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Asymptotic Behaviour of Solutions to Phase Field Models with Constraints

By

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Abstract. A phase field model with constraints is studied. The model is described as a coupled system of nonlinear parabolic PDEs governed by the absolute temperature and the order parameter. One of the PDEs includes a maximal monotone graph as a mathematical expression of a certain kind of constraints for the order parameter. In this paper we restrict the space dimension to one, and investigate the structure of the ω-limit set of any solution of the model as time $t \to +\infty$.

1. Introduction

We consider a model for solid-liquid phase transitions of the following form:

(1.1) $$[\rho(u)+\lambda(w)]_t - u_{xx} = f(t, x) \qquad \text{in } Q := (0, +\infty) \times (-L, L),$$

(1.2) $$w_t - \kappa w_{xx} + \xi + g(w) = \lambda'(w) u \qquad \text{in } Q,$$

(1.3) $$\xi \in \partial I_{[\sigma_*, \sigma^*]}(w) \qquad \text{in } Q,$$

(1.4) $$-u_x(t, -L) + n_0 u(t, -L) = h_-(t), \quad u_x(t, L) + n_0 u(t, L) = h_+(t) \quad \text{for } t > 0,$$

(1.5) $$w_x(t, -L) = w_x(t, L) = 0 \qquad \text{for } t > 0,$$

(1.6) $$u(0, x) = u_0(x), \qquad w(0, x) = w_0(x) \qquad \text{for } x \in (-L, L).$$

Here $L > 0$ is a positive number; $\kappa > 0$ and $n_0 > 0$ are constants; $\rho(u)$ is an increasing function of u, and $\lambda(w)$, $\lambda'(w) = \frac{d}{dw}\lambda(w)$, $g(w)$ are C^2-functions of w; $\partial I_{[\sigma_*, \sigma^*]}$ is the subdifferential of the indicator function $I_{[\sigma_*, \sigma^*]}$ of the interval $[\sigma_*, \sigma^*] \subset \boldsymbol{R}$; $f(t, x)$, $h_{\pm}(t)$, u_0 and w_0 are given data. We call this system "Phase-Field Model with Constraints" and denote it by (PFC).

In the context of solid-liquid phase transitions, $\theta := \rho(u)$ represents the absolute temperature and w the order parameter which indicates the physical situation of the system; the range of the order parameter w is assumed to be a compact interval $[\sigma_*, \sigma^*]$, and $w(t, x) = \sigma_*$ and $w(t, x) = \sigma^*$ mean respectively that the physical situation at (t, x) is of pure solid and pure liquid, while $\sigma_* < w(t, x) < \sigma^*$ means that the physical situation at (t, x) is mushy.

The system without term ξ in (1.2) was earlier introduced as a thermodynamically consistent model for phase transitions by Penrose & Fife [13] and analytically studied by Zheng [15] and Sprekels & Zheng [14]. Recently, system (1.1)–(1.6), which is derived from a class of free energy functionals involving non-smooth terms, was discussed by Kenmochi & Niezgódka [9–11], Blowey & Elliott [1] and Laurençot [12]. Also, for some related works we refer to [2, 4–8].

The most important characteristic of our model is in the nonlinear term ξ governed by (1.3). This term is called "constraints" for the order parameter w. In fact, it allows the coexistence of the pure phase $\Omega_l(t) := \{x \in (-L, L); w(t, x) = \sigma^*\}$, $\Omega_s(t) := \{x \in (-L, L); w(t, x) = \sigma_*\}$ and the mushy region $\Omega_m(t) := \{x \in (-L, L); \sigma_* < w(t, x) < \sigma^*\}$ which have positive linear measures in the dynamical process.

In this paper we are interested in the large time behaviour of the solution $\{u, w\}$ of (PFC). Assuming that $f(t, \cdot) \to 0$ and $h_\pm(t) \to h^\infty$ in appropriate senses as $t \to +\infty$, we consider the stationary problem $\{(1.7), (1.8)\}$:

$$(1.7) \qquad \begin{cases} -u^\infty_{xx} = 0 \quad \text{in } (-L, L), \\ -u^\infty_x(-L) + n_0 u^\infty(-L) = h^\infty, \qquad u^\infty_x(L) + n_0 u^\infty(L) = h^\infty, \end{cases}$$

$$(1.8) \qquad \begin{cases} -\kappa w^\infty_{xx} + \xi^\infty + g(w^\infty) - \lambda'(w^\infty)u^\infty = 0 \quad \text{in } (-L, L), \\ \xi^\infty \in \partial I_{[\sigma_*, \sigma^*]}(w^\infty) \quad \text{in } (-L, L), \\ w^\infty_x(-L) = w^\infty_x(L) = 0. \end{cases}$$

It is easy to see that u^∞ is uniquely determined by (1.7) and $u^\infty := \dfrac{h^\infty}{n_0}$ on $[-L, L]$, while there are infinitely many solutions of (1.8) in general. By the results established in [10, 11] we know that

(i) $u(t) \to u^\infty$ weakly in $H^1(-L, L)$ as $t \to +\infty$,

(ii) the ω-limit set $\omega(u_0, w_0) := \{z \in H^1(-L, L); w(t_n) \to z \text{ in } H^1(-L, L) \text{ for some } t_n \text{ with } t_n \uparrow +\infty\}$ is non-empty, bounded and closed in $H^2(-L, L)$ as well as connected in $H^1(-L, L)$, and any ω-limit point w^∞ is a solution of (1.8).

As is easily seen from a simple example, in general, system (1.8) has a continuum of solutions. Therefore, the above (i) and (ii) seem not enough as information of large time behaviour of solutions.

In this paper, further assuming that the function $g(w) - \lambda'(w)u^\infty$ is of the N-shape (with respect to w) and has exactly three zeros, say ζ_{0-}, ζ_0 and ζ_{0+} with $\zeta_{0-} < \zeta_0 < \zeta_{0+}$, we show that

(iii) the boundary values $w^\infty(-L)$, $w^\infty(L)$ and the number of the ζ_0-points of w^∞ are constant on $\omega(u_0, w_0)$, $\lim_{t\to+\infty} |w_x(t)|_{L^2(-L,L)} = |w_x^\infty|_{L^2(-L,L)}$.

This observation seems new, and says in solid-liquid systems that the number of the connected components of the interface is invariant for time t large enough, and in this sense, a pattern of the pure phases $\Omega_s(t)$, $\Omega_l(t)$ and the mushy region $\Omega_m(t)$ is formed as time t is large enough.

Notations. We use the following notations:

$(\cdot, \cdot)$: the standard inner product in $L^2(-L, L)$;

$$a(v, z) := \int_{-L}^{L} v_x z_x dx \qquad \text{for } v,\ z \in H^1(-L, L);$$

$H^1(-L, L)^*$: the dual space of $H^1(-L, L)$;

$\langle \cdot, \cdot \rangle$: the duality pairing between $H^1(-L, L)^*$ and $H^1(-L, L)$;

$$k_1 \wedge k_2 = \min\{k_1, k_2\}, \qquad k_1 \vee k_2 = \max\{k_1, k_2\}.$$

2. Known results

Evolution problem (PFC) is discussed under the following assumptions (A1)–(A4):

(A1) ρ is a maximal monotone graph in $\boldsymbol{R} \times \boldsymbol{R}$ whose domain $D(\rho)$ and range $R(\rho)$ are open in R, ρ is locally bi-Lipschitz continuous as a function from $D(\rho)$ onto $R(\rho)$ and there are constants $A_0 > 0$ and α with $1 \le \alpha < 2$ such that

$$|\rho(r_1) - \rho(r_2)| \ge \frac{A_0 |r_1 - r_2|}{|r_1 r_2|^\alpha + 1} \qquad \text{for all } r,\ r_2 \in D(\rho).$$

(A2) σ_*, σ^* are constants with $-\infty < \sigma_* < \sigma^* < +\infty$, and κ, n_0 are positive constants.

(A3) λ is a C^3-function from $\boldsymbol{R}$ into itself.

(A4) g is a C^2-function from $\boldsymbol{R}$ into itself.

Next, for the data f, $h_\pm$, u_0 and w_0 we suppose that

(H1) $f \in W^{1,2}_{loc}(\boldsymbol{R}_+; L^2(-L, L)) \cap L^2(\boldsymbol{R}_+; L^2(-L, L))$ such that

$$\sup_{t\ge 0} |f|_{W^{1,2}(t,t+1;L^2(-L,L))} < +\infty;$$

(H2) $h_\pm \in W^{1,2}_{loc}(\boldsymbol{R}_+)$ such that

$$\sup_{t\ge 0} \{|h_+|_{W^{1,2}(t,t+1)} + |h_-|_{W^{1,2}(t,t+1)}\} < +\infty,$$

and for some constant h^∞

$$h_\pm - h^\infty \in L^2(\boldsymbol{R}_+);$$

(H3) $\dfrac{h_\pm(t)}{n_0} \in \overline{D(\rho)}$ for all $t \geq 0$ and there are positive constants A_1 and A'_1 such that

$$\rho(r)(n_0 r - h_\pm(t)) \geq -A_1|r| - A'_1 \qquad \text{for all } r \in D(\rho) \text{ and all } t \geq 0;$$

(H4) $u_0 \in H^1(-L, L)$ with $\rho(u_0) \in L^2(-L, L)$, and $w_0 \in H^2(-L, L)$ such that

$$w_{0x}(-L) = w_{0x}(L) = 0, \qquad \sigma_* \leq w_0 \leq \sigma^* \qquad \text{on } [-L, L].$$

Remark 2.1. (1) From (1.3) it follows that the order parameter w satisfies $\sigma_* \leq w(t, x) \leq \sigma^*$, so that we may assume without loss of generality that the functions λ and g have compact supports.

(2) The typical examples of ρ and λ are given by

$$\rho(u) = -\frac{1}{u} \qquad \text{for } -\infty < u < 0,$$

and

$$\lambda(w) = -\frac{a_0}{2}w^2 + a_1 w + a_2 \qquad \text{for constants } a_0 > 0, \ a_1 \text{ and } a_2.$$

In this case, condition (H3) is automatically satisfied, if $h_\pm < 0$ on $\boldsymbol{R}_+$.

Now we give the variational formulation for (PFC). We say that a couple of functions u and w is a solution of (PFC), if the following conditions (w1)–(w3) are fulfilled:

(w1) For every finite $T > 0$, $u \in L^\infty(0, T; H^1(-L, L))$, $\rho(u) \in L^\infty(0, T; L^2(-L, L)) \cap W^{1,2}(0, T; H^1(-L, L)^*)$ and $w \in L^\infty(0, T; H^1(-L, L)) \cap W^{1,2}(0, T; L^2(-L, L))$.

(w2) For all $z \in H^1(-L, L)$ and for a.e. $t \geq 0$,

$$\langle \rho(u)'(t), z \rangle + (\lambda(w)'(t), z) + a(u(t), z) + (n_0 u(t, -L) - h_-(t))z(-L)$$
$$+ (n_0 u(t, L) - h_+(t))z(L) = (f(t), z)$$

and $u(0) = u_0$, where the prime "$'$" denotes the derivative $\dfrac{d}{dt}$.

(w3) For all $z \in H^1(-L, L)$ and for a.e. $t \geq 0$,

$$(w'(t), z) + \kappa a(w(t), z) + (\xi(t) + g(w(t)), z) = (\lambda'(w(t))u(t), z)$$

and $w(0) = w_0$, where $\xi \in L^2_{\mathrm{loc}}(\boldsymbol{R}_+; L^2(-L, L))$ is such that $\xi \in \partial I_{[\sigma_*, \sigma^*]}(w)$ a.e. in Q.

Next we give the variational formulation for (1.8). We say that for a given constant u^∞, a function v is a solution of $P(\sigma_*, \sigma^*; u^\infty)$, if $v \in H^2(-L, L)$ and

$$-\kappa v_{xx} + \gamma + g(v) - \lambda'(v)u^\infty = 0 \quad \text{a.e. in } (-L, L)\,, \tag{2.1}$$

$$\gamma \in L^2(-L, L)\,, \qquad \gamma \in \partial I_{[\sigma_*, \sigma^*]}(v) \quad \text{a.e. in } (-L, L)\,, \tag{2.2}$$

$$v_x(-L) = v_x(L) = 0\,. \tag{2.3}$$

We recall some results on (PFC) established in [11].

Theorem 2.1 (cf. [11; Theorem 2.1, 2.2]). *Under conditions* (A1)–(A4), (H1)–(H4) *and* $\frac{h^\infty}{n_0} \in D(\rho)$, *problem* (PFC) *admits one and only one solution* $\{u, w\}$ *which satisfies the following* (a)–(d):

(a) $u \in L^\infty(\boldsymbol{R}_+; H^1(-L, L))$, $w \in L^\infty(\boldsymbol{R}_+; H^2(-L, L))$ *and* $w' \in L^\infty(\boldsymbol{R}_+; L^2(-L, L))$.

(b) $u - u^\infty \in L^2(\boldsymbol{R}_+; H^1(-L, L))$ *with* $u^\infty = \frac{h^\infty}{n_0}$ *and* $w' \in L^2(\boldsymbol{R}_+; L^2(-L, L))$.

(c) $u(t) \to u^\infty$ *weakly in* $H^1(-L, L)$ *as* $t \to +\infty$.

(d) *Let* $\omega(u_0, w_0) := \{v \in H^1(-L, L); w(t_n) \to v$ *in* $H^1(-L, L)$ *for some* t_n *with* $t_n \to +\infty\}$. *Then the set* $\omega(u_0, w_0)$ *is non-empty, bounded and closed in* $H^2(-L, L)$ *as well as connected in* $H^1(-L, L)$. *Moreover, any function* $w^\infty \in \omega(u_0, w_0)$ *is a solution* v *of* $P(\sigma_*, \sigma^*; u^\infty)$.

In [11; Theorems 2.1, 2.2] the statements (c) and (d) are not explicitly mentioned, but they were actually proved there.

3. Some lemmas

In this section we assume that (A3) and (A4) are fulfilled for the functions λ and g, and u^∞ is a given constant.

In the study of the stationary problem $P(\sigma_*, \sigma^*; u^\infty)$ we consider a triple of u^∞, g and λ such that the function

$$q(u^\infty; w) := g(w) - \lambda'(w)u^\infty\,, \qquad w \in \boldsymbol{R}\,,$$

satisfies the following properties (q1), (q2):

(q1) The (algebraic) equation $q(u^\infty; w) = 0$ has exactly three roots $\zeta_{0-}, \zeta_0, \zeta_{0+}$ with $\zeta_{0-} < \zeta_0 < \zeta_{0+}$; of course they depend upon u^∞.

(q2) There are numbers ζ_{1-} and ζ_{1+}, depending on u^∞, such that

$$\zeta_{0-} < \zeta_{1-} < \zeta_0 < \zeta_{1+} < \zeta_{0+}\,,$$

$$q'(u^\infty; \cdot)\left(= \frac{d}{dw}q(u^\infty; \cdot)\right) > 0 \qquad \text{on } (-\infty, \zeta_{1-}) \cup (\zeta_{1+}, +\infty)\,,$$

$$q'(u^\infty; \zeta_{1\pm}) = 0\,,$$

$$q'(u^\infty; \cdot) < 0 \qquad \text{on } (\zeta_{1-}, \zeta_{1+})\,,$$

$$q''(u^\infty; \cdot)\left(= \frac{d^2}{dw^2} q(u^\infty; \cdot)\right) \leq 0 \qquad \text{on } (-\infty, \zeta_0)\,,$$

$$q''(u^\infty; \zeta_0) = 0\,,$$

$$q''(u^\infty; \cdot) \geq 0 \qquad \text{on } (\zeta_0, +\infty)\,.$$

As to the relationships between ζ_0, σ_* and σ^* various cases are possible, but we here suppose that

(3.1) $$\sigma_* < \zeta_0 < \sigma^*\,.$$

This restriction does not exclude physically relevant cases in phase transitions.

We denote by $G(u^\infty; w)$ the primitive of $q(u^\infty; w)$ satisfying $G(u^\infty; \zeta_0) = 0$, i.e.

$$G(u^\infty; w) := \int_{\zeta_0}^{u} q(u^\infty; s)ds\,, \qquad w \in \boldsymbol{R}\,,$$

and define $b_0(u^\infty)$ and $b_*(u^\infty)$ by

(3.2) $$b_0(u^\infty) := \max\{G(u^\infty; \zeta_{0-}), G(u^\infty; \zeta_{0+})\}\,,$$

(3.3) $$b_*(u^\infty) := \max\{G(u^\infty; \zeta_{0-} \vee \sigma_*), G(u^\infty; \zeta_{0+} \wedge \sigma^*)\}\,.$$

It is clear by (3.1), (q1) and (q2) that

$$b_0(u^\infty) \leq b_*(u^\infty) < 0\,.$$

In order to investigate the structure of the set of all solutions to $P(\sigma_*, \sigma^*; u^\infty)$, denoted by S^*, we decompose S^* into three disjoint classes S_c, S_0 and S_1, i.e.

(3.4) $$S^* = S_c + S_0 + S_1\,,$$

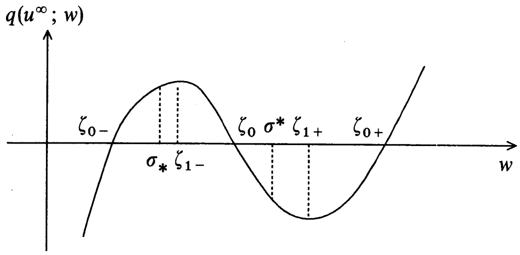

Fig. 1

where

(3.5) $S_c := \{v;\ v \text{ is a constant solution of } P(\sigma_*, \sigma^*; u^\infty)\}$,

(3.6) $S_0 := \{v \in H^2(-L, L);\ v \text{ is a non-constant solution of } P(\sigma_*, \sigma^*; u^\infty)$ such that $G(u^\infty; v(-L)) > b_*(u^\infty)\}$,

(3.7) $S_1 := \{v \in H^2(-L, L);\ v \text{ is a non-constant solution of } P(\sigma_*, \sigma^*; u^\infty)$ such that $G(u^\infty; v(-L)) = b_*(u^\infty)\}$;

as will be shown below, $G(u^\infty; v(-L)) \geq b_*(u^\infty)$ for any non-constant solution v of $P(\sigma_*, \sigma^*; u^\infty)$, so that (3.4) holds.

In the rest of this section we assume that conditions (q1), (q2) and (3.1) hold.

For the sake of simplicity, given a number b with

$$\min G(u^\infty; \cdot) \leq b < 0 ,$$

we denote by $\eta_+(b)$ and $\eta_-(b)$ the roots of $G(u^\infty; w) = b$ in the interval $(\zeta_0, \zeta_{0+}]$ and $[\zeta_{0-}, \zeta_0)$, respectively, if they exist. Note here that $G(u^\infty; w) = b$ has at most two roots within $[\zeta_{0-}, \zeta_{0+}]$. In fact, if $b_0(u^\infty) \leq b < 0$, then $G(u^\infty; w) = b$ has exactly two roots $\eta_\pm(b)$ with $\zeta_{0-} \leq \eta_-(b) < \zeta_0 < \eta_+(b) \leq \zeta_{0+}$. Also, if $G(u^\infty; \zeta_{0-}) \neq G(u^\infty; \zeta_{0+})$ and $\min G(u^\infty; \cdot) \leq b < b_0(u^\infty)$, then $G(u^\infty; w) = b$ has only one root in $[\zeta_{0-}, \zeta_{0+}]$; when $G(u^\infty; \zeta_{0+}) >$ (resp. $<$) $G(u^\infty; \zeta_{0-})$, the equation has a root $\eta_-(b) \in [\zeta_{0-}, \zeta_0)$ (resp. $\eta_+(b) \in (\zeta_0, \zeta_{0+}]$).

Lemma 3.1. *Let v be any solution of $P(\sigma_*, \sigma^*; u^\infty)$ and $b := G(u^\infty; v(-L))$. Then*:

(i) *$G(u^\infty; v(x)) \geq b$ for any $x \in [-L, L]$. $v_x(x) = 0$ if and only if $G(u^\infty; v(x)) = b$; hence $G(u^\infty; v(L)) = b$.*

(ii) *If $b = 0$, then $v \equiv \zeta_0$ on $[-L, L]$.*

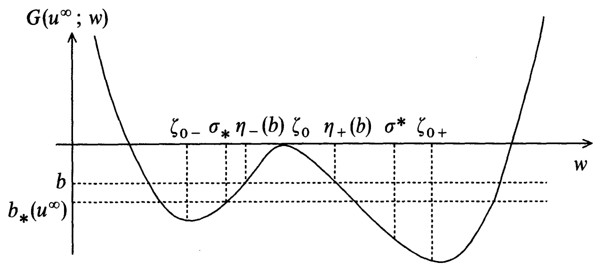

Fig. 2

(iii) *If v is non-constant on $[-L, L]$, then*

$$b_*(u^\infty) \le b < 0 .$$

(iv) *If v is non-constant on $[-L, L]$, then*

$$\eta_-(b) \le v \le \eta_+(b) \qquad on\ [-L, L] .$$

Proof. Multiplying (2.1) by v_x and integrating it over $[-L, x]$, we have

$$-\frac{\kappa}{2}|v_x(x)|^2 + G(u^\infty; v(x)) = b \qquad \text{for all } x \in [-L, L] . \tag{3.8}$$

Hence (i) holds.

Assume $b = 0$. Note that the equation $G(u^\infty; w) = 0$ has exactly three roots, say ζ_{2-}, ζ_0, ζ_{2+} with $\zeta_{2-} < \zeta_{0-} < \zeta_0 < \zeta_{0+} < \zeta_{2+}$. Taking account of (i), we see that one of the following three cases (α), (β) and (γ) holds:

$$(\alpha)\ v \le \zeta_{2-} \text{ on } [-L, L] , \quad (\beta)\ v \equiv \zeta_0 \text{ on } [-L, L] , \quad (\gamma)\ v \ge \zeta_{2+} \text{ on } [-L, L] .$$

In case (α) holds, $q(u^\infty; v) < 0$ and $v_x \le 0$ on $[-L, L]$. Therefore $v_x(L) \ne 0$ unless $v \equiv \zeta_{2-}$. Also $v \equiv \zeta_{2-}$ is not a solution, since $q(u^\infty; \zeta_{2-}) < 0$ and $\partial I_{[\sigma_*, \sigma^*]}(\zeta_{2-}) \subset \boldsymbol{R}_-$. Hence ($\alpha$) is excluded. The case (γ) is similarly excluded, too. Thus (β) must hold and (ii) is obtained.

Next, we show (iii) by contradiction. Clearly $\min G(u^\infty; \cdot) \le b \le 0$. Also, $b \ne 0$ by (ii), assuming v is non-constant. Since $\zeta_{0-} \vee \sigma_* = \zeta_{0-}$ or σ_*, $\zeta_{0+} \wedge \sigma^* = \zeta_{0+}$ or σ^* and $b_*(u^\infty) = G(u^\infty; \zeta_{0-} \vee \sigma_*)$ or $G(u^\infty; \zeta_{0+} \wedge \sigma^*)$, we have eight possible cases regarding $\zeta_{0-} \vee \sigma_*$, $\zeta_{0+} \wedge \sigma^*$ and $b_*(u^\infty)$. Assuming that

$$b_*(u^\infty) > b \ge \min G(u^\infty; \cdot) ,$$

we consider for instance

$$\zeta_{0-} \vee \sigma_* = \sigma_* , \qquad \zeta_{0+} \wedge \sigma^* = \zeta_{0+} , \qquad b_*(u^\infty) = G(u^\infty; \sigma_*) .$$

In this case, the equation $G(u^\infty; w) = b$ has at most two roots η_1, η_2 satisfying $\zeta_0 < \eta_1 \le \zeta_{0+} \le \eta_2$; note that $\eta_1 = \zeta_{0+} = \eta_2$ if $b = \min G(u^\infty; \cdot)(= G(u^\infty; \zeta_{0+}))$. This implies $v(-L) = \eta_1$ or η_2, and on account of (i) there are $x_1 \in [-L, L)$ and constant $\delta > 0$ such that

$$\sigma_* \le v \le \eta_1 - \delta \qquad \text{on } [x_1, L)$$

or

$$\sigma^* \ge v \ge \eta_2 + \delta \qquad \text{on } [x_1, L) .$$

Hence, for some constant $\delta_1 > 0$,

$$v_x \le -\delta_1 \quad \text{on } [x_1, L) \qquad \text{or} \qquad v_x \ge \delta_1 \quad \text{on } [x_1, L) ,$$

This implies that $v_x(L) \neq 0$, which contradicts the boundary condition $v_x(L) = 0$. In any other cases of $\zeta_{0-} \vee \sigma_*$, $\zeta_{0+} \wedge \sigma^*$ and $b_*(u^\infty)$, we have a similar contradiction. Thus (iii) is obtained.

Finally we show (iv). Note from (iii) that $b_*(u^\infty) \leq b < 0$, and the equation $G(u^\infty; w) = b$ has at most four roots, say $\tilde{\eta}_\pm(b)$ and $\eta_\pm(b)$, in $[\sigma_*, \sigma^*]$ such that

$$\tilde{\eta}_-(b) \leq \eta_-(b), \qquad \eta_+(b) \leq \tilde{\eta}_+(b)$$

$$\sigma_* \vee \zeta_{0-} \leq \eta_-(b) < \zeta_0 < \eta_+(b) \leq \zeta_{0+} \wedge \sigma^* .$$

We have by (i) the following three possibilities:

$$v \leq \tilde{\eta}_-(b) \qquad \text{on } [-L, L], \tag{3.9}$$

$$v \geq \tilde{\eta}_+(b) \qquad \text{on } [-L, L] \tag{3.10}$$

and

$$\eta_-(b) \leq v \leq \eta_+(b) \qquad \text{on } [-L, L]. \tag{3.11}$$

In a way similar to that of the proof of (iii) we can show that (3.9) and (3.10) are impossible. Accordingly (3.11) must be true. ◇

It is well-known that (cf. Chafee & Infante [3]) the problem $P(u^\infty)$ without constraints,

$$-\kappa v_{xx} + q(u^\infty; v) = 0 \qquad \text{in } (-L, L).$$

$$v_x(-L) = v_x(L) = 0 ,$$

has exactly three constant solutions ζ_{0-}, ζ_{0+}, ζ_0, and at most a finite number of non-constant solutions; of course the number of non-constant solutions depends upon u^∞. We define

$$\tilde{S}_0 := \{V_1, V_2, \ldots, V_p\} ,$$

where $V_1, V_2, \ldots, V_p$ are all non-constant solutions of $P(u^\infty)$ with

$$V_1(-L) < V_2(-L) < \cdots < V_p(-L) .$$

Lemma 3.2. (i) *$P(\sigma_*, \sigma^*; u^\infty)$ has exactly three constants solutions $\zeta_{0-} \vee \sigma_*$, $\zeta_{0+} \wedge \sigma^*$, ζ_0, that is, $S_c = \{\zeta_{0-} \vee \sigma_*, \zeta_{0+} \wedge \sigma^*, \zeta_0\}$.*

(ii) *Let S_0 be the set defined by* (3.6). *Then S_0 is a finite set and $S_0 \subset \tilde{S}_0$.*

Proof. Let c be any constant with $\sigma_* \leq c \leq \sigma^*$ and assume that $v \equiv c$ is a solution of $P(\sigma_*, \sigma^*; u^\infty)$. Then $\partial I_{[\sigma_*, \sigma^*]}(c) + q(u^\infty; c) \ni 0$. First, assume that $\sigma_* < c < \sigma^*$. Then it follows that $q(u^\infty; c) = 0$, i.e. $c \in \{\zeta_{0-}, \zeta_{0+}, \zeta_0\}$, since $\partial I_{[\sigma_*, \sigma^*]}(c) = \{0\}$. Hence c is equal to one of $\zeta_{0-} \vee \sigma_*$, ζ_0 and $\zeta_{0+} \wedge \sigma^*$. Next, assume $c = \sigma_*$ (resp. σ^*). Then $q(u^\infty; c) \geq 0$ (resp. ≤ 0), so that $\zeta_{0-} \leq c =$

$\sigma_* < \zeta_0$ (resp. $\zeta_0 < c = \sigma^* \leq \zeta_{0+}$). Hence $c = \zeta_{0-} \vee \sigma_*$ (resp. $\zeta_{0+} \wedge \sigma^*$). Thus we have seen that if v is a constant solution of $P(\sigma_*, \sigma^*; u^\infty)$, then v is equal to one of $\zeta_{0-} \vee \sigma_*$, ζ_0, $\zeta_{0+} \wedge \sigma^*$. Conversely, $\zeta_{0-} \vee \sigma_*$, ζ_0 and $\zeta_{0+} \wedge \sigma^*$ are constant solutions. Hence (i) is obtained.

We show (ii). Let v be any non-constant solution of $P(\sigma_*, \sigma^*; u^\infty)$ such that $b := G(u^\infty; v(-L)) > b_*(u^\infty)$. Then, by Lemma 3.1 (iv) we see that

$$\eta_-(b) \leq v \leq \eta_+(b) \qquad \text{on } [-L, L].$$

By assumption, we have $\sigma_* < \eta_-(b) < \eta_+(b) < \sigma^*$, so that $\gamma \equiv 0$. This implies that v is a non-constant solution of $P(u^\infty)$, hence $v \in \tilde{S}_0$. ◇

Lemma 3.3 (cf. Chafee & Infante [3]). *For each b with $b_0(u^\infty) < b < 0$, define*

$$I(b) := \int_{\eta_-(b)}^{\eta_+(b)} \frac{dv}{\{G(u^\infty; v) - b\}^{1/2}}.$$

Then $I(b)$ is continuous and strictly decreasing with respect to $b \in (b_0(u^\infty), 0)$, and

$$\lim_{b \uparrow 0} I(b) = \frac{\sqrt{2}\pi}{|q'(u^\infty; \zeta_0)|^{1/2}}, \qquad \lim_{b \downarrow b_0(u^\infty)} I(b) = +\infty.$$

Lemma 3.4. *Let v be a non-constant solution of $P(\sigma_*, \sigma^*; u^\infty)$ and $b := G(u^\infty; v(-L))$. Also, let (x_1, x_2) be any connected component of the set $\{x \in [-L, L];\ v_x(x) \neq 0\}$. Then $v(x_1) = \eta_-(b)$ or $\eta_+(b)$. Furthermore, if $v(x_1) = \eta_-(b)$ (resp. $\eta_+(b)$), then*

$$v(x_2) = \eta_+(b) \ \ (\textit{resp. } \eta_-(b)), \qquad v_x > 0 \ \ (\textit{resp. } v_x < 0) \ \ \textit{on } (x_1, x_2), \tag{3.12}$$

$$v_x(x_1) = v_x(x_2) = 0, \tag{3.13}$$

$$x_2 - x_1 = \left(\frac{\kappa}{2}\right)^{1/2} I(b), \tag{3.14}$$

and there is a unique point x_0 with $x_1 < x_0 < x_2$ such that

$$v(x_0) = \zeta_0, \qquad v_x(x_0) = \left(\frac{-2b}{\kappa}\right)^{1/2} \left(\textit{resp. } -\left(\frac{-2b}{\kappa}\right)^{1/2}\right). \tag{3.15}$$

Proof. We see (3.13) and $v(x_1) = \eta_-(b)$ or $\eta_+(b)$ by Lemma 3.1 (i). Assume $v(x_1) = \eta_-(b)$. Then it follows that v is increasing on (x_1, x_2), so that (3.12) holds. Besides from (3.8) we see that

$$\left(\frac{\kappa}{2}\right)^{1/2} \frac{v_x(x)}{\{G(u^\infty; v(x)) - b\}^{1/2}} = 1,$$

whence

$$x_1 - x_2 = \left(\frac{\kappa}{2}\right)^{1/2} \int_{x_1}^{x_2} \frac{v_x}{\{G(u^\infty; v) - b\}^{1/2}} dx$$
$$= \left(\frac{\kappa}{2}\right)^{1/2} \int_{\eta_-(b)}^{\eta_+(b)} \frac{dv}{\{G(u^\infty; v) - b\}^{1/2}} = \left(\frac{\kappa}{2}\right)^{1/2} I(b).$$

Thus (3.14) holds. Finally, by (3.12) and (3.13) there is a unique point x_0 in (x_1, x_2) such that $v(x_0) = \zeta_0$. Since $G(u^\infty; v(x_0)) = 0$, we have $-\frac{\kappa}{2}|v_x(x_0)|^2 = b$. Hence (3.15) holds. ◇

4. Structure of the solution set S^*

In this section we study the structure of $S^* = S_c + S_0 + S_1$. The sets S_c and S_0 have been investigated in Lemma 3.2, so we study below the structure of S_1. In order to do so we introduce the notion of the principal parts of the soluton to $P(u^\infty)$.

Throughout this section, assume that (q1), (q2) and (3.1) are fulfilled, and let $b_0(u^\infty)$ and $b_*(u^\infty)$ be as the numbers defined by (3.2) and (3.3), respectively, and use the same notations for $\eta_\pm(b)$, etc., as in the previous section.

Lemma 4.1. *Suppose that $b_0(u^\infty) < b < 0$. Then there exists a unique pair of a bounded interval (x_-^b, x_+^b) with $x_-^b < 0 < x_+^b$ and a function $v^b \in C^4([x_-^b, x_+^b])$ such that*

$$-\kappa v_{xx}^b + q(u^\infty; v^b) = 0 \quad \textit{in } (x_-^b, x_+^b), \tag{4.1}$$

$$v^b(0) = \zeta_0, \quad v_x^b(0) = \left(\frac{-2b}{\kappa}\right)^{1/2} \tag{4.2}$$

with

$$v_x^b > 0 \quad \textit{on } (x_-^b, x_+^b), \quad v_x^b(x_-^b) = v_x(x_+^b) = 0. \tag{4.3}$$

In this case, we have

$$v^b(x_-^b) = \eta_-(b), \quad v^b(x_+^b) = \eta_+(b) \tag{4.4}$$

and

$$x_+^b - x_-^b = \left(\frac{\kappa}{2}\right)^{1/2} I(b). \tag{4.5}$$

Proof. Consider the Cauchy problem to find a function $p = p(x) \in C^4(\boldsymbol{R})$ such that

$$(4.6) \qquad -\kappa p_{xx} + q(u^\infty; p) = 0 \qquad \text{in } \boldsymbol{R},$$

$$(4.7) \qquad p(0) = \zeta_0, \qquad p_x(0) = \left(\frac{-2b}{\kappa}\right)^{1/2}.$$

From the general theory of ODEs we see that problem (4.6)–(4.7) has one and only one global solution p. Now, let (x_-, x_+) be the maximal interval in $\boldsymbol{R}$ such that $x_- < 0 < x_+$ and $p_x > 0$ on (x_-, x_+). Then, multiplying (4.6) by p_x and integrating it over $[0, x)$ or $(x, 0]$, we have by (4.7)

$$(4.8) \qquad -\frac{\kappa}{2} p_x(x)^2 + G(u^\infty; p(x)) = b \qquad \text{for all } x \in (x_-, x_+).$$

From (4.8) and assumption $b_0(u^\infty) < b < 0$ it follows immediately that $-\infty < x_- < 0 < x_+ < +\infty$ and

$$p(x_-) = \eta_-(b), \qquad p(x_+) = \eta_+(b), \qquad p_x(x_-) = p_x(x_+) = 0.$$

Moreover, just as in the proof of Lemma 3.4,

$$x_+ - x_- = \left(\frac{\kappa}{2}\right)^{1/2} I(b) < +\infty.$$

Thus the interval $(x_-^b, x_+^b) := (x_-, x_+)$ and the function $v^b := p$ satisfy (4.1)–(4.5). It is easy to see that such a pair is uniquely determined. ◇

For each b with $b_0(u^\infty) < b < 0$, let $v^b \in C^4([x_-^b, x_+^b])$ be the function constructed in Lemma 4.1, and define $\bar{v}^b \in C^4([\bar{x}_-^b, \bar{x}_+^b])$ by

$$\bar{x}_-^b = -x_+^b, \qquad \bar{x}_+^b = -x_-^b$$

and

$$\bar{v}^b(x) = v^b(-x) \qquad \text{for } x \in [\bar{x}_-^b, \bar{x}_+^b].$$

We then say that v^b and $\bar{v}^b$ are the principal parts of solutions to $P(u^\infty)$. With this terminology Lemma 3.4 is mentioned as follows. Let v be any solution of $P(\sigma_*, \sigma^*; u^\infty)$ and put $b := G(u^\infty; v(-L))$. Then any non-constant part of v coincides with a translation of the principal part v^b or $\bar{v}^b$.

Regarding the structure of S_1 we have the following theorem.

Theorem 4.1. *Suppose that* (A3), (A4), (q1), (q2) *and* (3.1) *hold, and further that*

$$(4.9) \qquad b_0(u^\infty) < b_*(u^\infty) =: b$$

and

$$(4.10) \qquad \left(\frac{\kappa}{2}\right)^{1/2} I(b) \le 2L.$$

Then:

(1) *$P(\sigma_*, \sigma^*; u^\infty)$ has at least one non-constant solution v with $G(u^\infty; v(-L)) = b$, i.e. $S_1 \neq \emptyset$.*

(2) *Let v be any non-constant solution with $b = G(u^\infty; v(-L))$, namely, $v \in S_1$. Then there is a partition*

$$-L \le x_L^1 < x_R^1 \le x_L^2 < x_R^2 \le \cdots \le x_L^l < x_R^l \le L \tag{4.11}$$

of the interval $[-L, L]$ such that

$$x_R^k = x_L^k + \left(\frac{\kappa}{2}\right)^{1/2} I(b), \qquad k = 1, 2, \ldots, l, \tag{4.12}$$

and one of the following (4.13) *and* (4.14) *holds*:

$$\begin{cases} v = \eta_-(b) \quad \text{on } [-L, x_L^1], \\ v = v^b(x_-^b - x_L^1 + \cdot) \quad \text{on } (x_L^1, x_R^1), \\ v = \eta_+(b) \quad \text{on } [x_R^1, x_L^2], \\ v = \bar{v}^b(\bar{x}_-^b - x_L^2 + \cdot) \quad \text{on } (x_L^2, x_R^2), \\ v = \eta_-(b) \quad \text{on } [x_R^2, x_L^3], \\ \quad \cdots\cdots \\ v = \begin{cases} \bar{v}^b(\bar{x}_-^b - x_L^l + \cdot) \quad \text{on } (x_L^l, x_R^l) & \text{if } l \text{ is even}, \\ v^b(x_-^b - x_L^l + \cdot) \quad \text{on } (x_L^l, x_R^l) & \text{if } l \text{ is odd}, \end{cases} \\ v = \begin{cases} \eta_-(b) \quad \text{on } [x_R^l, L] & \text{if } l \text{ is even}, \\ \eta_+(b) \quad \text{on } [x_R^l, L] & \text{if } l \text{ is odd}, \end{cases} \end{cases} \tag{4.13}$$

$$\begin{cases} v = \eta_+(b) \quad \text{on } [-L, x_L^1], \\ v = \bar{v}^b(\bar{x}_-^b - x_L^1 + \cdot) \quad \text{on } (x_L^1, x_R^1), \\ v = \eta_-(b) \quad \text{on } [x_R^1, x_L^2], \\ v = v^b(x_-^b - x_L^2 + \cdot) \quad \text{on } (x_L^2, x_R^2), \\ v = \eta_+(b) \quad \text{on } [x_R^2, x_L^3], \\ \quad \cdots\cdots \\ v = \begin{cases} v^b(x_-^b - x_L^l + \cdot) \quad \text{on } (x_L^l, x_R^l) & \text{if } l \text{ is even}, \\ \bar{v}^b(\bar{x}_-^b - x_L^l + \cdot) \quad \text{on } (x_L^l, x_R^l) & \text{if } l \text{ is odd}, \end{cases} \\ v = \begin{cases} \eta_+(b) \quad \text{on } [x_R^l, L] & \text{if } l \text{ is even}, \\ \eta_-(b) \quad \text{on } [x_R^l, L] & \text{if } l \text{ is odd}. \end{cases} \end{cases} \tag{4.14}$$

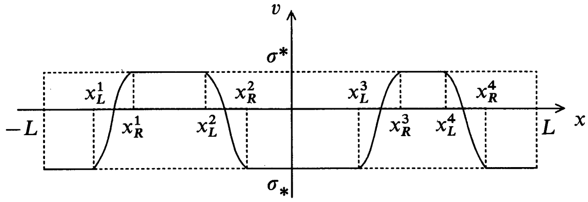

Fig. 3

(3) *Let* v *be a function defined by* (4.13) *or* (4.14) *with* (4.11) *and* (4.12). *Then* v *is a solution of* $P(\sigma_*, \sigma^*; u^\infty)$, *that is,* $v \in S_1$.

Proof. By (4.9), we have $\eta_-(b) = \sigma_*$ or $\eta_+(b) = \sigma^*$. Without loss of generality we may assume that $\eta_+(b) = \sigma^*$. Consider now the function

$$v(x) = \begin{cases} v^b(x_-^b + L + x) & \text{for } x \in \left[-L, -L + \left(\frac{\kappa}{2}\right)^{1/2} I(b)\right), \\ \sigma^* & \text{for } x \in \left[-L + \left(\frac{\kappa}{2}\right)^{1/2} I(b), L\right], \end{cases}$$

which is well-defined by (4.10) and (4.12). Then, since $\sigma_* \le \eta_-(b) < \zeta_0 < \eta_+(b) = \sigma^*$, $q(u^\infty; \sigma^*) \le 0$ and $\boldsymbol{R}_+ = \partial I_{[\sigma_*, \sigma^*]}(\sigma^*)$, it follows that v is a solution of $P(\sigma_*, \sigma^*; u^\infty)$. Thus (1) is obtained.

Let v be any non-constant solution with $b = G(u^\infty; v(-L))$ of $P(\sigma_*, \sigma^*; u^\infty)$, and let (x_L^k, x_R^k), $k = 1, 2, \dots, l$, be all of the connected components of the set $\{x \in [-L, L];\ v_x(x) \neq 0\}$ in $[-L, L]$. Then it follows from Lemma 3.4 that $x_R^k - x_L^k = \left(\frac{\kappa}{2}\right)^{1/2} I(b)$ for all k, l is a positive integer with $l\left(\frac{\kappa}{2}\right)^{1/2} I(b) \le 2L$, and moreover (4.13) or (4.14) with (4.11) and (4.12) holds. ◇

Corollary to Theorem 4.1. *Under the same assumptions as in Theorem* 4.1, *let* v *be any non-constant solution of* $P(\sigma_*, \sigma^*; u^\infty)$ *with* $b = G(u^\infty; v(-L))$. *Then there exists an positive integer* l *such that*

$$|v_x|^2_{L^2(-L,L)} = l|v_x|^2_{L^2(x_-^b, x_+^b)} .$$

In this case, the number of all points $x \in [-L, L]$ *with* $v(x) = \zeta_0$ *is* l.

Remark 4.1. (i) In Theorem 4.1, if $b_*(u^\infty) = G(u^\infty; \sigma^*) > G(u^\infty; \zeta_{0-} \vee \sigma_*)$, then

$$(4.15) \qquad -L = x_L^1, \qquad x_R^2 = x_L^3, \qquad x_R^4 = x_L^5, \dots .$$

in (4.13) and

(4.16) $$x_R^1 = x_L^2, \qquad x_R^2 = x_L^4, \ \ldots .$$

in (4.14). If $b_*(u^\infty) = G(u^\infty; \sigma_*) > G(u^\infty; \zeta_{0+} \wedge \sigma^*)$, then (4.16) and (4.15) hold in (4.13) and (4.14), respectively.

(ii) From the construction of the principal parts v^b and $\bar{v}^b$ it is easily seen that any non-constant solution v of $P(u^\infty)$ has expression (4.13) or (4.14) with a partition (4.11) satisfying (4.12) and

$$-L = x_L^1, \qquad x_R^1 = x_L^2, \qquad x_R^2 = x_L^3, \qquad x_R^3 = x_L^4, \ \ldots, \qquad x_R^l = L.$$

In this case, $l\left(\frac{\kappa}{2}\right)^{1/2} I(b) = 2L$ for $b = G(u^\infty; v(-L))$.

(iii) If $\left(\frac{\kappa}{2}\right)^{1/2} I(b) > 2L$, then there is no non-constant solution v with $b = G(u^\infty; v(-L))$ of $P(\sigma_*, \sigma^*; u^\infty)$.

According to Theorem 4.1 and Remark 4.1 the solution set S^* of $P(\sigma_*, \sigma^*; u^\infty)$ has a finite number of connected components and is able to be written as their direct sum:

(4.17) $$S^* = \sum_{k=1}^{3} S_c(k) + \sum_{i=i_0}^{p} S_0(i) + \sum_{l=1}^{l_0} \{S_1(l) + \bar{S}_1(l)\},$$

where

$$S_c(1) = \{\zeta_{0-} \vee \sigma_*\}, \qquad S_c(2) = \{\zeta_{0+} \wedge \sigma^*\}, \qquad S_c(3) = \{\zeta_0\};$$

i_0 is the smallest integer of all i with $1 \le i \le p$ such that $G(u^\infty; V_i(-L)) > b_*(u^\infty)$ and $S_0(i) = \{V_i\}$ for $i_0 \le i \le p$; l_0 is the largest integer of all l such that $l\left(\frac{\kappa}{2}\right)^{1/2} I(b_*(u^\infty)) \le 2L$, and for each $l = 1, 2, \ldots, l_0$, $S_1(l)$ (resp. $\bar{S}_1(l)$) is the set of all non-constant solutions v with $b := G(u^\infty; v(-L)) = b_*(u^\infty)$ of $P(\sigma_*, \sigma^*; u^\infty)$ having expression (4.13) (resp. (4.14)) for partition (4.11) with (4.12). It is easy to see that $S_c(k)$ $(k = 1, 2, 3)$, $S_0(i)$ $(i = i_0, i_0 + 1, \ldots, p)$, $S_1(l)$ and $\bar{S}_1(l)$ $(l = 1, 2, \ldots, l_0)$ are connected and mutually disjoint in $H^1(-L, L)$.

Remark 4.2. If $l_0\left(\frac{\kappa}{2}\right)^{1/2} I(b_*(u^\infty)) = 2L$, then $S_1(l_0)$ and $\bar{S}_1(l_0)$ are singletons consisting of solutions to $P(u^\infty)$. Also, if

$$b_0(u^\infty) < b_*(u^\infty), \qquad G(u^\infty; \zeta_{0-} \vee \sigma_*) > (\text{resp.} <)\ G(u^\infty; \zeta_{0+} \wedge \sigma^*),$$

then $S_1(1)$, $\bar{S}_1(1)$ and $\bar{S}_1(2)$ (resp. $S_1(2)$) are singletons.

5. ω-limit set $\omega(u_0, w_0)$

In this section we consider the ω-limit set $\omega(u_0, w_0)$ of the order parameter w as $t \to +\infty$.

Theorem 5.1. *Suppose that* (A1)–(A4), (H1)–(H4), (q1), (q2), $\frac{h^\infty}{n_0} \in D(\rho)$ *and* (3.1) *hold. Further suppose that*

$$b_*(u^\infty) = b_0(u^\infty) \tag{5.1}$$

Then the ω-limit set $\omega(u_0, w_0)$ is a singleton $\{w^\infty\}$, and therefore the order parameter $w(t)$ converges to w^∞ in $H^1(-L, L)$ as $t \to +\infty$. Moreover, unless w^∞ is constant on $[-L, L]$, w^∞ is a solution of $P(u^\infty)$.

Proof. Under (5.1), assume that $S_1 \neq \varnothing$. Let $v \in S_1$. Then $b := G(u^\infty; v(-L)) = b_0(u^\infty)$. But, by Lemmas 3.3 and 3.4, we have $2L \geq \left(\frac{\kappa}{2}\right)^{1/2} I(b) = +\infty$, which is a contradiction. Thus $S_1 = \varnothing$ must hold, and $S^* = S_c + S_0$ is a finite set by Lemma 3.2. Besides, by Theorem 2.1 (d), $\omega(u_0, w_0) \subset S^*$, so that $\omega(u_0, w_0)$ is a singleton $\{w^\infty\}$ with $w^\infty \in S_c$ or S_0. ◇

The above theorem says that under (5.1) the behaviour of any order parameter w has no influence of the constraints $\sigma_* \leq w(t, x) \leq \sigma^*$, when t is large enough.

Theorem 5.2. *Suppose that* (A1)–(A4), (H1)–(H4), (q1), (q2), $\frac{h^\infty}{n_0} \in D(\rho)$ *and* (3.1) *hold. Further suppose that*

$$b_0(u^\infty) < b_*(u^\infty) \ (< 0)\,. \tag{5.2}$$

Then, for the ω-limit set $\omega(u_0, w_0)$, there are the following two possibilities (1) *and* (2):

(1) *$\omega(u_0, w_0)$ is a singleton $\{w^\infty\}$. In this case, the order parameter $w(t)$ converges to w^∞ in $H^1(-L, L)$ as $t \to +\infty$. Moreover, if $G(u^\infty; w^\infty(-L)) > b_*(u^\infty)$ and w^∞ is non-constant on $[-L, L]$, then w^∞ is a solution of $P(u^\infty)$.*

(2) *$\omega(u_0, w_0)$ contains a continuum of solutions to $P(\sigma_*, \sigma^*; u^\infty)$. In this case, the following statements* (e1)–(e4) *hold:*

 (e1) *$G(u^\infty; v(-L)) = G(u^\infty; v(L)) = b_*(u^\infty)$ for all $v \in \omega(u_0, w_0)$.*

 (e2) *The boundary values $v(-L)$ and $v(L)$ are independent of the choice of $v \in \omega(u_0, w_0)$, and they are in $\{\eta_-(b_*(u^\infty)), \eta_+(b_*(u^\infty))\}$.*

 (e3) *For the order parameter $w(t)$ it holds that*

$$\lim_{t \to +\infty} |w_x(t)|_{L^2(-L,L)} = |v_x|_{L^2(-L,L)}$$

and

$$\lim_{t \to +\infty} \int_{-L}^{L} G(u^\infty; w(t, x))dx = \int_{-L}^{L} G(u^\infty; v(x))dx$$

for all $v \in \omega(u_0, w_0)$.

(e4) *The number of all points* $x \in [-L, L]$ *with* $v(x) = \zeta_0$ *is finite and independent of the choice of* $v \in \omega(u_0, w_0)$.

Proof. Suppose that $\omega(u_0, w_0)$ contains more than one element. Then by the connectedness of $\omega(u_0, w_0)$ in $H^1(-L, L)$ and (4.17), $\omega(u_0, w_0)$ contains a continuum of solutions to $P(\sigma_*, \sigma^*; u^\infty)$ and there exists a positive integer l such that

$$\omega(u_0, w_0) \subset S_1(l) \quad \text{or} \quad \omega(u_0, w_0) \subset \bar{S}_1(l)\,. \tag{5.3}$$

This together with the corollary to Theorem 4.1 proves (e1), (e2), (e4) and that

$$|v_x|_{L^2(-L,L)} = l|v_x^b|_{L^2(x_-^b, x_+^b)} \quad \text{for all } v \in \omega(u_0, w_0)\,, \tag{5.4}$$

where $b := G(u^\infty; v(-L)) = b_*(u^\infty)$. Since (5.4) implies that $|v_x|_{L^2(-L,L)}$ is constant on $\omega(u_0, w_0)$, it follows from global estimates (a) of Theorem 2.1 that

$$\lim_{t\to+\infty} |w_x(t)|_{L^2(-L,L)} = l|v_x^b|_{L^2(x_-^b, x_+^b)}\,. \tag{5.5}$$

Now we use the convergence result

$$\lim_{t\to+\infty} \left\{\frac{\kappa}{2}|w_x(t)|^2_{L^2(-L,L)} + \int_{-L}^{L} G(u^\infty; w(t,x))ds\right\} = \frac{\kappa}{2}|v_x|^2_{L^2(-L,L)} + \int_{-L}^{L} G(u^\infty; v(x))dx\,. \tag{5.6}$$

$$\text{for all } v \in \omega(u_0, w_0)\,;$$

the proof of (5.6) is found in [11; section 6]. Combining (5.6) with (5.5), we obtain (e3). ◇

According to Theorem 5.2 (e3)–(e4), under condition (5.2), the number of points x with $w(t, x) = \zeta_0$ does not change for large time t. In such a sense, the number of the connected components of the interface is constant for large time t; in other words, a pattern of phases is formed for large time t. As some numerical experiences show, there is actually a process in which $\omega(u_0, w_0)$ is a continuum of solutions of $P(\sigma_*, \sigma^*; u^\infty)$. This can be explained as follows. In general, for the order parameter w we see that $w_t \in L^2(\boldsymbol{R}_+; L^2(-L, L))$, but $w_t \notin L^1(\boldsymbol{R}_+; L^2(-L, L))$, so that the point $x(t) \in [-L, L]$ with $w(t, x(t)) = \zeta_0$ may oscillate as $t \to +\infty$.

Remark 5.1. In this paper, $q(u^\infty; \cdot)$ is supposed to be defined on the whole real line $\boldsymbol{R}$ and to satisfy (q1) and (q2). But, as is easily checked, we have the same results as Theorems 4.1, 5.1 and 5.2, as long as $q(u^\infty; \cdot)$ is a function defined on an open interval (ζ_*, ζ^*) such that

$$\lim_{w \downarrow \zeta_*} q(u^\infty; w) = -\infty, \qquad \lim_{w \uparrow \zeta^*} q(u^\infty; \cdot) = +\infty,$$

$$\zeta_* \le \sigma_* < \sigma^* \le \zeta^*$$

and (q1) is satisfied as well as the following (q2)′:

(q2)′ There are numbers ζ_{1-} and ζ_{1+}, depending on u^∞, such that

$$\zeta_* < \zeta_{0-} < \zeta_{1-} < \zeta_0 < \zeta_{1+} < \zeta_{0+} < \zeta^*,$$

$$q'(u^\infty; \cdot) > 0 \qquad \text{on } (\zeta_*, \zeta_{1-}) \cup (\zeta_{1+}, \zeta^*),$$

$$q'(u^\infty; \zeta_{1\pm}) = 0,$$

$$q'(u^\infty; \cdot) < 0 \qquad \text{on } (\zeta_{1-}, \zeta_{1+}),$$

$$q''(u^\infty; \cdot) \le 0 \qquad \text{on } (\zeta_*, \zeta_0),$$

$$q''(u^\infty; \zeta_0) = 0,$$

$$q''(u^\infty; \cdot) \ge 0 \qquad \text{on } (\zeta_0, \zeta^*).$$

In the case where $\zeta_* = \sigma_*$ and $\zeta^* = \sigma^*$, there are actually no constraints in the formulation of our problem.

Example 5.1. Consider the case where

$$g(w) = w^3 - w, \qquad \lambda(w) = -\frac{1}{2}w^2.$$

Then, for constants $u^\infty < 0$ and σ_*, σ^* with $\sigma_* < 0 < \sigma^*$, we see that $q(u^\infty; w) = 0$ has three roots $-(1 - u^\infty)^{1/2}$, 0, $(1 - u^\infty)^{1/2}$, and (q1), (q2) are fulfilled. It is easy to check that (5.1) holds if

$$1 - \sigma_*^2 \wedge \sigma^{*2} \le u^\infty < 0,$$

while (5.2) otherwise.

Example 5.2. Consider the case where

$$g(w) = w^3 + w, \qquad \lambda(w) = -\frac{1}{2}w^2.$$

Then, for constrants $u^\infty < -1$ and σ_*, σ^* with $\sigma_* < 0 < \sigma^*$, we see that $q(u^\infty; w) = 0$ has three roots $-(-u^\infty - 1)^{1/2}$, 0, $(-u^\infty - 1)^{1/2}$. Also, (q1) and (q2) are fulfilled if $u^\infty < -1$, while $q(u^\infty; w) = 0$ has exactly one root 0 and $G(u^\infty; w)$ is strictly convex if $-1 \le u^\infty < 0$. In the latter case of u^∞, problem $P(\sigma_*, \sigma^*; u^\infty)$ has a unique solution 0, and the order parameter $w(t)$ converges to 0 in $H^1(-L, L)$ as $t \to +\infty$. In the former case of u^∞, we have both of possibilities (5.1) and (5.2).

Example 5.3. We consider, for instance, a mean-field model for the Ising ferromagnet (cf. [13]), which is of the form:

$$\left(-\frac{1}{u}-\frac{a_0}{2}w^2\right)_t - u_{xx} = f(t,x) \qquad \text{in } Q := (0,+\infty)\times(-L,L),$$

and

$$w_t + \kappa w_{xx} + a_1 \log\frac{1+w}{1-w} - a_2 w + a_3 + a_0 wu = 0 \qquad \text{in } Q,$$

where a_0, a_1, a_2 are positive constants and a_3 is a constant. Under similar initial and boundary conditions to (1.4)–(1.6), this system has a unique global solution $\{u, w\}$ (cf. [8, 11, 12]) and the corresponding stationary problem is of the form

$$-\kappa v_{xx} + a_1 \log\frac{1+v}{1-v} - a_2 v + a_3 + a_0 v u^\infty = 0 \qquad \text{in } (-L, L)$$

with homogeneous Neumann boundary condition, where $u^\infty := \lim_{t\to+\infty} u(t,x)$ ($=$ const. < 0). Since the nonlinear term

$$q(u^\infty; v) := a_1 \log\frac{1+v}{1-v} - a_2 v + a_3 + a_0 v u^\infty$$

satisfies (q1) and (q2)$'$ with $\zeta_* = \sigma_* = -1$ and $\zeta^* = \sigma^* = 1$, we can extensively apply Theorem 5.1 and conclude that the stationary problem has at most a finite number of solutions and the function $w(t)$ converges in $H^1(-L,L)$ to one of them as $t \to +\infty$.

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