v_\beta$, $u > 0$.

In particular Proposition 5.2 gives a positive solution u to (P_+) such that $u(x) \to 0$ as $|x| \to \infty$. We recall that for $\beta > 1/(2N)$ hypothesis (q_9) allows q to assume negative values. Also we note that when $\beta < N/(2+2\alpha)$, q can satisfy (q_9) without satisfying (q_8), so Proposition 5.4 is not contained in Theorem 5.2.

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nuna adreso:

M. Badiale
Dipartimento di Matematica
Pura e Applicata
Università degli Studi di Padova
via Belzoni 7
35131 Padova
Italia

F. Dobarro
Instituto Argentino de Matemáticas
Conicet Viamonte 1634
Capital Federal
Argentina

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