

## Viscosity Solutions of Nonlinear Second Order Elliptic PDEs Associated with Impulse Control Problems II

By

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Dedicated to Professor Sadakazu Aizawa on the occasion of his 60th birthday

### §1. Introduction

This paper is concerned with the uniqueness and existence of viscosity solutions of nonlinear second order elliptic partial differential equations (PDEs) with an implicit obstacle.

Let  $\Omega \subset \mathbf{R}^N$  be a bounded domain. For any function  $u: \bar{\Omega} \rightarrow \mathbf{R}$ , we define the operator  $M$  as the following:

$$Mu(x) = \inf_{\substack{\xi \geq 0 \\ x + \xi \in \bar{\Omega}}} \{k(\xi) + u(x + \xi)\},$$

where  $k(\xi)$  is a nonnegative and continuous function on  $(\mathbf{R}^+)^N$  and  $\xi \geq 0$  means  $\xi \in (\mathbf{R}^+)^N$ . We consider the nonlinear elliptic PDEs of the form:

$$(1.1) \quad \begin{cases} \max \{F(x, u, Du, D^2u), u - Mu\} = 0 & \text{in } \Omega, \\ \max \{u - g, u - Mu\} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here the  $g$  is a given function and the  $F$  is a second order degenerate elliptic operator. The problem (1.1) is associated with the impulse control problems for certain diffusion processes. For the formal derivation of (1.1) and some results on the impulse control problems, see A. Bensoussan–J. L. Lions [3], J. L. Menaldi [19], B. Perthame [20] and G. Barles [1] etc.

In the case where  $F$  is nondegenerate, we can interpret the boundary condition in (1.1) in the “classical” sense. In [3] the existence and uniqueness of solutions of (1.1) in  $H_0^1(\Omega) \cap C(\bar{\Omega})$  is discussed from the viewpoint of quasi-variational inequality when  $F$  is linear and  $g \equiv 0$  on  $\partial\Omega$ . B. Perthame [21] obtained the existence and uniqueness of solutions of (1.1) in  $W_{loc}^{2,\infty}(\Omega) \cap C(\bar{\Omega})$ . After introducing the notion of viscosity solutions, B. Perthame [22] and the author [11] showed the uniqueness and existence of viscosity solutions of (1.1).

However, in the case  $F$  is degenerate (especially on  $\partial\Omega$ ), we cannot interpret the boundary condition in the classical sense. Then H. Ishii [8] pointed out that in the degenerate case we should interpret the boundary condition

in the “viscosity” sense and proved the comparison principle and existence of viscosity solutions of first order Hamilton-Jacobi equations by analytical methods. (Also see M. G. Crandall–H. Ishii–P. L. Lions [4] and references therein). In order to get the comparison principle, he assumed the continuity of viscosity sub- and supersolutions near  $\partial\Omega$ . Recently M. A. Katsoulakis [12] and [13] have obtained the comparison principle of viscosity solutions of nonlinear second order degenerate elliptic PDEs. To show the comparison principle he has assumed the nontangential semicontinuity of viscosity sub- and supersolutions, which is a weaker assumption than that in [8]. Moreover in [12] and [13] he has established the existence of such solutions by probabilistic arguments. As to the systems of elliptic PDEs, see S. Koike [15] and M. A. Katsoulakis–S. Koike [14].

Our main purpose here is to get the comparison principle and existence of viscosity solutions of the problem (1.1). Since we deal with the case where  $F$  is degenerate, we consider the boundary condition in the viscosity sense.

This paper is organized in the following way. In Section 2 we give the definition of viscosity solutions of (1.1) and the equivalent propositions. In Section 3 we prove the comparison principle of viscosity solutions of (1.1). We remark that its proof is improved as compared with that of [11; Theorem 3.1]. Sections 4 and 5 provide the existence of continuous viscosity solutions of (1.1). Since it is difficult to discuss it for general elliptic operators, we consider only the case  $F$  is the Hamilton-Jacobi-Bellman operator in these sections. In Section 4 we apply the iterative approximation scheme by B. Hanouzet–J. L. Joly [7] to obtain the existence result, assuming the existence of continuous viscosity solutions of the usual obstacle problems. In Section 5 we show it by using the results in [13]. In Section 6 we prove that the unique solution of (1.1) obtained in Section 4 can be represented as the optimal cost function associated with the impulse control problem. In Section 7 we treat the boundary value problem of oblique type involving the operator  $M$ . For the related problems, see P. L. Lions–B. Perthame [18], P. Dupuis–H. Ishii [5], [6] and H. Ishii [9].

In what follows we suppress the term “viscosity” since we are mainly concerned with viscosity sub-, super- and solutions.

## §2. Definitions of solutions

In this section we give the definitions of solutions of (1.1) and the equivalent propositions.

We begin by preparing some notations.

$\langle \cdot, \cdot \rangle$  = the Euclidean inner product in  $R^N$ .

$B(x, r)$  = the open ball of radius  $r$  centered at  $x$ .

$$K(r, s, n) = \bigcup_{0 < t \leq r} \overline{B(tn, st)} \text{ for } r, s > 0 \text{ and } n \in \mathbf{R}^N \text{ with } |n| = 1.$$

$\mathbf{S}^N$  = the set of all  $N \times N$  real symmetric matrices.

$I$  = the identity matrix.

Let  $\mathcal{O} \subset \mathbf{R}^N$ . For any function  $u: \mathcal{O} \rightarrow \mathbf{R}$ , we define

$$u^*(x) = \limsup_{r \rightarrow 0} \{u(y) \mid y \in B(x, r) \cap \mathcal{O}\},$$

$$u_*(x) = \liminf_{r \rightarrow 0} \{u(y) \mid y \in B(x, r) \cap \mathcal{O}\},$$

$$J_{\mathcal{O}}^{2,+} u(x) = \{(p, X) \in \mathbf{R}^N \times \mathbf{S}^N \mid u(x+h) \leq u(x) + \langle p, h \rangle + \frac{1}{2} \langle Xh, h \rangle + o(|h|^2) \text{ as } x+h \in \mathcal{O} \text{ and } h \rightarrow 0\},$$

$$J_{\mathcal{O}}^{2,-} u(x) = \{(p, X) \in \mathbf{R}^N \times \mathbf{S}^N \mid u(x+h) \geq u(x) + \langle p, h \rangle + \frac{1}{2} \langle Xh, h \rangle + o(|h|^2) \text{ as } x+h \in \mathcal{O} \text{ and } h \rightarrow 0\}.$$

$$\begin{aligned} \bar{J}_{\mathcal{O}}^{2,+} u(x) &= \{(p, X) \in \mathbf{R}^N \times \mathbf{S}^N \mid \\ &\text{there exist } \{x_n\}_{n \in \mathbf{N}} \subset \mathcal{O} \text{ and } (p_n, X_n) \in J^{2,+} u(x_n) \\ &\text{such that } (x_n, u(x_n), p_n, X_n) \rightarrow (x, u(x), p, X) \text{ as } n \rightarrow +\infty\}, \end{aligned}$$

$$\begin{aligned} \bar{J}_{\mathcal{O}}^{2,-} u(x) &= \{(p, X) \in \mathbf{R}^N \times \mathbf{S}^N \mid \\ &\text{there exist } \{x_n\}_{n \in \mathbf{N}} \subset \mathcal{O} \text{ and } (p_n, X_n) \in J^{2,-} u(x_n) \\ &\text{such that } (x_n, u(x_n), p_n, X_n) \rightarrow (x, u(x), p, X) \text{ as } n \rightarrow +\infty\}, \end{aligned}$$

$$USC(\mathcal{O}) = \{u: \mathcal{O} \rightarrow \mathbf{R} \cup \{\pm\infty\}: \text{u.s.c.}\},$$

$$LSC(\mathcal{O}) = \{u: \mathcal{O} \rightarrow \mathbf{R} \cup \{\pm\infty\}: \text{l.s.c.}\}.$$

In order to give the definition, we set

$$G^*(x, r, p, X, m) = \begin{cases} \max \{F(x, r, p, X), r - m\} & (x \in \Omega), \\ \max \{ \max \{F(x, r, p, X), r - m\}, \\ \max \{r - g(x), r - m\} \} & (x \in \partial\Omega), \end{cases}$$

$$G_*(x, r, p, X, m) = \begin{cases} \max \{F(x, r, p, X), r - m\} & (x \in \Omega), \\ \min \{ \max \{F(x, r, p, X), r - m\}, \\ \max \{r - g(x), r - m\} \} & (x \in \partial\Omega), \end{cases}$$

where  $F \in C(\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N)$  satisfies the degenerate ellipticity condition:

$$F(x, r, p, X + Y) \leq F(x, r, p, X)$$

for all  $x \in \bar{\Omega}$ ,  $r \in \mathbf{R}$ ,  $p \in \mathbf{R}^N$ ,  $X, Y \in \mathbf{S}^N$  and  $Y \geq O$ .

**Definition 2.1.** Let  $u: \bar{\Omega} \rightarrow \mathbf{R}$ .

- (1) We say  $u$  is a subsolution of (1.1) provided  $u^* < +\infty$  on  $\bar{\Omega}$  and for any  $\varphi \in C^2(\bar{\Omega})$ , if  $u^* - \varphi$  attains a local maximum at  $x \in \bar{\Omega}$ , then

$$G_*(x, u^*(x), D\varphi(x), D^2\varphi(x), Mu^*(x)) \leq 0.$$

- (2) We say  $u$  is a supersolution of (1.1) provided  $u_* > -\infty$  on  $\bar{\Omega}$  and for any  $\varphi \in C^2(\bar{\Omega})$ , if  $u_* - \varphi$  attains a local minimum at  $x \in \bar{\Omega}$ , then

$$G^*(x, u_*(x), D\varphi(x), D^2\varphi(x), Mu_*(x)) \geq 0.$$

- (3) We say  $u$  is a solution of (1.1) provided  $u$  is both a sub- and a supersolution of (1.1).

Next we state the equivalent propositions of Definition 2.1. We refer the reader to M. G. Crandall–H. Ishii–P. L. Lions [4; Section 7] for general elliptic PDEs.

**Proposition 2.2.** Let  $u: \bar{\Omega} \rightarrow \mathbf{R}$ .

- (1)  $u$  is a subsolution of (1.1) if and only if  $u^* < +\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in \bar{J}_\Omega^{2,+} u^*(x)$ ,  $u^*$  satisfies

$$G_*(x, u^*(x), p, X, Mu^*(x)) \leq 0.$$

- (2)  $u$  is a supersolution of (1.1) if and only if  $u_* > -\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in \bar{J}_\Omega^{2,-} u_*(x)$ ,  $u_*$  satisfies

$$G^*(x, u_*(x), p, X, Mu_*(x)) \geq 0.$$

We note that, when  $F \in C(\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N)$  and  $g \in C(\bar{\Omega})$ ,  $G^*$  (resp.,  $G_*$ ) is the u.s.c. (resp., l.s.c.) envelope of the function  $G$ :

$$G(x, r, p, X, m) = \begin{cases} \max\{F(x, r, p, X), r - m\} & (x \in \Omega), \\ \max\{r - g(x), r - m\} & (x \in \partial\Omega). \end{cases}$$

**Proposition 2.3.** Assume  $M: USC(\bar{\Omega}) \rightarrow USC(\bar{\Omega})$  and  $M: LSC(\bar{\Omega}) \rightarrow LSC(\bar{\Omega})$ .

Let  $u: \bar{\Omega} \rightarrow \mathbf{R}$ .

- (1)  $u$  is a subsolution of (1.1) if and only if  $u^* < +\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in \bar{J}_\Omega^{2,+} u^*(x)$ ,  $u^*$  satisfies

$$G_*(x, u^*(x), p, X, Mu^*(x)) \leq 0.$$

- (2)  $u$  is a supersolution of (1.1) if and only if  $u_* > -\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in \bar{J}_\Omega^{2,-} u_*(x)$ ,  $u_*$  satisfies

$$G^*(x, u_*(x), p, X, Mu_*(x)) \geq 0.$$

Since the proofs of the above propositions are similar to those in [4], we leave them to the reader.

Finally, we state the definition and the equivalent propositions for the usual obstacle problems we treat in Sections 4 and 5.

$$(2.1) \quad \begin{cases} \max \{F(x, u, Du, D^2u), u - \psi\} = 0 & \text{in } \Omega, \\ \max \{u - g, u - \psi\} = 0 & \text{on } \partial\Omega. \end{cases}$$

**Definition 2.4.** Let  $u: \bar{\Omega} \rightarrow \mathbf{R}$ .

- (1) We say  $u$  is a subsolution of (2.1) provided  $u^* < +\infty$  on  $\bar{\Omega}$  and for any  $\varphi \in C^2(\bar{\Omega})$ , if  $u^* - \varphi$  attains a local maximum at  $x \in \bar{\Omega}$ , then

$$G_*(x, u^*(x), D\varphi(x), D^2\varphi(x), \psi^*(x)) \leq 0.$$

- (2) We say  $u$  is a supersolution of (2.1) provided  $u_* > -\infty$  on  $\bar{\Omega}$  and for any  $\varphi \in C^2(\bar{\Omega})$ , if  $u_* - \varphi$  attains a local minimum at  $x \in \bar{\Omega}$ , then

$$G^*(x, u_*(x), D\varphi(x), D^2\varphi(x), \psi_*(x)) \geq 0.$$

- (3) We say  $u$  is a solution of (2.1) provided  $u$  is both a sub- and a supersolution of (2.1).

**Proposition 2.5.** Let  $u: \bar{\Omega} \rightarrow \mathbf{R}$ .

- (1)  $u$  is a subsolution of (2.1) if and only if  $u^* < +\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in \bar{J}_\Omega^{2,+} u^*(x)$ ,  $u^*$  satisfies

$$G_*(x, u^*(x), p, X, \psi^*(x)) \leq 0.$$

- (2)  $u$  is a supersolution of (2.1) if and only if  $u_* > -\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in \bar{J}_\Omega^{2,-} u_*(x)$ ,  $u_*$  satisfies

$$G^*(x, u_*(x), p, X, \psi_*(x)) \geq 0.$$

**Proposition 2.6.** Assume  $\psi \in C(\bar{\Omega})$ . Let  $u: \bar{\Omega} \rightarrow \mathbf{R}$ .

- (1)  $u$  is a subsolution of (2.1) if and only if  $u^* < +\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in \bar{J}_\Omega^{2,+} u^*(x)$ ,  $u^*$  satisfies

$$G_*(x, u^*(x), p, X, \psi(x)) \leq 0.$$

- (2)  $u$  is a supersolution of (2.1) if and only if  $u_* > -\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in \bar{J}_\Omega^{2,-} u_*(x)$ ,  $u_*$  satisfies

$$G^*(x, u_*(x), p, X, \psi(x)) \geq 0.$$

We omit the proofs of the above propositions. See [4; Section 7].

### §3. Comparison principle of solutions

In this section we prove the comparison principle of solutions of the problem (1.1). To do so, we use the similar techniques to those in H. M. Soner [23], H. Ishii [8] and M. A. Katsoulakis–S. Koike [14].

We make the following assumptions.

(A.1)  $\Omega \subset \mathbf{R}^N$  is a bounded domain.

(A.2) There exist constants  $r, s, t > 0$  and a mapping  $n \in C(\bar{\Omega}; \mathbf{R}^N)$  with  $|n| = 1$  on  $\partial\Omega$  such that

$$z + K(r, s, n(z)) \subset \Omega \quad \text{for all } z \in \partial\Omega,$$

$$y + K\left(r, t, \frac{x}{|x|}\right) \subset \Omega \quad \text{for all } x \in K(r, s, n(z)), y \in B(z, r) \cap \bar{\Omega}.$$

(A.3) There exists a mapping  $P: \bar{\Omega} \times (\mathbf{R}^+)^N \rightarrow (\mathbf{R}^+)^N$  satisfying

$$x + P(x, \xi) \in \bar{\Omega} \quad \text{for all } (x, \xi) \in \bar{\Omega} \times (\mathbf{R}^+)^N,$$

$$P(x, \xi) = \xi \quad \text{if } x + \xi \in \bar{\Omega},$$

$$P(\cdot, \xi) \in C(\bar{\Omega}) \quad \text{for each } \xi \geq 0.$$

(A.4)  $F \in C(\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N)$ .

(A.5) There exists a function  $\omega_1 \in C(\mathbf{R}^+)$  such that  $\omega_1(0) = 0$  for which

$$F(y, r, p, Y) - F(x, r, p, X) \leq \omega_1(\alpha|x - y|^2 + |x - y|(|p| + 1))$$

$$\text{if } -3\alpha \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

for all  $x, y \in \bar{\Omega}$ ,  $p \in \mathbf{R}^N$ ,  $\alpha > 1$  and  $X, Y \in \mathbf{S}^N$ .

(A.6) There exists a function  $\omega_2 \in C(\mathbf{R}^+)$  such that  $\omega_2(0) = 0$  for which

$$|F(x, r, p, X) - F(x, r, q, X)| \leq \omega_2(|p - q|)$$

for all  $x \in \bar{\Omega}$ ,  $r \in \mathbf{R}$ ,  $p, q \in \mathbf{R}^N$  and  $X \in \mathbf{S}^N$ .

(A.7) There exists a constant  $\lambda > 0$  such that

$$F(x, r, p, X) - F(x, s, p, X) \leq \lambda(r - s) \quad \text{if } r \leq s$$

for all  $x \in \bar{\Omega}$ ,  $r, s \in \mathbf{R}$ ,  $p \in \mathbf{R}^N$ ,  $X \in \mathbf{S}^N$ .

(A.8)  $k \in C((\mathbf{R}^+)^N)$  and there exists a constant  $k_0 > 0$  such that  $k(\xi) \geq k_0$  for all  $\xi \geq 0$ .

(A.9)  $g \in C(\bar{\Omega})$ .

*Remark 3.1.* (1) When  $\partial\Omega$  is of class  $C^2$ , we take  $r = s = t > 0$  sufficiently small and  $n \in C(\bar{\Omega}; \mathbf{R}^N)$  such that  $n(x)$  is the inner normal to  $\Omega$  at  $x \in \partial\Omega$ . Then it is easily verified that (A.2) is satisfied.

(2) The assumption (A.3) is not trivial. When  $\Omega$  is convex and regular, we can take  $P(x, \xi)$  as the projection of  $\xi$  on  $(\mathbf{R}^+)^N \cap \bar{\Omega} - \{x\}$ . See A. Benssousan–J. L. Lions [3; Chapter 4, Remark 1.7] and J. L. Menaldi [19, Section 1].

(3) If (A.6) holds, then the operator  $F$  is degenerate elliptic. (cf. [4; Remark 3.4].)

(4) A typical example of  $F$  satisfying (A.4)–(A.7) is the Hamilton-Jacobi-Bellman operator treated in Sections 4 and 5.

We recall the properties of the operator  $M$ .

**Proposition 3.2.** *Assume (A.1), (A.3) and (A.8) hold. Let  $u, v: \bar{\Omega} \rightarrow \mathbf{R}$ . Then we have the following properties.*

- (1) *If  $u \leq v$  on  $\bar{\Omega}$ , then  $Mu \leq Mv$  on  $\bar{\Omega}$ .*
- (2)  *$M(tu + (1-t)v) \geq tMu + (1-t)Mv$  for all  $t \in [0, 1]$ .*
- (3)  *$M(u + c) = Mu + c$  for all  $c \in \mathbf{R}$ .*
- (4) *If  $u \in LSC(\bar{\Omega})$ , then  $Mu \in LSC(\bar{\Omega})$ .*
- (5) *If  $u \in USC(\bar{\Omega})$ , then  $Mu \in USC(\bar{\Omega})$ .*
- (6)  *$\|Mu - Mv\|_{C(\bar{\Omega})} \leq \|u - v\|_{C(\bar{\Omega})}$  for all  $u, v \in C(\bar{\Omega})$ .*

*Proof.* We only show (4) and (5) because it is obvious by the definition of  $M$  that (1)–(3) and (6) hold.

(4) We take  $\{x_n\}_{n \in \mathbf{N}} \subset \bar{\Omega}$ ,  $x \in \bar{\Omega}$  such that  $x_n \rightarrow x$  ( $n \rightarrow +\infty$ ). The condition  $u \in LSC(\bar{\Omega})$  implies that for each  $x_n$  there exists a  $\xi_n \geq 0$  such that

$$x_n + \xi_n \in \bar{\Omega}, \quad Mu(x_n) = k(\xi_n) + u(x_n + \xi_n).$$

Since  $\{\xi_n\}_{n \in \mathbf{N}}$  is bounded, by taking a subsequence, if necessary, we may consider that  $\lim_{n \rightarrow +\infty} \xi_n = \xi \geq 0$  such that  $x + \xi \in \bar{\Omega}$ . Hence we have

$$\begin{aligned} \liminf_{n \rightarrow +\infty} Mu(x_n) &\geq \lim_{n \rightarrow +\infty} k(\xi_n) + \liminf_{n \rightarrow +\infty} u(x_n + \xi_n) \\ &\geq k(\xi) + u(x + \xi) \\ &\geq Mu(x), \end{aligned}$$

that is,  $Mu \in LSC(\bar{\Omega})$ .

(5) We take  $\{x_n\}_{n \in \mathbf{N}} \subset \bar{\Omega}$  and  $x \in \bar{\Omega}$  as in the proof of (4) and fix  $\xi \geq 0$  such that  $x + \xi \in \bar{\Omega}$ . Then we have by (A.3),

$$Mu(x_n) \leq k(P(x_n, \xi)) + u(x_n + P(x_n, \xi)).$$

Thus we get

$$\begin{aligned} \limsup_{n \rightarrow +\infty} Mu(x_n) &\leq \lim_{n \rightarrow +\infty} k(P(x_n, \xi)) + \limsup_{n \rightarrow +\infty} u(x_n + P(x_n, \xi)) \\ &\leq k(\xi) + u(x + \xi). \end{aligned}$$

Taking the infimum with respect to  $\xi \geq 0$  satisfying  $x + \xi \in \bar{\Omega}$ ,  $Mu \in USC(\bar{\Omega})$  is proved. ■

The comparison principle of solutions of (1.1) is stated as follows.

**Theorem 3.3.** *Assume (A.1)–(A.9) hold. Let  $u$  and  $v$ , respectively, be a subsolution and a supersolution of (1.1). For each  $z \in \partial\Omega$ , let  $K_z = z + K(r, s, n(z))$ . If any one of the following holds, then  $u^* \leq v_*$  on  $\bar{\Omega}$ .*

- (1)  $\limsup_{K_z \ni x \rightarrow z} u^*(x) = u^*(z)$  and  $\liminf_{K_z \ni x \rightarrow z} v_*(x) = v_*(z)$  for each  $z \in \partial\Omega$ .
- (2)  $\limsup_{K_z \ni x \rightarrow z} u^*(x) = u^*(z)$  and  $u^*(z) \leq g(z)$  for each  $z \in \partial\Omega$ .
- (3)  $\liminf_{K_z \ni x \rightarrow z} v_*(x) = v_*(z)$  and  $v_*(z) \geq g(z)$  for each  $z \in \partial\Omega$ .

*Remark 3.4.* We call the properties in Theorem 3.3 (1) nontangential upper- and lower semicontinuity, respectively. See [12], [13] and [14].

We need the following lemma to deal with the term  $u - Mu$ .

**Lemma 3.5.** *Let  $\mathcal{O} \subset \mathbf{R}^N$  be compact and  $u \in USC(\mathcal{O})$ . Then, for a.a.  $q \in \mathbf{R}^N$ , the function  $u(x) + \langle q, x \rangle$  takes its strict maximum on  $\mathcal{O}$ .*

For the proof, see. H. Ishii–S. Koike [10; Lemma 3.3.].

*Proof of Theorem 3.3.* We may assume  $u \in USC(\bar{\Omega})$  and  $v \in LSC(\bar{\Omega})$ . We easily observe that  $u \leq Mu$  on  $\bar{\Omega}$ . First let the condition (1) hold.

We suppose  $\sup_{x \in \bar{\Omega}} (u - v) = 5\theta > 0$  and shall get a contradiction. Let  $L = \sup_{x \in \bar{\Omega}} |x|$  and let  $\{e_i\}_{1 \leq i \leq N}$  be the standard basis for  $\mathbf{R}^N$ . We take  $q \geq 0$  such that

$$(3.1) \quad 0 < |q| \leq \theta/L, \quad 0 \leq \omega_2(|q|) \leq \lambda\theta,$$

$$(3.2) \quad \langle q, e_i \rangle > 0 \quad \text{for each } i = 1, \dots, N$$

and fix it. Then by Lemma 3.5 the function  $u(x) - v(x) + \langle q, x \rangle$  attains its strict maximum at  $z (= z_q) \in \bar{\Omega}$ . We easily see

$$(3.3) \quad u(z) - v(z) + \langle q, z \rangle \geq 4\theta, \quad u(z) > v(z).$$

We claim

$$(3.4) \quad v(z) < Mv(z).$$

To prove this, suppose  $v(z) \geq Mv(z)$ . Since  $v \in LSC(\bar{\Omega})$ , using the definition of  $M$  and (A.8), we can find  $\xi_z \geq 0$  satisfying  $\xi_z \neq 0$ ,  $z + \xi_z \in \bar{\Omega}$  and  $Mv(z) = k(\xi_z) + v(z + \xi_z)$ . Thus  $u(z) \leq Mu(z)$  and  $v(z) \geq Mv(z)$  imply

$$u(z) - v(z) + \langle q, z \rangle \leq u(z + \xi_z) - v(z + \xi_z) + \langle q, z + \xi_z \rangle - \langle q, \xi_z \rangle.$$

Then we obtain a contradiction because  $\langle q, \xi_z \rangle > 0$  by (3.2). Therefore we get the claim (3.4).

We divide our consideration into three cases.

Case 1.  $z \in \partial\Omega$  and  $v(z) < g(z)$ .

Let  $\{z_n\}_{n \in \mathbb{N}} \subset K_z$  be a sequence such that

$$z_n \rightarrow z, \quad u^*(z_n) \rightarrow u^*(z) \quad (n \rightarrow +\infty).$$

We define the function  $\Phi(x, y)$  on  $\bar{\Omega} \times \bar{\Omega}$  by

$$\Phi(x, y) = u(x) - v(y) + \langle q, y \rangle - \frac{\alpha_n}{2} |x - y - z_n + z|^2,$$

where  $\alpha_n = s_0^2 / |z_n - z|^2$  and  $s_0 > 0$  satisfies  $\omega_1(s_0^2) < \lambda\theta$ .

Let  $(x_n, y_n) \in \bar{\Omega} \times \bar{\Omega}$  be a maximum point of  $\Phi$ . Since  $\Phi(z_n, z) \leq \Phi(x_n, y_n)$ , we get

$$\begin{aligned} (3.5) \quad u(z_n) - v(z) + \langle q, z \rangle &\leq u(z_n) - v(z) + \langle q, z \rangle + \frac{\alpha_n}{2} |x_n - y_n - z_n + z|^2 \\ &\leq u(x_n) - v(y_n) + \langle q, y_n \rangle. \end{aligned}$$

The function  $u(x) - v(y)$  is bounded above on  $\bar{\Omega} \times \bar{\Omega}$  because  $u, -v \in USC(\bar{\Omega})$  and  $\bar{\Omega} \times \bar{\Omega}$  is compact in  $\mathbb{R}^{2N}$ . Hence (3.5) implies  $|x_n - y_n - z_n + z| \rightarrow 0$  as  $n \rightarrow +\infty$ . Moreover we easily observe  $|x_n - y_n| \rightarrow 0$  as  $n \rightarrow +\infty$ . Then there exist a sequence  $\{n_k\} \subset \mathbb{N}$  and a point  $\bar{z} \in \bar{\Omega}$  such that  $x_{n_k}, y_{n_k} \rightarrow \bar{z}$  as  $k \rightarrow +\infty$ . It follows from this, (3.5) and the semicontinuity of  $u$  and  $v$  that

$$u(z) - v(z) + \langle q, z \rangle \leq u(\bar{z}) - v(\bar{z}) + \langle q, \bar{z} \rangle$$

Since  $z$  is a unique maximum point of the function  $u(x) - v(x) + \langle q, x \rangle$  on  $\bar{\Omega}$ , it follows from this inequality that  $\bar{z} = z$  and

$$(3.6) \quad x_n, y_n \rightarrow z \quad (n \rightarrow +\infty).$$

Thus, by (3.5) we get

$$\lim_{n \rightarrow \infty} (u(x_n) - v(y_n)) = u(z) - v(z).$$

Using (3.5), this equality and the semicontinuity of  $u$  and  $v$ , we have

$$(3.7) \quad u(x_n) \rightarrow u(z), \quad v(y_n) \rightarrow v(z), \quad \alpha_n |x_n - y_n - z_n + z|^2 \rightarrow 0 \quad (n \rightarrow +\infty).$$

We may consider  $x_n \in \Omega$  for sufficiently large  $n \in N$  because (3.7) implies  $|x_n - y_n - z_n + z| < t|z_n - z|$  for large  $n \in N$ , where  $t$  is the constant in (A.2). Furthermore, it is observed by the definition of  $\alpha_n$  and (3.7) that

$$(3.8) \quad \sqrt{\alpha_n} |x_n - y_n| \rightarrow s_0 \quad (n \rightarrow +\infty).$$

We can apply the maximum principle for semicontinuous functions to obtain  $X, Y \in \mathbf{S}^N$  satisfying

$$\begin{aligned} (p_n, X) &\in \bar{J}^{2,+} u(x_n), \\ (p_n + q, Y) &\in \bar{J}^{2,+} v(y_n), \end{aligned}$$

and

$$-3\alpha_n \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq 3\alpha_n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

where  $p_n = \alpha_n(x_n - y_n - z_n + z)$ . Using the fact that  $u$  and  $v$  are respectively, a subsolution and a supersolution of (1.1), we obtain the following inequalities:

$$(3.9) \quad G_*(x_n, u(x_n), p_n, X, Mu(x_n)) \leq 0,$$

$$(3.10) \quad G^*(y_n, v(y_n), p_n + q, Y, Mv(y_n)) \geq 0.$$

We note  $v(y_n) < g(y_n)$  for large  $n \in N$  by (A.9), (3.6) and  $v(z) < g(z)$ . Moreover, since  $Mv \in LSC(\bar{\Omega})$  by Proposition 3.2 (3), using (3.4) we get

$$\limsup_{n \rightarrow +\infty} (v(y_n) - Mv(y_n)) \leq v(z) - Mv(z) < 0$$

and conclude that  $v(y_n) - Mv(y_n) < 0$  for sufficiently large  $n \in N$ . Therefore, by (3.10) we obtain

$$(3.11) \quad F(y_n, v(y_n), p_n + q, Y) \geq 0.$$

From (3.9) and  $x_n \in \Omega$  for large  $n \in N$ , we have

$$(3.12) \quad F(x_n, u(x_n), p_n, X) \leq 0.$$

Subtracting (3.12) from (3.11) and using (A.5), (A.6), (A.7) and (3.3), we obtain

$$\begin{aligned} 4\lambda\theta &\leq \lambda(u(x_n) - v(y_n) + \langle q, y_n \rangle) \\ &\leq F(y_n, u(x_n), p_n + q, Y) - F(x_n, u(x_n), p_n, X) + \lambda \langle q, y_n \rangle \\ &\leq \omega_1(\alpha_n |x_n - y_n|^2 + |x_n - y_n|(|p_n| + 1)) + \omega_2(|q|) + \lambda L|q|. \end{aligned}$$

Recalling (3.1), (3.7) and (3.8) and letting  $n \rightarrow +\infty$ , we get

$$4\lambda\theta \leq \omega_1(s_0^2) + \omega_2(|q|) + \lambda\theta \leq 3\lambda\theta,$$

which is a contradiction.

Case 2.  $z \in \partial\Omega$ ,  $u(z) > g(z)$ .

As in Case 1, we define the function  $\Phi$  by

$$\Phi(x, y) = u(x) - v(y) + \langle q, x \rangle - \frac{\alpha_n}{2} |x - y + z_n - z|^2 \quad \text{on } \bar{\Omega} \times \bar{\Omega}.$$

We can prove the remainder similarly to the above.

Case 3.  $z \in \Omega$ .

For  $\alpha > 0$ , we consider the function

$$\Phi(x, y) = u(x) - v(y) + \langle q, x \rangle - \frac{\alpha}{2} |x - y|^2 \quad \text{on } \bar{\Omega} \times \bar{\Omega}.$$

In this case the proof is standard. See [8; Section 3].

When the condition (2) (resp., (3)) holds, it is sufficient to consider only Case 2, 3 (resp., Case 1, 3) in the above proof. Thus we obtain the result. ■

*Remark 3.6.* As compared with the author [11; Theorem 3.1], the proof of Theorem 3.3 is improved on the point that we do not need the uniform continuity in the variable  $X \in \mathbf{S}^N$  and the convexity in  $(r, p, X) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$ .

We conclude this section by stating the comparison principle of solutions of the usual obstacle problem (2.1). We omit the proof because it is similar to that of Theorem 3.3.

**Theorem 3.7.** Assume (A.1), (A.2), (A.4)–(A.7), (A.9) and  $\psi \in C(\bar{\Omega})$ . Let  $u, v$  be, respectively, a subsolution and a supersolution of (2.1). For each  $z \in \partial\Omega$ , let  $K_z = z + K(r, s, n(z))$ . If any one of the following holds, then  $u^* \leq v_*$  on  $\bar{\Omega}$ .

- (1)  $\limsup_{K_z \ni x \rightarrow z} u^*(x) = u^*(z)$  and  $\liminf_{K_z \ni x \rightarrow z} v_*(x) = v_*(z)$  for each  $z \in \partial\Omega$ .
- (2)  $\limsup_{K_z \ni x \rightarrow z} u^*(x) = u^*(z)$  and  $u^*(z) \leq g(z)$  for each  $z \in \partial\Omega$ ,
- (3)  $\liminf_{K_z \ni x \rightarrow z} v_*(x) = v_*(z)$  and  $v_*(z) \geq g(z)$  for each  $z \in \partial\Omega$ ,

*Remark 3.8.* Of course, in Theorems 3.3 and 3.7, if  $u, v \in C(\bar{\Omega})$ , then  $u \leq v$  on  $\bar{\Omega}$ .

#### §4. Existence of continuous solutions

In this and the next section we establish the existence of continuous solutions of (1.1). As mentioned in Section 1, it is difficult to show it for the general elliptic operator case. Hence in these sections we treat the case  $F$  is the Hamilton-Jacobi-Bellman operator:

$$F(x, r, p, X) = \sup_{\alpha \in A} \{ -\text{tr}(\sigma(x, \alpha)\sigma(x, \alpha)X) + \langle b(x, \alpha), p \rangle + c(x, \alpha)r - f(x, \alpha) \},$$

where  $\mathcal{A}$  is a compact metric space and  $tr A$  and  ${}^tA$  denote, respectively, the trace and the transposed matrix of  $A$ . In this and the next sections we assume  $\partial\Omega$  is of class  $C^2$ . Then we note (A.2) is satisfied. Let  $\rho(x) = dist(x, \Omega^c)$ . We make the assumptions of the coefficients of  $F$  as follows.

(C.1)  $\sup_{\alpha \in \mathcal{A}} \{ \|\sigma(\cdot, \alpha)\|_{W^{1,\infty}(\bar{\Omega})}, \|b(\cdot, \alpha)\|_{W^{1,\infty}(\bar{\Omega})}, \|c(\cdot, \alpha)\|_{C(\bar{\Omega})}, \|f(\cdot, \alpha)\|_{C(\bar{\Omega})} \} < +\infty.$

(C.2)  $\inf\{c(x, \alpha) \mid x \in \bar{\Omega}, \alpha \in \mathcal{A}\} \geq c_0$  for some  $c_0 > 0$ .

(C.3) There exists a function  $\alpha \in W^{1,\infty}(\bar{\Omega})$  satisfying

(i)  $tr({}^t\sigma(x, \alpha(x))\sigma(x, \alpha(x))D^2\rho(x)) - \langle b(x, \alpha(x)), D\rho(x) \rangle \geq \eta$  for some  $\eta > 0$ ,

(ii)  $\langle {}^t\sigma(x, \alpha(x))\sigma(x, \alpha(x))D\rho(x), D\rho(x) \rangle = 0$ ,

(iii) There are unit vectors  $\{\hat{e}_i\}_{1 \leq i \leq N-1} \subset \mathbf{R}^N$  by which the tangent plane at  $x$  is spanned such that

$$\langle {}^t\sigma(x, \alpha(x))\sigma(x, \alpha(x))\hat{e}_i, \hat{e}_i \rangle = 0$$

except at most two vectors  $\{\hat{e}_{i_1}, \hat{e}_{i_2}\}$ .

for all  $x \in \partial\Omega$ .

(C.4) There exist a constant  $\eta > 0$  and a function  $\beta \in W^{1,\infty}(\bar{\Omega})$  satisfying either

(i)  $tr({}^t\sigma(x, \beta(x))\sigma(x, \beta(x))D^2\rho(x)) - \langle b(x, \beta(x)), D\rho(x) \rangle \leq -\eta$

or

(ii)  $\langle {}^t\sigma(x, \beta(x))\sigma(x, \beta(x))D\rho(x), D\rho(x) \rangle \geq \eta$ .

for all  $x \in \partial\Omega$ .

*Remark 4.1.* (1) As to the assumption (C.3), see M. A. Katsoulakis [12] and [13].

(2) We consider the following operator:

$$F(x, r, p, X) = \max\{-tr X + r - f^1(x), \langle b(x), p \rangle + r - f^2(x)\}.$$

Here  $b \in W^{1,\infty}(\bar{\Omega})$ ,  $b = -v$  on  $\partial\Omega$  and  $f^1, f^2 \in C(\bar{\Omega})$ . Then the above  $F$  satisfies the assumptions (C.1)–(C.4).

(3) In the case  $\sigma(x, \alpha) \equiv 0$  for all  $x \in \bar{\Omega}$ ,  $\alpha \in \mathcal{A}$ , the existence of solutions was proved by H. Ishii [8; Section 4].

(4) In the case only (C.4) holds for all  $x \in \partial\Omega$  and  $\alpha \in \mathcal{A}$ , we have already proved the existence of solutions of (1.1) by Perron’s method. See the author [11; Section 4].

Under the assumptions (A.1), (A.3), (A.8), (A.9), (C.1) and (C.2), Theorems 3.3 and 3.7 hold. We get the following theorem.

**Theorem 4.2.** *Assume (A.1), (A.3), (A.8), (A.9) and (C.1)–(C.4). Then there exists a unique solution  $u \in C(\bar{\Omega})$  of the problem (1.1).*

In order to show this theorem, the following proposition plays an important role.

**Proposition 4.3.** *Assume (A.1), (A.9) and (C.1)–(C.4). Then, for each  $\psi \in C(\bar{\Omega})$ , there exists a unique solution  $u_\psi \in C(\bar{\Omega})$  of the problem (2.1).*

Here we admit Proposition 4.3 is true and prove Theorem 4.2. We give the proof of Proposition 4.3 in the next section.

*Proof of Theorem 4.2.* We adopt the iterative approximation scheme introduced in B. Honouzet–J. L. Joly [18].

Let  $C_1 = \max \{ \sup_{\alpha \in A} \{ \|f(\cdot, \alpha)\|_{C(\bar{\Omega})}, \|g\|_{C(\bar{\Omega})} \}, \|g\|_{C(\bar{\Omega})} \}$ . By replacing  $f(\cdot, \alpha), g$  with  $f(\cdot, \alpha) + C_1, g + C_1$ , respectively, we may assume  $f(\cdot, \alpha) \geq 0$  ( $\alpha \in A$ ),  $g \geq 0$  on  $\bar{\Omega}$ . Using the results in Katsoulakis [13], there exists a unique solution  $u_0 \in C(\bar{\Omega})$  of

$$(4.1) \quad \begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u - g = 0 & \text{on } \partial\Omega. \end{cases}$$

Since  $Mu_0 \in C(\bar{\Omega})$  by Proposition 3.2 (4), (5), there exists a unique solution  $u_1 \in C(\bar{\Omega})$  of

$$(4.2)_1 \quad \begin{cases} \max \{ F(x, u, Du, D^2u), u - Mu_0 \} = 0 & \text{in } \Omega, \\ \max \{ u - g, u - Mu_0 \} = 0 & \text{on } \partial\Omega. \end{cases}$$

by Proposition 4.3. For  $n = 2, 3, \dots$ , we denote by  $u_n \in C(\bar{\Omega})$  a unique solution of

$$(4.2)_n \quad \begin{cases} \max \{ F(x, u, Du, D^2u), u - Mu_{n-1} \} = 0 & \text{in } \Omega, \\ \max \{ u - g, u - Mu_{n-1} \} = 0 & \text{on } \partial\Omega. \end{cases}$$

(It follows from Proposition 3.2 (4), (5) that  $Mu_{n-1} \in C(\bar{\Omega})$ .) Since  $u_1$  is a subsolution of (4.1), we obtain  $u_1 \leq u_0$  on  $\bar{\Omega}$  by Theorem 3.7. It is easily seen that  $\underline{u} \equiv 0$  on  $\bar{\Omega}$  is a subsolution of (4.2)<sub>1</sub>. Thus Theorem 3.7 implies  $u_1 \geq 0$  on  $\bar{\Omega}$ . Since  $0 \leq Mu_1 \leq Mu_0$  on  $\bar{\Omega}$  by  $0 \leq u_1 \leq u_0$  on  $\bar{\Omega}$  and Proposition 3.2 (1),  $u_2$  is a subsolution of (4.2)<sub>1</sub>. Then we get  $u_2 \leq u_1$  by using Theorem 3.7. In the similar way to the above, we have  $u_2 \geq 0$  on  $\bar{\Omega}$ . Continuing these processes, we conclude

$$(4.3) \quad 0 \leq \dots \leq u_n \leq \dots \leq u_2 \leq u_1 \leq u_0 \quad \text{on } \bar{\Omega}.$$

Next we show an upper estimate. We take  $\mu \in (0, 1)$  such that  $\mu \|u_0\|_{C(\bar{\Omega})} \leq k_0$ . For each  $n \in \mathbb{N}$ , there exists a  $\theta_n \in (0, 1]$  such that

$$(4.4) \quad u_n - u_{n+1} \leq \theta_n u_n \quad \text{on } \bar{\Omega}.$$

It is observed by (A.8) and Proposition 3.2 (2) in that

$$(4.5) \quad \begin{aligned} (1 - \theta_n)Mu_n + \theta_n k_0 &\leq (1 - \theta_n)Mu_n + \theta_n M0 \\ &\leq M(1 - \theta_n)u_n \leq Mu_{n+1} \quad \text{on } \bar{\Omega}. \end{aligned}$$

We define  $\psi$ ,  $w$  and  $v_0$  as follows:

$$\psi = (1 - \theta_n)Mu_n + \theta_n k_0 (\in C(\bar{\Omega})),$$

$w \in C(\bar{\Omega})$ : a unique solution of

$$(4.6) \quad \begin{cases} \max \{F(x, u, Du, D^2u), u - \psi\} = 0 & \text{in } \Omega, \\ \max \{u - g, u - \psi\} = 0 & \text{on } \partial\Omega, \end{cases}$$

$v \in C(\bar{\Omega})$ : a unique solution of

$$(4.7) \quad \begin{cases} \max \{F(x, u, Du, D^2u), u - k_0\} = 0 & \text{in } \Omega, \\ \max \{u - g, u - k_0\} = 0 & \text{on } \partial\Omega. \end{cases}$$

Noting  $\psi \leq Mu_{n+1}$  on  $\bar{\Omega}$ , we see that  $w$  is a subsolution of (4.2) $_{n+2}$ . Hence we get  $w \leq u_{n+2}$  on  $\bar{\Omega}$  by Theorem 3.7. It is observed by  $f(\cdot, \alpha) \geq 0$ ,  $g \geq 0$  on  $\bar{\Omega}$  and (4.7) that  $\theta_n v_0$  is a solution of

$$(4.8) \quad \begin{cases} \max \{F_{\theta_n}(x, u, Du, D^2u), u - \theta_n k_0\} = 0 & \text{in } \Omega, \\ \max \{u - \theta_n g, u - \theta_n k_0\} = 0 & \text{on } \partial\Omega. \end{cases}$$

where

$$F_{\theta}(x, r, p, X) = \sup_{\alpha \in \Lambda} \{-tr({}^t\sigma(x, \alpha)\sigma(x, \alpha)X) + \langle b(x, \alpha), p \rangle + c(x, \alpha) - \theta f(x, \alpha)\}.$$

It follows from  $f(\cdot, \alpha) \geq 0$ ,  $g \geq 0$ ,  $\psi \geq \theta_n k_0$  on  $\bar{\Omega}$  and (4.6) that  $\theta_n w$  is a supersolution of (4.8). Thus, using Theorem 3.7, we have  $\theta_n v_0 \leq \theta_n w$  on  $\bar{\Omega}$ . Moreover, we easily see that  $(1 - \theta_n)u_{n+1}$  and  $(1 - \theta_n)w$  are, respectively, a subsolution and a solution of

$$\begin{cases} \max \{F_{1-\theta_n}(x, u, Du, D^2u), u - (1 - \theta_n)\psi\} = 0 & \text{in } \Omega, \\ \max \{u - (1 - \theta_n)g, u - (1 - \theta_n)\psi\} = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore we obtain  $(1 - \theta_n)u_{n+1} \leq (1 - \theta_n)w$  on  $\bar{\Omega}$  by Theorem 3.7. Consequently, we get

$$(4.9) \quad (1 - \theta_n)u_{n+1} + \theta_n v_0 \leq u_{n+2} \quad \text{on } \bar{\Omega}.$$

From  $\mu \|u_0\|_{C(\bar{\Omega})} \leq k_0$  and  $f(\cdot, \alpha) \geq 0$ ,  $g \geq 0$  on  $\bar{\Omega}$ , we observe that  $\mu u_{n+1}$  is a subsolution of (4.7). Thus, by Theorem 3.6 we have  $\mu u_{n+1} \leq v_0$  on  $\bar{\Omega}$ . Hence (4.9) implies

$$(4.10) \quad u_{n+1} - u_{n+2} \leq \theta_n(1 - \mu)u_{n+1} \quad \text{on } \bar{\Omega}.$$

By the way, since  $u_1 - u_2 \leq u_1$  on  $\bar{\Omega}$ , we obtain  $u_2 - u_3 \leq (1 - \mu)u_2$  on  $\bar{\Omega}$ . Therefore we can take  $\theta_2 = 1 - \mu$  in (4.4) when  $n = 2$ . Then it is observed by (4.10)  $u_3 - u_4 \leq (1 - \mu)^2 u_3$  on  $\bar{\Omega}$ . Therefore, using the above argument inductively, we conclude

$$(4.11) \quad u_{n+1} - u_{n+2} \leq (1 - \mu)^n u_{n+1} \leq (1 - \mu)^n \|u_0\|_{C(\bar{\Omega})} \quad \text{on } \bar{\Omega},$$

which is our desired estimate.

Combining (4.3) with (4.11), we can find a function  $u \in C(\bar{\Omega})$  such that  $\|u_n - u\|_{C(\bar{\Omega})} \rightarrow 0$  as  $n \rightarrow +\infty$ . By Proposition 3.2 (6) and the stability of solutions (cf. P. L. Lions [17; Proposition I.3]), we conclude that  $u$  is a solution of (1.1). The uniqueness follows from Theorem 3.3. ■

### §5. Proof of Proposition 4.3

In this section we show Proposition 4.3. We always assume the assumptions in Proposition 4.3. We prepare some notations.

$W_t$  = standard  $N$ -dimensional Brownian motion.

$\mathcal{A} = \{\alpha_t: [0, +\infty) \rightarrow A: \text{progressively measurable}\}$ .

$\mathcal{B} = \{\theta: \text{stopping time}\}$ .

$X_t$ : solution of

$$\begin{cases} dX_t = -b(X_t, \alpha_t) dt + \sqrt{2}\sigma(X_t, \alpha_t) dW_t, t > 0, \\ X_0 = x \in \bar{\Omega}. \end{cases}$$

$\tau = \inf \{t \geq 0 \mid X_t \notin \bar{\Omega}\}$ .

$1_A$  = characteristic function for  $A$ .

Let  $\tilde{g} = \min \{g, \psi\}$  on  $\partial\Omega$ . We consider the penalized problem for (2.1).

$$(5.1) \quad \begin{cases} F(x, u_n, Du_n, D^2u_n) + n(u_n - \psi)^+ = 0 & \text{in } \Omega, \\ u_n = \tilde{g} & \text{on } \partial\Omega, \end{cases}$$

where  $n \in N$  and  $r^+ = \max \{r, 0\}$ .

Noting  $r^+ = \sup \{\gamma r \mid 0 \leq \gamma \leq 1\}$ , it is easily seen that (5.1) is equivalent to the following PDE:

$$(5.2) \quad \begin{cases} \sup_{\substack{a \in A \\ \gamma \in [0, 1]}} \{-tr({}^t\sigma(x, \alpha)\sigma(x, \alpha)Du_n) + \langle b(x, \alpha), Du_n \rangle \\ \quad + (c(x, \alpha) + n\gamma)u_n - f(x, \alpha) - n\gamma\psi\} = 0 & \text{in } \Omega, \\ u_n = \tilde{g} & \text{on } \partial\Omega. \end{cases}$$

Then applying the results in M. A. Katsoulakis [13], for each  $n \in N$ , there exists a unique solution  $u_n \in C(\bar{\Omega})$  of (5.2).

Next we consider the following problem:

$$(5.3) \quad \begin{cases} F(x, v_n, Dv_n, D^2v_n) + n(u_n - \psi)^+ = 0 & \text{in } \Omega, \\ v_n = \tilde{g} & \text{on } \partial\Omega, \end{cases}$$

where  $u_n$  is the function obtained above. Using the results in [13] again, for each  $n \in N$ , there exists a unique solution  $v_n \in C(\bar{\Omega})$  of (5.3) and it is characterized as follows:

$$v_n(x) = \inf_{\alpha \in \mathcal{A}} E_x \left\{ \int_0^\tau (f(X_t, \alpha_t) - n(u_n(X_t) - \psi(X_t))^+) \cdot \exp\left(-\int_0^t c(X_s, \alpha_s) ds\right) dt + \tilde{g}(X_\tau) \exp\left(-\int_0^\tau c(X_s, \alpha_s) ds\right) \right\}.$$

Since (5.1) and (5.2) are equivalent to each other and the uniqueness of solutions of (5.1) holds in the class  $C(\bar{\Omega})$ , we get

$$(5.4) \quad u_n(x) = \inf_{\alpha \in \mathcal{A}} E_x \left\{ \int_0^\tau (f(X_t, \alpha_t) - n(u_n(X_t) - \psi(X_t))^+) \cdot \exp\left(-\int_0^t c(X_s, \alpha_s) ds\right) dt + \tilde{g}(X_\tau) \exp\left(-\int_0^\tau c(X_s, \alpha_s) ds\right) \right\}.$$

Using (C.4) and the barrier argument, we have

$$(5.5) \quad u_n \leq \tilde{g} \quad \text{on } \partial\Omega \text{ for all } n \in N.$$

Since the operator  $nr^+$  is monotone with respect to  $n \in N$  and  $u_n \geq -C$  for large  $C > 0$ , we obtain

$$(5.6) \quad -C \leq \dots \leq u_n \leq \dots \leq u_2 \leq u_1 \quad \text{on } \bar{\Omega}$$

by the comparison principle of solutions of (5.1). (cf. M. G. Crandall–H. Ishii–P. L. Lions [4; Theorem 7.9].) Hence we can define the function  $u$  by

$$(5.7) \quad u(x) = \lim_{n \rightarrow +\infty} u_n(x) \left( = \lim_{\substack{n \rightarrow +\infty \\ y \rightarrow x}} \sup u_n(y) \right).$$

Then we get the following lemma.

**Lemma 5.1.** *The above function  $u$  is a u.s.c. subsolution of (2.1).*

*Proof.* Since the sequence  $\{u_n\}_{n \in N}$  is decreasing by (5.6), we easily observe  $u \in USC(\bar{\Omega})$ . Using (5.5) and letting  $n \rightarrow +\infty$ , we have  $u \leq \tilde{g}$  on  $\partial\Omega$ .

For any  $\varphi \in C^2(\bar{\Omega})$ , we assume that  $u - \varphi$  attains a local maximum at  $x_0 \in \bar{\Omega}$ . We may consider  $x_0 \in \Omega$  and that  $x_0$  is a strict local maximum point of  $u - \varphi$ . Then there exists  $\delta > 0$  such that

$$(5.8) \quad u(x_0) - \varphi(x_0) > u(x) - \varphi(x) \quad \text{for all } x \in \overline{B(x_0, \delta)} \cap \Omega, \quad x \neq x_0.$$

Let  $x_n$  be a maximum point of  $u_n - \varphi$  on  $\overline{B(x_0, \delta)}$ . Then there exists a subsequence  $\{x_{n_k}\}_{k \in N} \subset \{x_n\}_{n \in N}$  such that

$$x_{n_k} \rightarrow \bar{x} \in \overline{B(x_0, \delta)}, \quad u_{n_k}(x_{n_k}) \rightarrow \beta \in \mathbf{R} \quad (k \rightarrow +\infty).$$

Since

$$u_{n_k}(x) - \varphi(x) \leq u_{n_k}(x_{n_k}) - \varphi(x_{n_k}) \quad \text{for all } x \in B(x_0, \delta),$$

we get

$$\begin{aligned} u(x_0) - \varphi(x_0) &\leq \limsup_{\substack{k \rightarrow +\infty \\ x \rightarrow x_0}} (u_{n_k}(x) - \varphi(x)) \\ &\leq \limsup_{k \rightarrow +\infty} (u_{n_k}(x_{n_k}) - \varphi(x_{n_k})) \\ &= \beta - \varphi(\bar{x}) \\ &\leq \limsup_{\substack{k \rightarrow +\infty \\ x \rightarrow \bar{x}}} (u_{n_k}(x) - \varphi(x)) \\ &= u(\bar{x}) - \varphi(\bar{x}). \end{aligned}$$

Therefore using (5.8) and the above inequality, we obtain

$$(5.9) \quad x_n \rightarrow x_0, \quad u_n(x_n) \rightarrow u(x_0) \quad (n \rightarrow +\infty).$$

(cf. G. Barles–B. Perthame [2; Lemma A.3].) Since  $u_n$  is a subsolution of (5.1), we get

$$(5.10) \quad F(x_n, u_n(x_n), D\varphi(x_n), D^2\varphi(x_n)) + n(u_n(x_n) - \psi(x_n))^+ \leq 0.$$

It follows from (C.1) and (5.9) that there exists a constant  $C > 0$  such that

$$n(u_n(x_n) - \psi(x_n))^+ \leq C \quad \text{for all } n \in \mathbf{N}.$$

Thus passing to the limit as  $n \rightarrow +\infty$ , we have

$$u(x_0) - \psi(x_0) \leq 0.$$

Moreover, (5.10) implies  $F(x_n, u_n(x_n), D\varphi(x_n), D^2\varphi(x_n)) \leq 0$ . Sending  $n \rightarrow +\infty$ , we obtain

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

Therefore we have completed the proof. ■

*Remark 5.2.* We notice that we cannot apply the results for the limit operations in [4; Section 5] to (2.1) and (5.1) directly since the term  $n(r - \psi(x))^+$  does not converge to 0 locally uniformly on  $\bar{\Omega} \times \mathbf{R}$  as  $n \rightarrow +\infty$ .

We return to the formula (5.4). According to N. V. Krylov [16; p. 37], we get the following lemma.

**Lemma 5.3.** *The formula (5.4) can be rewritten as follows.*

$$(5.11) \quad u_n(x) = \inf_{\substack{\alpha \in \mathcal{A} \\ \theta \in \mathcal{B}}} E_x \left\{ \int_0^{\tau \wedge \theta} f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + 1_{\theta < \tau} \psi_n(X_\theta) \exp \left( - \int_0^\theta c(X_s, \alpha_s) ds \right) \right. \\ \left. + 1_{\theta \geq \tau} \tilde{g}(X_\tau) \exp \left( - \int_0^\tau c(X_s, \alpha_s) ds \right) \right\},$$

where  $a \wedge b = \min(a, b)$  and  $\psi_n = \psi + (u_n - \psi)^+$ .

*Proof.* We remark that the function  $u_n$  satisfies the dynamic programming principle:

$$u_n(x) = \inf_{\alpha \in \mathcal{A}} E_x \left\{ \int_0^{\tau \wedge \theta} (f(X_t, \alpha_t) - n(u_n(X_t) - \psi(X_t))^+) \cdot \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + 1_{\theta < \tau} u_n(X_\theta) \exp \left( - \int_0^\theta c(X_s, \alpha_s) ds \right) \right. \\ \left. + 1_{\theta \geq \tau} \tilde{g}(X_\tau) \exp \left( - \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}$$

for any  $\theta \in \mathcal{B}$ . Since  $u_n \leq \psi_n$  on  $\bar{Q}$ , we get

$$(5.12) \quad u_n(x) \leq \inf_{\substack{\alpha \in \mathcal{A} \\ \theta \in \mathcal{B}}} E_x \left\{ \int_0^{\tau \wedge \theta} f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + 1_{\theta < \tau} \psi_n(X_\theta) \exp \left( - \int_0^\theta c(X_s, \alpha_s) ds \right) \right. \\ \left. + 1_{\theta \geq \tau} \tilde{g}(X_\tau) \exp \left( - \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}.$$

Let  $\theta^n = \theta^{n,\alpha} = \inf \{t \geq 0 \mid u_n(X_t) \geq \psi(X_t)\}$ . Then using  $u_n(X_t) < \psi(X_t)$  ( $0 \leq t < \theta^n$ ) and  $u_n(X_{\theta^n}) = \psi(X_{\theta^n})$  with probability 1, we have

$$u_n(x) = \inf_{\alpha \in \mathcal{A}} E_x \left\{ \int_0^{\tau \wedge \theta^n} f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + 1_{\theta^n < \tau} \psi_n(X_{\theta^n}) \exp \left( - \int_0^{\theta^n} c(X_s, \alpha_s) ds \right) \right. \\ \left. + 1_{\theta^n \geq \tau} \tilde{g}(X_\tau) \exp \left( - \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}.$$

Combining this with (5.12), we obtain the result. ■

Using this lemma, we have

**Lemma 5.4.**  $u_n \rightrightarrows u$  on  $\bar{\Omega}$  as  $n \rightarrow +\infty$  and the function  $u$  is represented as

$$\begin{aligned} u(x) = \inf_{\substack{\alpha \in \mathcal{A} \\ \theta \in \mathcal{B}}} E_x \left\{ \int_0^{\tau \wedge \theta} f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + 1_{\theta < \tau} \psi(X_\theta) \exp \left( - \int_0^\theta c(X_s, \alpha_s) ds \right) \right. \\ \left. + 1_{\theta \geq \tau} \tilde{g}(X_\tau) \exp \left( - \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}. \end{aligned}$$

*Proof.* It is easily seen by Lemma 5.1 that  $u \leq \psi$  on  $\bar{\Omega}$ . We observe that  $\psi_n \in C(\bar{\Omega})$  for all  $n \in \mathbb{N}$  and

$$\psi_n(x) \searrow \psi(x) \quad (n \rightarrow +\infty) \quad \text{for each } x \in \bar{\Omega}$$

by (5.6) and (5.7). Hence, using Dini's Theorem, we get

$$\psi_n \rightrightarrows \psi \quad \text{on } \bar{\Omega} \quad (n \rightarrow +\infty).$$

Letting  $n \rightarrow +\infty$ , we conclude that

$$\begin{aligned} \text{RHS of (5.11)} \rightrightarrows \inf_{\substack{\alpha \in \mathcal{A} \\ \theta \in \mathcal{B}}} E_x \left\{ \int_0^{\tau \wedge \theta} f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + 1_{\theta < \tau} \psi(X_\theta) \exp \left( - \int_0^\theta c(X_s, \alpha_s) ds \right) \right. \\ \left. + 1_{\theta \geq \tau} \tilde{g}(X_\tau) \exp \left( - \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}. \end{aligned}$$

On the other hand, we have already obtained  $u_n(x) \rightarrow u(x)$  as  $n \rightarrow +\infty$  for each  $x \in \bar{\Omega}$  by (5.7). Thus we have the result. ■

We are now in a position to prove Proposition 4.3.

*Proof of Proposition 4.3.* We have only to show that  $u$  is a supersolution of (2.1).

For any  $\varphi \in C^2(\bar{\Omega})$ , we assume  $u - \varphi$  takes a strict local minimum at  $x_0 \in \bar{\Omega}$ . We consider the case  $x_0 \in \partial\Omega$ . Then we may assume  $u(x_0) < \tilde{g}(x_0)$ , because, if otherwise, we get  $u(x_0) = g(x_0)$  or  $u(x_0) = \psi(x_0)$  and have nothing to prove. Since  $u \in C(\bar{\Omega})$  by Lemma 5.4, there exists a  $\delta > 0$  satisfying

$$\begin{aligned} u(x) < g(x) & \quad x \in \overline{B(x_0, \delta)} \cap \partial\Omega, \\ u(x) < \psi(x) & \quad x \in \overline{B(x_0, \delta)} \cap \bar{\Omega}. \end{aligned}$$

Moreover, Lemma 5.4 implies there exists an  $n_0 \in N$  satisfying, for all  $n > n_0$ ,

$$(5.13) \quad u_n(x) < g(x) \quad x \in \overline{B(x_0, \delta)} \cap \partial\Omega,$$

$$(5.14) \quad u_n(x) < \psi(x) \quad x \in \overline{B(x_0, \delta)} \cap \overline{\Omega}.$$

Let  $x_n \in \overline{B(x_0, \delta)} \cap \overline{\Omega}$  be a minimum point of  $u_n - \varphi$  on  $\overline{B(x_0, \delta)} \cap \overline{\Omega}$ . By the same argument as in the proof of Lemma 5.1, we have

$$x_n \rightarrow x_0, \quad u_n(x_n) \rightarrow u(x_0) \quad (n \rightarrow +\infty).$$

Therefore, using (5.13), (5.14) and the fact that  $u_n$  is a supersolution of (5.1), we obtain

$$F(x_n, u_n(x_n), D\varphi(x_n), D^2\varphi(x_n)) \geq 0.$$

Sending  $n \rightarrow +\infty$ , we get

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0.$$

Thus the proof is completed. ■

## §6. Stochastic representation of solutions

In this section we prove that the unique solution of (1.1) is represented as the optimal cost function for the impulse control problem.

We call a collection  $(\alpha, \theta, \xi)$  an impulse control if

$$\alpha \in \mathcal{A},$$

$$\theta = \{\theta_i\}_{i=1}^{+\infty} \subset \mathcal{B} \text{ satisfies}$$

$$\theta_1 < \theta_2 < \cdots < \theta_n < \cdots \rightarrow +\infty \quad (n \rightarrow +\infty),$$

$$\xi = \{\xi_i\}_{i=1}^{+\infty}: \text{a sequence of random variables taking values on } (\mathbf{R}^+)^N, \\ \text{adapted with respect to } \{\theta_i\}_{i=1}^{+\infty}.$$

The  $\mathcal{C}$  denotes the set of all impulse controls.

We define the sequence of diffusions  $\{X_t^n\}_{n=1}^{+\infty}$  with jumps by the Ito equation:

$$\begin{cases} dX_t^0 = -b(X_t^0, \alpha_t) dt + \sqrt{2}\sigma(X_t^0, \alpha_t) dW_t, & t \geq 0, \\ X_0^0 = x \in \overline{\Omega}, \end{cases}$$

$$\begin{cases} dX_t^n = -b(X_t^n, \alpha_t) dt + \sqrt{2}\sigma(X_t^n, \alpha_t) dW_t, & t > \theta_n, \\ X_0^n = X_{\theta_n}^{n-1} + 1_{t=\theta_n} \xi_n & t \leq \theta_n. \end{cases}$$

We set

$$X_t = \lim_{n \rightarrow +\infty} X_t^n, \quad t \geq 0.$$

Then the process  $X_t$ , which is right continuous and has left limits, satisfies the following stochastic differential equation:

$$\begin{cases} dX_t = -b(X_t, \alpha_t)dt + \sqrt{2}\sigma(X_t, \alpha_t)dW_t + \sum_{i=1}^{+\infty} \xi_i \delta(t - \theta_i)dt, & t \geq 0, \\ X_0 = x, \end{cases}$$

where  $\delta(t)$  is the Dirac measure. We put

$$\tau = \inf \{t \geq 0 | X_t \notin \bar{\Omega}\}.$$

We call a collection  $(\alpha, \theta, \xi) \in \mathcal{C}$  an admissible impulse control if it satisfies

$$X_\tau \in \bar{\Omega} \quad \text{a.s. on } \{\tau < +\infty\},$$

that is, no jump of the process  $X_t$  is outside of  $\bar{\Omega}$  before  $\tau$ . We denote by  $\mathcal{C}_0$  the set of all admissible impulse controls.

Now, we can define the cost function for this system:

$$\begin{aligned} K &= (\alpha, \theta, \xi), \\ J(x, K) &= E_x \left\{ \int_0^\tau f(X_t, \alpha_t) \exp\left(-\int_0^t c(X_s, \alpha_s) ds\right) dt \right. \\ &\quad + \sum_{i=1}^{+\infty} 1_{\theta_i < +\infty} k(\xi_i) \exp\left(-\int_0^{\theta_i} c(X_s, \alpha_s) ds\right) \\ &\quad \left. + g(X_\tau) \exp\left(-\int_0^\tau c(X_s, \alpha_s) ds\right) \right\} \end{aligned}$$

and the optimal cost function:

$$w(x) = \inf_{K \in \mathcal{C}_0} J(x, K).$$

Then we have the following theorem.

**Theorem 6.1.** Assume (A.1), (A.3), (A.8), (A.9), (C.1)–(C.4) and  $\partial\Omega$  is of class  $C^2$ . Let  $u$  be a unique solution of (1.1). Then  $u = w$  on  $\bar{\Omega}$ .

We state some properties of the sequence  $\{u_n\}_{n \in \mathbb{N}}$  of solutions of (5.1).

**Lemma 6.2.** For each  $n \in \mathbb{N}$ , we have

$$\begin{aligned} (6.1) \quad u_n(x) &= \inf_{K \in \mathcal{C}^n} E_x \left\{ \int_0^\tau f(X_t, \alpha_t) \exp\left(-\int_0^t c(X_s, \alpha_s) ds\right) dt \right. \\ &\quad + \sum_{i=1}^n 1_{\theta_i < +\infty} k(\xi_i) \exp\left(-\int_0^{\theta_i} c(X_s, \alpha_s) ds\right) dt \\ &\quad \left. + g(X_\tau) \exp\left(-\int_0^\tau c(X_s, \alpha_s) ds\right) dt \right\}. \end{aligned}$$

where  $\mathcal{C}^n = \{(\alpha, \{\theta_i\}_{i=1}^{+\infty}, \{\xi_i\}_{i=1}^{+\infty}) \in \mathcal{C}_0 | \theta_i = +\infty \text{ for } i \geq n + 1\}$ .

*Proof.* By Lemma 5.4 the function  $u_n$  can be represented as the following:

$$\begin{aligned} u_n(x) = & \inf_{\substack{\alpha \in \mathcal{A} \\ \theta \in \mathcal{B}}} E_x \left\{ \int_0^{\tau \wedge \theta} f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ & + 1_{\theta < \tau} M u_{n-1}(X_\theta) \exp \left( - \int_0^\theta c(X_s, \alpha_s) ds \right) \\ & \left. + 1_{\theta \geq \tau} \tilde{g}(X_\tau) \exp \left( - \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}. \end{aligned}$$

We prove the assertion by induction. Let  $w_n = \text{RHS of (6.1)}$ .

For  $n = 1$ , it is trivial. We assume  $u_n = w_n$  on  $\bar{\Omega}$  for  $n \geq 1$  and show  $u_{n+1} = w_{n+1}$  on  $\bar{\Omega}$ .

Fix  $x \in \bar{\Omega}$  and  $K \in \mathcal{C}^{n+1}$ . We may consider  $\theta_1 < \tau$ , because, if otherwise, we have the result. It is clear that

$$\begin{aligned} w_{n+1}(x) \leq & E_x \left\{ \int_0^\tau f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ & + \sum_{i=1}^{n+1} 1_{\theta_i < +\infty} k(\xi_i) \exp \left( - \int_0^{\theta_i} c(X_s, \alpha_s) ds \right) dt \\ & \left. + g(X_\tau) \exp \left( - \int_0^\tau c(X_s, \alpha_s) ds \right) dt \right\}. \end{aligned}$$

We observe

$$\begin{aligned} w_{n+1}(x) \leq & E_x \left[ \int_0^{\theta_1} f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ & + E_{X_{\theta_1 - 0} + \xi_1} \left\{ \int_0^{\tau - \theta_1} f(X_{t+\theta_1}, \alpha_{t+\theta_1}) \exp \left( - \int_0^t c(X_{s+\theta_1}, \alpha_{s+\theta_1}) ds \right) dt \right. \\ & + \sum_{i=2}^{n+1} 1_{\theta_i < +\infty} k(\xi_i) \exp \left( - \int_0^{\theta_i - \theta_1} c(X_{s+\theta_1}, \alpha_{s+\theta_1}) ds \right) dt \\ & \left. + g(X_\tau) \exp \left( - \int_0^{\tau - \theta_1} c(X_{s+\theta_1}, \alpha_{s+\theta_1}) ds \right) dt + k(\xi_1) \right\} \\ & \left. \cdot \exp \left( - \int_0^{\theta_1} c(X_s, \alpha_s) ds \right) dt \right]. \end{aligned}$$

Since  $(\alpha_{t+\theta_1}, \{\theta_i - \theta_1\}_{i=2}^{n+1}, \{\xi_i\}_{i=2}^{n+1}) \in \mathcal{C}^n$ , we take the infimum with respect to admissible controls in  $\mathcal{C}^n$  to obtain

$$w_{n+1}(x) \leq E_x \left\{ \int_0^{\theta_1} f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + (u_n(X_{\theta_1-0} + \xi_1) + k(\xi_1)) \exp \left( - \int_0^{\theta_1} c(X_s, \alpha_s) ds \right) dt \right\}.$$

Moreover, taking the infimum with respect to  $\xi_1 \geq 0$  satisfying  $X_{\theta_1-0} + \xi \in \bar{\Omega}$ , we have

$$w_{n+1}(x) \leq E_x \left\{ \int_0^{\theta_1} f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + Mu_n(X_{\theta_1-0}) \exp \left( - \int_0^{\theta_1} c(X_s, \alpha_s) ds \right) dt \right\}.$$

Hence by taking the infimum with respect to  $(\alpha, \theta_1) \in \mathcal{A} \times \mathcal{B}$  we get  $w_{n+1}(x) \leq u_{n+1}(x)$ .

Next we prove the opposite inequality. For each  $\varepsilon > 0$ , there exists an impulse control  $K = (\alpha, \theta, \xi) \in \mathcal{C}^{n+1}$  such that

$$w(x) + \varepsilon \geq J(x, K).$$

We calculate

$$w_{n+1}(x) + \varepsilon \geq E_x \left\{ \int_0^{\theta_1} f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + (u_n(X_{\theta_1-0} + \xi_1) + k(\xi_1)) \exp \left( - \int_0^{\theta_1} c(X_s, \alpha_s) ds \right) dt \right\} \\ \geq E_x \left\{ \int_0^{\theta_1} f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + Mu_n(X_{\theta_1-0}) \exp \left( - \int_0^{\theta_1} c(X_s, \alpha_s) ds \right) dt \right\} \\ \geq u_{n+1}(x).$$

Letting  $\varepsilon \rightarrow 0$ , we have  $w_{n+1}(x) \geq u_{n+1}(x)$ . Thus we have completed the proof. ■

*Remark 6.3.* We can show that the function  $w$  satisfies

$$w(x) = \inf_{\substack{\alpha \in \mathcal{A} \\ \theta \in \mathcal{B}}} E_x \left\{ \int_0^\tau f(X_t, \alpha_t) \exp \left( - \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + 1_{\theta < \tau} Mw(X_\theta) \exp \left( - \int_0^\theta c(X_s, \alpha_s) ds \right) dt \right. \\ \left. + g(X_\tau) \exp \left( - \int_0^\tau c(X_s, \alpha_s) ds \right) dt \right\}$$

in a similar way. See G. Barles [1; Theorem 2.1].

**Lemma 6.4.** *We have*

$$(6.2) \quad \|u_n - u\|_{L^\infty(\Omega)} \leq \frac{C}{k_0 n} \quad \text{for some } C > 0.$$

*Proof.* First we remark that  $w \leq \dots \leq u_n \leq \dots \leq u_1 \leq u_0$  on  $\bar{\Omega}$ . Let  $K = (\alpha, \theta, \xi) \in \mathcal{C}_0$ . We set

$$\theta_i^n = \begin{cases} \theta_i & \text{if } i \leq n, \\ +\infty & \text{if } i \geq n+1, \end{cases}$$

$$\theta^n = \{\theta_i^n\}_{i=1}^{+\infty},$$

$$K^n = (\alpha, \theta^n, \xi) \in \mathcal{C}^n.$$

Let  $X_t^n$  be the process associated with  $K^n$  and  $\tau^n = \inf\{t \geq 0 \mid X_t^n \notin \bar{\Omega}\}$ . Then we note that if  $\tau < \theta_{n+1}$  or  $\theta_{n+1} = +\infty$ , then  $\tau^n = \tau$ . Hence we get

$$\begin{aligned} J(x, K) - J(x, K^n) &\geq E_x \left[ \left\{ \int_{\theta_n}^{\tau} f(X_t, \alpha_t) \exp\left(-\int_0^t c(X_s, \alpha_s) ds\right) dt \right. \right. \\ &\quad + g(X_\tau) \exp\left(-\int_0^{\tau} c(X_s, \alpha_s) ds\right) \\ &\quad - \int_{\theta_n}^{\tau^n} f(X_t^n, \alpha_t) \exp\left(-\int_0^t c(X_s^n, \alpha_s) ds\right) dt \\ &\quad \left. \left. + g(X_{\tau^n}^n) \exp\left(-\int_0^{\tau^n} c(X_s^n, \alpha_s) ds\right) dt \right\} 1_{\theta_{n+1} < +\infty} 1_{\theta_{n+1} \leq \tau} \right] \\ &\geq E_x \left[ 1_{\theta_n < +\infty} u_0(X_{\theta_n}) \exp\left(-\int_0^{\theta_n} c(X_s, \alpha_s) ds\right) dt \right. \\ &\quad - 1_{\theta_n < +\infty} \frac{1}{c_0} (\|f(\cdot, \alpha)\|_{C(\bar{\Omega})} + \|g\|_{C(\bar{\Omega})}) \\ &\quad \left. \cdot \exp\left(-\int_0^{\theta_n} c(X_s^n, \alpha_s) ds\right) \right] \\ &\geq -CE_x \left\{ \exp\left(-\int_0^{\theta_n} c(X_s^n, \alpha_s) ds\right) 1_{\theta_n < +\infty} \right\}, \end{aligned}$$

where  $C = \|u_0\|_{C(\bar{\Omega})} + (\sup_{\alpha \in \mathcal{A}} \|f(\cdot, \alpha)\|_{C(\bar{\Omega})} + \|g\|_{C(\bar{\Omega})})/c_0$ .

On the other hand, we may consider  $J(x, K) \leq \|u_0\|_{C(\bar{\Omega})}$  for all  $K \in \mathcal{C}_0$ . Therefore we have

$$E_x \left\{ \sum_{i=1}^{+\infty} 1_{\theta_i < +\infty} k(\xi_i) \exp\left(-\int_0^{\theta_i} c(X_s, \alpha_s) ds\right) \right\} \leq C.$$

Thus it is observed by the above inequality and (A.8) that

$$k_0 n E_x \left\{ 1_{\theta_n < +\infty} \exp \left( - \int_0^{\theta_n} c(X_s, \alpha_s) ds \right) \right\} \leq C.$$

Hence we obtain

$$\begin{aligned} J(x, K) &\geq J(x, K^n) - \frac{C}{k_0 n} \\ &\geq u_n - \frac{C}{k_0 n}. \end{aligned}$$

Taking the infimum with respect to  $K \in \mathcal{C}_0$ , we get

$$u(x) \geq u_n(x) - \frac{C}{k_0 n} \quad \text{for all } x \in \bar{\Omega}.$$

Thus we obtain (6.2). ■

*Proof of Theorem 6.1.* Lemmas 6.2 and 6.4 imply  $u_n \rightrightarrows w$  on  $\bar{\Omega}$  as  $n \rightarrow +\infty$ . Hence it is clear that  $u = w$  on  $\bar{\Omega}$ . ■

### §7. Boundary value problem of oblique type

In this section we treat the boundary value problem of oblique type:

$$(7.1) \quad \begin{cases} \max \{F(x, u, Du, D^2u), u - Mu\} = 0 & \text{in } \Omega, \\ \max \left\{ \frac{\partial u}{\partial \gamma}, u - Mu \right\} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here  $\partial\Omega$  is smooth and  $\gamma$  is a vector field on  $\mathbf{R}^N$  “oblique” to  $\partial\Omega$ . The problem (7.1) is derived from the impulse control problem for the diffusion processes reflecting at the boundary  $\partial\Omega$ . See P. L. Lions–B. Perthame [18] for the related problems. P. Dupuis–H. Ishii [5], [6] and H. Ishii [9] have obtained the uniqueness and existence of solutions of some oblique derivative problems. But they do not contain the problem (7.1).

Here we prove the comparison principle and existence of solutions of (7.1) by the similar arguments to those in [5], [6] and [9]. Instead of (A.6) we assume the uniform continuity of the  $F$  with respect to the variable  $(p, X) \in \mathbf{R}^N \times \mathbf{S}^N$ .

(A.6)' There exists a function  $\omega_3 \in C(\mathbf{R}^+)$  such that  $\omega_3(0) = 0$  for which

$$|F(x, r, p, X) - F(x, r, q, Y)| \leq \omega_3(|p - q| + \|X - Y\|)$$

for all  $x \in \bar{\Omega}$ ,  $r \in \mathbf{R}$ ,  $p, q \in \mathbf{R}^N$ ,  $X, Y \in \mathbf{S}^N$ , where  $\|X\|$  is the norm of  $X \in \mathbf{S}^N$  as a self-adjoint operator.

Besides, we put the following assumption.

- (A.10)  $\gamma \in C^2(\partial\Omega)$  and there exists a constant  $\eta > 0$  such that  $\langle v(x), \gamma(x) \rangle \geq \eta$  for all  $x \in \partial\Omega$ .

In order to give the definition of solutions of (7.1), we set

$$H^*(x, r, p, X, m) = \begin{cases} \max \{F(x, r, p, X), r - m\} & (x \in \Omega), \\ \max \{ \max \{F(x, r, p, X), r - m\}, \\ \max \{ \langle p, \gamma(x) \rangle, r - m \} \} & (x \in \partial\Omega), \end{cases}$$

$$H_*(x, r, p, X, m) = \begin{cases} \max \{F(x, r, p, X), r - m\} & (x \in \Omega), \\ \min \{ \max \{F(x, r, p, X), r - m\}, \\ \max \{ \langle p, \gamma(x) \rangle, r - m \} \} & (x \in \partial\Omega), \end{cases}$$

where  $F$  is the same function as in Section 2.

**Definition 7.1.** Let  $u: \bar{\Omega} \rightarrow \mathbf{R}$ .

- (1) We say  $u$  is a subsolution of (7.1) provided  $u^* < +\infty$  on  $\bar{\Omega}$  and for any  $\varphi \in C^2(\bar{\Omega})$ , if  $u^* - \varphi$  attains a local maximum at  $x_0 \in \bar{\Omega}$ , then

$$H_*(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0), Mu^*(x_0)) \leq 0.$$

- (2) We say  $u$  is a supersolution of (7.1) provided  $u_* > -\infty$  on  $\bar{\Omega}$  and for any  $\varphi \in C^2(\bar{\Omega})$ , if  $u_* - \varphi$  attains a local minimum at  $x_0 \in \bar{\Omega}$ , then

$$H^*(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0), Mu_*(x_0)) \geq 0.$$

- (3) We say  $u$  is a solution of (7.1) provided  $u$  is both a sub- and a supersolution of (7.1).

We mention the equivalent propositions of Definition 7.1 without their proofs.

**Proposition 7.2.** Let  $u: \bar{\Omega} \rightarrow \mathbf{R}$ .

- (1)  $u$  is a subsolution of (7.1) if and only if  $u^* < +\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in J_{\bar{\Omega}}^{2,+} u^*(x)$ ,  $u^*$  satisfies

$$H_*(x, u^*(x), p, X, Mu^*(x)) \leq 0.$$

- (2)  $u$  is a supersolution of (7.1) if and only if  $u_* > -\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in J_{\bar{\Omega}}^{2,-} u_*(x)$ ,  $u_*$  satisfies

$$H^*(x, u_*(x), p, X, Mu_*(x)) \geq 0.$$

**Proposition 7.3.** Assume  $M: USC(\bar{\Omega}) \rightarrow USC(\bar{\Omega})$  and  $M: LSC(\bar{\Omega}) \rightarrow LSC(\bar{\Omega})$ . Let  $u: \bar{\Omega} \rightarrow \mathbf{R}$ .

- (1)  $u$  is a subsolution of (7.1) if and only if  $u^* < +\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in \bar{J}_{\Omega}^{2,+} u^*(x)$ ,  $u^*$  satisfies

$$H_*(x, u^*(x), p, X, Mu^*(x)) \leq 0.$$

- (2)  $u$  is a supersolution of (7.1) if and only if  $u_* > -\infty$  on  $\bar{\Omega}$  and for all  $x \in \bar{\Omega}$  and  $(p, X) \in \bar{J}_{\Omega}^{2,-} u_*(x)$ ,  $u_*$  satisfies

$$H^*(x, u_*(x), p, X, Mu_*(x)) \geq 0.$$

Now, we state our main results in this section.

**Theorem 7.4.** Assume (A.1), (A.3)–(A.5), (A.6)', (A.7), (A.8), (A.10) and  $\partial\Omega$  is smooth. Let  $u, v$  be, respectively, a subsolution and a supersolution of (7.1). Then  $u^* \leq v_*$  on  $\bar{\Omega}$ .

**Theorem 7.5.** Under the same assumptions as in Theorem 7.4, there exists a unique solution  $u$  of (7.1). Moreover  $u \in C(\bar{\Omega})$ .

We need some lemmas to prove these theorems.

**Lemma 7.6.** Assume (A.10). Let  $z \in \partial\Omega$ . Then there exist  $\delta > 0, C_0 > 0$  and  $\{w_\alpha\}_{\alpha>0}: C^{1,1}$ -functions on  $\bar{B}(z, \delta) \times \bar{B}(z, \delta)$  satisfying the following properties:

$$\begin{aligned} w_\alpha(x, x) &\leq \frac{1}{\alpha} && \text{on } \bar{B}(z, \delta), \\ w_\alpha(x, y) &\geq \frac{\alpha}{8}|x - y|^2 && \text{on } \bar{B}(z, \delta) \times \bar{B}(z, \delta), \\ \langle D_x w_\alpha(x, y), \gamma(x) \rangle &\geq -C_0 \delta_\alpha && \text{if } x \in \partial\Omega \text{ and } y \in \bar{B}(z, \delta), \\ \langle -D_y w_\alpha(x, y), \gamma(y) \rangle &\leq C_0 \delta_\alpha && \text{if } y \in \partial\Omega \text{ and } x \in \bar{B}(z, \delta), \\ |D_x w_\alpha(x, y)| &\leq C_0(\alpha|x - y| + 1), \\ |D_x w_\alpha(x, y) + D_y w_\alpha(x, y)| &\leq C_0 \delta_\alpha, \\ \left( D w_\alpha(x, y), \alpha C_0 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_0 \delta_\alpha \begin{pmatrix} I & O \\ O & I \end{pmatrix} \right) &\in \bar{J}^{2,+} w_\alpha(x, y) \\ &&& \text{for } \alpha > 0, x, y \in \bar{B}(z, \delta) \end{aligned}$$

where  $\delta_\alpha = \alpha|x - y|^2 + 1/\alpha$ .

The above lemma is proved in [9; Section 4]. Hence we omit the proof.

**Lemma 7.7.** Assume (A.1), (A.3) and (A.8). Let  $u \in USC(\bar{\Omega})$  and  $v \in LSC(\bar{\Omega})$ . If  $u \leq Mu$  on  $\bar{\Omega}$ , then there exists a maximum point  $z \in \bar{\Omega}$  of the function  $u - v$  on  $\bar{\Omega}$  such that  $v(z) - Mv(z) < 0$ .

This lemma is mentioned in [18; Section 5] without its proof. For the sake of completeness we give the proof.

*Proof.* Let  $z_0 \in \bar{\Omega}$  be any maximum point of the function  $u - v$  on  $\bar{\Omega}$ . If the assertion in this lemma holds at  $z_0$ , we have nothing to prove. We suppose  $v(z_0) - Mv(z_0) \geq 0$ . Then there exists a  $\xi_0 \geq 0$  such that  $\xi_0 \neq 0$ ,  $z_0 + \xi_0 \in \bar{\Omega}$  and  $Mv(z_0) = k(\xi_0) + v(z_0 + \xi_0)$  by (A.8). Since  $u \leq Mu$  on  $\bar{\Omega}$ , we obtain

$$(7.2) \quad u(z_0) - v(z_0) \leq u(z_0 + \xi_0) - v(z_0 + \xi_0)$$

Hence  $z_0 + \xi_0$  is a maximum point of  $u - v$ . If the assertion holds at  $z_0 + \xi_0$ , the proof is completed. Here we suppose that the above process can be repeated infinitely, that is,  $\{z_n\}_{n \in N}$  is a sequence of maximum points of  $u - v$  on  $\bar{\Omega}$  such that

$$z_1 = z_0 + \xi_0, \quad z_n = z_{n-1} + \xi_{n-1}, \quad v(z_n) \geq Mv(z_n) \quad (n \in N),$$

where  $\xi_n \geq 0$  satisfies

$$\begin{aligned} \xi_n &\neq 0, \quad z_n + \xi_n \in \bar{\Omega}, \\ Mv(z_n) &= k(\xi_n) + v(z_n + \xi_n). \end{aligned}$$

Then we obtain

$$z_n \rightarrow \bar{z} \quad (n \rightarrow +\infty) \quad \text{for some } \bar{z} \in \bar{\Omega},$$

because  $\bar{\Omega}$  is compact and  $z_0 \leq z_1 \leq \cdots \leq z_n \leq \cdots$  by the definition of  $\{z_n\}_{n \in N}$ . ( $z_n \geq z_{n-1}$  means  $z_n - z_{n-1} \in (\mathbf{R}^+)^N$ .) Since the inequality (7.2) holds at  $z_n + \xi_n$  in place of  $z_0 + \xi_0$  and  $u - v \in USC(\bar{\Omega})$ , we have

$$u(z_0) - v(z_0) = \lim_{n \rightarrow +\infty} (u(z_n) - v(z_n)) = u(\bar{z}) - v(\bar{z}).$$

Thus it follows from the above equality and semicontinuity of  $u$  and  $v$  that  $v(z_n) \rightarrow v(\bar{z})$  as  $n \rightarrow +\infty$ . Using the definition of  $\{z_n\}_{n \in N}$  and (A.8), we conclude that

$$\begin{aligned} v(\bar{z}) &= \lim_{n \rightarrow +\infty} v(z_n) \geq \lim_{n \rightarrow +\infty} Mv(z_n) \\ &= \lim_{n \rightarrow +\infty} (k(\xi_n) + v(z_n + \xi_n)) \\ &\geq k_0 + v(\bar{z}), \end{aligned}$$

which contradicts the fact  $k_0 > 0$ . Therefore we can find an  $n_0 \in N$  such that  $v(z_{n_0}) - Mv(z_{n_0}) < 0$ . ■

Now, we can prove Theorem 7.4.

*Proof of Theorem 7.4.* We may assume  $u \in USC(\bar{\Omega})$  and  $v \in LSC(\bar{\Omega})$ . We suppose  $\sup_{\bar{\Omega}}(u - v) = \theta > 0$  and get a contradiction. Since  $u$  is a subsolution of (7.1), we get  $u \leq Mu$  on  $\bar{\Omega}$ . It is seen from Lemma 7.7 that there exists a maximum point  $z \in \bar{\Omega}$  of  $u - v$  satisfying  $v(z) - Mv(z) < 0$ . We divide our consideration into two cases.

Case 1.  $z \in \partial\Omega$ .

For simplicity we consider  $|\gamma(x)| = 1$ . Let  $\varphi \in C^2(\bar{\Omega})$  be a function such that

$$\varphi = 0 \text{ on } \partial\Omega, \quad \varphi > 0 \text{ in } \Omega, \quad \text{and } \langle D\varphi, \gamma \rangle \geq \eta_0 \text{ on } \partial\Omega.$$

for some  $\eta_0 > 0$ . (cf. M. G. Crandall–H. Ishii–P. L. Lions [8; Section 7].) For each  $\beta > 0$ , the function  $u(x) - v(x) - \beta(|x - z|^2 + 2\varphi(x))$  attains a strict maximum on  $\bar{\Omega}$  at  $z$ . Thus we may restrict this function on  $\bar{B}(z, \delta) \cap \bar{\Omega} (= W)$ . For any  $\alpha > 0$ , we define the function  $\Phi(x, y)$  on  $W \times W$  by

$$\Phi(x, y) = u(x) - v(y) - w_\alpha(x, y) - \beta(|x - z|^2 + \varphi(x) + \varphi(y)),$$

where  $w_\alpha$  is the function in Lemma 7.6. Let  $(\bar{x}, \bar{y}) \in W \times W$  be a maximum point of  $\Phi$ . By  $\Phi(z, z) \leq \Phi(\bar{x}, \bar{y})$  and (7.4) we get

$$\theta - \frac{1}{\alpha} \leq u(\bar{x}) - v(\bar{y}) - \frac{\alpha}{8}|x - y|^2.$$

Thus we have

$$|\bar{x} - \bar{y}| \rightarrow 0 \quad (\alpha \rightarrow +\infty).$$

As in the proof of Theorem 3.3, we obtain the behaviors of  $\bar{x}, \bar{y}, u(\bar{x}), v(\bar{y})$  as  $\alpha \rightarrow +\infty$ :

$$(7.4) \quad \bar{x}, \bar{y} \rightarrow z, \quad u(\bar{x}) \rightarrow u(z), \quad v(\bar{y}) \rightarrow v(z), \quad \alpha|\bar{x} - \bar{y}|^2 \rightarrow 0.$$

Moreover,  $\Phi(x, y) \leq \Phi(\bar{x}, \bar{y})$  on  $W$  implies, as  $(x, y) \rightarrow (\bar{x}, \bar{y})$ ,

$$\begin{aligned} u(x) - u(y) &\leq u(\bar{x}) - v(\bar{y}) - \beta|\bar{x} - z|^2 + \beta|x - z|^2 \\ &\quad - \beta(\varphi(\bar{x}) - \varphi(x)) - \beta(\varphi(\bar{y}) - \varphi(y)) - w_\alpha(\bar{x}, \bar{y}) - w_\alpha(x, y) \\ &\leq u(\bar{x}) - v(\bar{y}) + 2\beta\langle \bar{x} - z, x - \bar{x} \rangle + \beta\langle x - \bar{x}, x - \bar{x} \rangle \\ &\quad + \beta \left\{ \langle D_\varphi(\bar{x}), x - \bar{x} \rangle + \frac{1}{2} \langle D^2\varphi(\bar{x})(x - \bar{x}), x - \bar{x} \rangle \right\} \\ &\quad + \beta \left\{ \langle D\varphi(\bar{y}), y - \bar{y} \rangle + \frac{1}{2} \langle D^2\varphi(\bar{y})(y - \bar{y}), y - \bar{y} \rangle \right\} \end{aligned}$$

$$\begin{aligned}
& + \langle D_x w_\alpha(\bar{x}, \bar{y}), x - \bar{x} \rangle + \langle D_y w_\alpha(\bar{x}, \bar{y}), y - \bar{y} \rangle \\
& + \frac{1}{2} \alpha C_0 |(x - \bar{x}) - (y - \bar{y})|^2 \\
& + \delta_\alpha (|x - \bar{x}|^2 + |y - \bar{y}|^2) \\
& + o(|x - \bar{x}|^2 + |y - \bar{y}|^2).
\end{aligned}$$

Thus we conclude

$$\begin{aligned}
& \left( \begin{pmatrix} 2\beta(\bar{x} - z) + D_x w_\alpha(\bar{x}, \bar{y}) + \beta D\varphi(\bar{x}) \\ D_y w_\alpha(\bar{x}, \bar{y}) + \beta D\varphi(\bar{y}) \end{pmatrix}, \alpha C_0 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \right) \\
& + \delta_\alpha \begin{pmatrix} I & O \\ O & I \end{pmatrix} + \beta \begin{pmatrix} 2I + D^2\varphi(\bar{x}) & O \\ O & D^2\varphi(\bar{y}) \end{pmatrix} \in J^{2,+}(u(\bar{x}) - v(\bar{y})).
\end{aligned}$$

Therefore by the maximum principle, there exist  $X, Y \in \mathbb{S}^N$  such that

$$\begin{aligned}
& (2\beta(\bar{x} - z) + D_x w_\alpha(\bar{x}, \bar{y}) + \beta D\varphi(\bar{x}), X) \in \bar{J}^{2,+}u(\bar{x}), \\
& (-D_y w_\alpha(\bar{x}, \bar{y}) - \beta D\varphi(\bar{y}), Y) \in \bar{J}^{2,-}v(\bar{y}), \\
& -3\alpha C_0 \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X - \delta_\alpha I & O \\ O & -Y - \delta_\alpha I \end{pmatrix} \\
& \leq 3\alpha C_0 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \beta \begin{pmatrix} 2I + D^2\varphi(\bar{x}) & O \\ O & D^2\varphi(\bar{y}) \end{pmatrix}.
\end{aligned}$$

In the case  $\bar{x} \in \partial\Omega$ , we have

$$(7.5) \quad \langle 2\beta(\bar{x} - z) + D_x w_\alpha(\bar{x}, \bar{y}) + \beta D\varphi(\bar{x}), \gamma(\bar{x}) \rangle \geq 2\beta \langle \bar{x} - z, \gamma(\bar{x}) \rangle - \delta_\alpha + \beta\eta_0 > 0$$

for sufficiently large  $\alpha > 0$ . Similarly, in the case  $\bar{y} \in \partial\Omega$ , we get

$$(7.6) \quad \langle -D_y w_\alpha(\bar{x}, \bar{y}) - \beta D\varphi(\bar{y}), \gamma(\bar{y}) \rangle \leq \delta_\alpha - \beta\eta_0 < 0$$

for sufficiently large  $\alpha > 0$ . Moreover, since it is easily observed from (7.4),  $Mv \in LSC(\bar{\Omega})$  and Lemma 7.7 that

$$(7.7) \quad \limsup_{\alpha \rightarrow +\infty} (v(\bar{y}) - Mv(\bar{y})) \leq v(z) - Mv(z) < 0,$$

we obtain  $v(\bar{y}) - Mv(\bar{y}) < 0$  for large  $\alpha > 0$ . Hence using (7.5), (7.6), (7.7) and the fact that  $u$  and  $v$  are, respectively, a subsolution and a supersolution of (7.1), we obtain the following inequalities:

$$F(\bar{x}, u(\bar{x}), 2\beta(\bar{x} - z) + D_x w_\alpha(\bar{x}, \bar{y}) + \beta D\varphi(\bar{x}), X) \leq 0,$$

$$F(\bar{y}, v(\bar{y}), -D_y w_\alpha(\bar{x}, \bar{y}) - \beta D\varphi(\bar{y}), Y) \geq 0.$$

By (A.5), (A.6)', (A.7), (7.2) and Lemma 7.6 we observe

$$\begin{aligned}
 \lambda\theta &\leq \lambda(u(\bar{x}) - v(\bar{y})) \\
 &\leq F(\bar{y}, u(\bar{x}), -D_y w_\varepsilon(\bar{x}, \bar{y}) - \beta D\varphi(\bar{y}), Y) \\
 &\quad - F(\bar{x}, u(\bar{x}), 2\beta(\bar{x} - z) + D_x w_\varepsilon(\bar{x}, \bar{y}) + \beta D\varphi(\bar{x}), X) \\
 &\leq F(\bar{y}, u(\bar{x}), -D_y w_\alpha(\bar{x}, \bar{y}), Y + \delta_\alpha I + \beta D^2\varphi(\bar{y})) \\
 &\quad - F(\bar{x}, u(\bar{x}), -D_y w_\alpha(\bar{x}, \bar{y}), X - \delta_\alpha I - \beta(2I + D^2\varphi(\bar{x}))) \\
 &\quad + \omega_3(\delta_\alpha + \beta(|D\varphi| + \|D^2\varphi\|)) \\
 &\quad + \omega_3(2\delta_\alpha + \beta(2 + 2|\bar{x} - z| + |D\varphi| + \|D^2\varphi\|)) \\
 &\leq \omega_1(C_0|\bar{x} - \bar{y}|(\alpha|\bar{x} - \bar{y}| + 1) + \alpha C_0|\bar{x} - \bar{y}|^2) \\
 &\quad + 2\omega_3(2\delta_\alpha + \beta(2 + 2|\bar{x} - z| + |D\varphi| + \|D^2\varphi\|)).
 \end{aligned}$$

Letting  $\alpha \rightarrow +\infty$  and then  $\beta \rightarrow 0$ , we obtain a contradiction.

Case 2.  $z \in \Omega$

We define the function  $\Phi(x, y)$  on  $\bar{\Omega} \times \bar{\Omega}$  by

$$\Phi(x, y) = u(x) - v(y) - \frac{\alpha}{2}|x - y|^2 - |x - z|^4.$$

By the same calculation as in Case 1 with this function we also get a contradiction.

Thus we have completed the proof. ■

*Proof of Theorem 7.5.* Let  $C = \sup_{\bar{\Omega}} |F(x, 0, 0, 0)|$ . Then it is easily verified that  $\underline{u}(x) \equiv -C$  and  $\bar{u}(x) \equiv C$  are, respectively, a subsolution and a supersolution of (7.1). Thus by Perron's method and Theorem 7.4 we can show the existence of a unique solution  $u$  of (7.1) and  $u \in C(\bar{\Omega})$ . ■

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