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Abstract Second Order Differential Equations with Applications

By

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Dedicated to Professor Hiroki Tanabe on the occasion of his 60th birthday

0. Introduction

Of concern in this paper is the second order differential equation in Banach space

$$(\mathscr{E}) \qquad \frac{d}{dt}(Cu') + Bu' + Au = f,$$

where u' denotes the derivative du/dt with respect to t and invertibility of the closed linear operator C is not assumed. Here A, B, C are closed linear operators from the complex Banach space Y into the Banach space X.

There is a large literature on the subject and we refer to Carroll-Showalter [3], Showalter [13] both for a wide bibliography and for various methods of approaching $(\mathscr{E})$.

Usually $(\mathscr{E})$ is transformed into a first order equation in a suitable product space and then the general results on evolution problems are applied.

Much attention has been reserved in particular to the case where the derived problem is parabolic. In this context the most immediate statements on solvability to $(\mathscr{E})$ are obtained when the domain $D(B)$ of the operator B is contained in the domain of A. On the other hand, if C is the identity operator, $-B$, $-AB^{-1}$ generate analytic semigroups, and AB^{-1} is large with respect to B in some sense, then it is shown in Krein [10], Yakubov [15] that a new system obtained by the previous one applying a certain operator-matrix to it, is parabolic. We notice that some important applications of this trick are described in the book by Rodman [12].

In this paper we shall study $(\mathscr{E})$ correspondingly and to this end we shall use results and methods of our works Favini-Yagi [6, 7]. In the first paper [6] we extended the theory of infinitely differentiable semigroups of linear operators to multivalued operators in order to handle first order equations of the type

$$(*) \qquad \frac{dv}{dt} + Av \ni f,$$

whereas more classical tools, related to real interpolation spaces and the operational methods of Da Prato-Grisvard [4], were employed in [7] to obtain regular (in time) solutions to

$$(**) \qquad \frac{d}{dt}(Mu) + Lu = f.$$

The three sections of the paper are subdivided as follows.

Paragraph 1 contains a summary of the main definitions and results of [6, 7] on $(*)$, $(**)$, with slight changes, in order to adapt them more directly to $(\mathscr{E})$.

The second section concerns $(\mathscr{E})$ in the case $D(B) \subseteq D(A)$. Then $(\mathscr{E})$ is easily reduced to a problem of type $(**)$ in the space $D(B) \times X$.

Three examples of applications are given, even if many other could be added. In particular, in Example 3 we show that our theory permits to treat some very interesting cases of second order euations in Hilbert space.

Paragraph 3 studies the more complicated case $D(A) \subset D(B)$. At first glance it could seem a hard task to deal with this problem by the methods in section 2. In order to single out the type of hypotheses to be made on the operators involved, we first generalize the approach of Krein [10] and Yakubov [15], transforming $(\mathscr{E})$ into a certain equation $(*)$ in $X \times X$ to which the results of Favini-Yagi [6] are applicable, provided that one strict relation between the operators A, B and C holds. But then we are able to show that under the indicated assumptions the first order system to which $(\mathscr{E})$ was reduced when $D(B) \subseteq D(A)$ is directly solvable in the same space $D(B) \times X$! We illustrate how the relative abstract statements work by two examples of partial differential equations.

Some of the main results of this paper were announced in Favini-Yagi [8].

Notation. By $\| \cdot ; E \|$ we shall denote the norm in the Banach space E.

If E, F are Banach spaces, $L(E, F)$ denotes the space of bounded linear operators from E into F, with the sup norm. If $E = F$, we use $L(E)$ for $L(E, E)$.

If S is a closed linear operator in E, $D(S)$ and $R(S)$ denote, respectively, the domain of S and the range of S.

If $0 < T < \infty$, $C([0, T]; E)$ is the space of all E-valued strongly continuous functions on $[0, T]$, and for $\sigma \in (0, 1)$, $C^{\sigma}([0, T]; E)$ denotes the space of Hölder-continuous functions from $[0, T]$ into E with exponent σ.

$C^1([0, T]; E)$ is the space of all continuously differentiable E-valued functions on $[0, T]$. We shall also use L^p spaces and Sobolev spaces for which we refer to Triebel [14].

1. Preliminaries

Let A be a multivalued linear operator in the complex Banach space X. Given a continuous function f: $[0, T] \to X$ and $v_0 \in D(A)$, we seek a function v from $[0, T]$ into X satisfying

$$\text{(P)} \qquad \begin{cases} v'(t) + Av(t) \ni f(t), & 0 < t \le T, \\ v(0) = v_0. \end{cases}$$

under the assumptions

$$\text{(a)} \qquad \begin{aligned} \rho(A) &= \{z \in \boldsymbol{C}: \exists (z + A)^{-1} \in L(X)\} \\ &\supseteq \textstyle\sum_\eta = \{z: \operatorname{Re} z \ge -c(1 + |\operatorname{Im} z|)^\eta\}, \ c > 0, \end{aligned}$$

and

$$\text{(b)} \qquad \|(z + A)^{-1}; L(X)\| \le M(1 + |z|)^{-\beta}, \ z \in \textstyle\sum_\eta,$$

where

$$0 < \beta \le \eta \le 1.$$

The sense to be given to a solution of (P) is as follows:

Definition 1. A function v: $[0, T] \to X$ is a classical solution to (P) if $v \in C^1((0, T]; X)$, $v(t) \in \mathscr{D}(A)$, $0 < t \le T$, and (P) holds.

We have (Favini-Yagi [6]):

Proposition 1. *Assume* (a)-(b), *and* $2\eta + \beta > 2$. *Let*

$$(2 - \eta - \beta)/\eta < \sigma \le 1, \qquad v_0 \in D(A).$$

Then for any $f \in C^\sigma([0, T]; X)$, *Problem* (P) *has a unique classical solution* v, *with*

$$v(t) = \exp(-tA)v_0 + \int_0^t \exp(-(t - s)A)f(s)ds, \qquad 0 < t \le T.$$

We remark that if (b) is true in the halfplane Re $z \ge 0$, then $(z + A)^{-1}$ can be continued to a region $\sum_\eta$ with $\eta = \beta$ and a similar estimate.

As an application of Proposition 1 we can handle some degenerate Cauchy problems. Precisely, let L and M be two densely defined univalent closed linear operators in X such that $D(L) \subseteq D(M)$ and $L^{-1} \in L(X)$. Let $f \in C([0, T]; X)$, $u_0 \in D(L)$; then u: $(0, T] \to X$ is the unknown function of the problem

$$\text{(E)} \qquad \begin{cases} \dfrac{d}{dt}(Mu(t)) + Lu(t) = f(t), & 0 < t \le T, \\ \\ Mu(0) = Mu_0, \end{cases}$$

according to

Definition 2. The function u is a classical solution of (E) if $u(t) \in D(L)$ for all $t \in (0, T]$, $Mu \in C^1((0, T]; X)$, $Lu \in C((0, T]; X)$ and (E) is verified, where the initial condition signifies $\lim_{t \to 0+} \| Mu(t) - Mu_0; X \| = 0$.

If we put $Mu = v$, then (E) is reduced to (P) with the (multivalued) operator $A = LM^{-1}$ and $v_0 = Mu_0 \in D(A)$, since $u_0 \in D(L)$.

Let us formulate the assumptions corresponding to (a), (b) in this case.

(a)′ $\rho(M, L) = \{z \in C; zM + L \text{ has a bounded inverse}\} \supseteq \sum_\eta$

(b)′ $\| M(zM + L)^{-1}; L(X) \| \leq C(1 + |z|)^{-\beta}$, $z \in \sum_\eta$, $0 < \beta \leq \eta \leq 1$.

Clearly, (b)′ is equivalent to the estimate

(b)″ $\| L(zM + L)^{-1}; L(X) \| \leq C(1 + |z|)^{1-\beta}$, $z \in \sum_\eta$.

One then has

Proposition 2. *Under* (a)′, (b)′, *with* $2\eta + \beta > 2$, *for any* $f \in C^\sigma([0, T]; X)$, $(2 - \eta - \beta)/\eta < \sigma \leq 1$, *and each* $u_0 \in D(L)$ *there is a unique classical solution to Problem* (E).

The case $\eta = 1$ is obviously the best one when more regularity of the solution to (E) is desiderated. To this regard, we introduce a suitable definition.

Definition 3. A function $u: [0, T] \to X$ is a strict solution to (E) if $u(t) \in D(L)$ for $0 \leq t \leq T$, $Lu \in C([0, T]; X)$, $Mu \in C^1([0, T]; X)$, the equation in (E) is satisfied on the closed interval $[0, T]$ and the initial condition $Mu(0) = Mu_0$ holds.

We then have the following result (Favini-Yagi [7, Theorem 9]).

Proposition 3. *Assume* (a)′, (b)′, *with* $\eta = 1$. *Given* $\sigma \in (1 - \beta, 1)$, *then for all* $f \in C^\sigma([0, T]; X)$ *and* $u_0 \in D(L)$ *satisfying the compatibility relation* $f(0) - Lu_0 \in R(ML^{-1})$, *there is a unique strict solution* u *to* (E) *for which* Lu, $\frac{d}{dt} Mu(\cdot) \in C^{\sigma+\beta-1}([0, T]; X)$.

A further refinement of the preceding Proposition could be obtained when $\beta = \eta = 1$, using real interpolation spaces. Notice that our conclusions remain true also when L and M act between two different Banach spaces.

We also observe that under (b)″ the condition $D(L) \subseteq D(M)$ is not restrictive. In fact, the change of variable $u(t) = e^{kt} v(t)$, $k > 0$, leads to a problem of type (E) with the pair $(M, kM + L)$ instead of (M, L) and nevertheless $(a)'$, $(b)'$ hold for the new operator pencil.

2. The equation ($\mathscr{E}$) in the case $D(B) \subseteq D(A)$

We shall study existence and uniqueness of the solution (in some senses) to the initial value problem

$$\text{(Q)} \qquad \begin{cases} \dfrac{d}{dt}(Cu') + Bu' + Au = f = f(t), & 0 < t \leq T, \\ u(0) = u_0, \\ Cu'(0) = Cu_1, \end{cases}$$

where A, B, C are closed linear operators from the complex Banach space Y into another Banach space X, with $D(B) \subseteq D(A)$. Without loss of generality, we shall always suppose that B has a bounded inverse. Moreover, since the change of variable $u(t) = e^{kt}v(t)$, $k > 0$, transforms the equation of (Q) into

$$\frac{d}{dt}(Cv') + (B + 2kC)v' + (A + kB + k^2C)v = e^{-kt}f(t),$$

the assumption $D(B) \subseteq D(C)$, that we shall always make, is not restrictive. Putting $u' = v$, (Q) transforms into the problem

$$\begin{cases} \dfrac{d}{dt}(Mz(t)) + Lz(t) = F(t), & 0 < t \leq T, \\ Mz(0) = Mz_0, \end{cases}$$

where $z(t) = (u(t), v(t))$, $z_0 = (u_0, u_1)$, $F(t) = (0, f(t))$, $0 < t \leq T$, and

$$M = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}, \qquad L = \begin{bmatrix} 0 & -I \\ A & B \end{bmatrix}.$$

We shall choose a good product space in which all our previous results can be applied. In fact, under the actual assumptions the space $D(B) \times X$ works, where of course $D(B)$ is endowed with the graph norm $\|x; D(B)\| = \|Bx; X\|$.

In order to solve $(zM + L)(u, v) = f = (f_1, f_2)$, $f_1 \in D(B)$, $f_2 \in X$, and then estimate the norm of $M(zM + L)^{-1}$, we have to solve

$$\begin{cases} v = zu - f_1, \\ Au + z(zC + B)u = (zC + B)f_1 + f_2. \end{cases}$$

Let us introduce

$$P(z) = z^2C + zB + A$$

and assume that $P(z)$ has a bounded inverse for all $z \in \sum_\eta$. Then, formally, the preceding system has the unique solution

$$\begin{cases} u = P(z)^{-1}(zC + B)f_1 + P(z)^{-1}f_2, \\ v = -P(z)^{-1}Af_1 + zP(z)^{-1}f_2. \end{cases}$$

Hence

$$(zM + L)^{-1} = [A_{ij}(z)], \qquad i, j = 1, 2,$$

and

$$M(zM + L)^{-1} = [B_{ij}(z)], \qquad i, j = 1, 2,$$

where

$$\begin{aligned} &A_{11}(z) = z^{-1} - z^{-1}P(z)^{-1}, && A_{12}(z) = P(z)^{-1}, \\ &A_{21}(z) = -P(z)^{-1}A, && A_{22}(z) = zP(z)^{-1}, \\ &B_{11}(z) = z^{-1} - z^{-1}P(z)^{-1}A, && B_{12}(z) = P(z)^{-1}, \\ &B_{21}(z) = -CP(z)^{-1}A, && B_{22}(z) = zCP(z)^{-1}. \end{aligned}$$

We are then reduced to give some conditions on the operators implying that $P(z)^{-1} \in L(X)$ with suitable growth bounds for its norm.

To this end, we suppose that B, C fulfill

$$\text{(H)} \qquad \begin{aligned} &zC + B \text{ has a bounded inverse for all } z \in \textstyle\sum_\eta \text{ and} \\ &\|C(zC + B)^{-1}; L(X)\| \le k(1 + |z|)^{-\beta}, \ z \in \textstyle\sum_\eta, \\ &\text{where } 0 < \beta \le \eta \le 1. \end{aligned}$$

Then, from

$$P(z) = (z + AB^{-1})(zC + B) - zAB^{-1}C,$$

we infer that for all $z \in \sum_\eta$, $|z|$ sufficiently large, (notice that AB^{-1} is bounded),

$$P(z) = (z + AB^{-1})Q(z)(zC + B),$$

where

$$Q(z) = I - z(z + AB^{-1})^{-1}AB^{-1}C(zC + B)^{-1}.$$

On the other hand, for these z's,

$$\|z(z + AB^{-1})^{-1}AB^{-1}C(zC + B)^{-1}; L(X)\| \le m(1 + |z|)^{-\beta}.$$

Hence, for such complex numbers z the inverse $P(z)^{-1}$ exists and belongs to $L(X)$. Furthermore,

$$\|BP(z)^{-1}; L(X)\| \le m_1(1 + |z|)^{-\beta},$$

$$\|CP(z)^{-1}; L(X)\| \le m_2(1 + |z|)^{-\beta-1}.$$

This implies that

$$\begin{aligned}&\|z^{-1}B(f_1 - P(z)^{-1}Af_1) + BP(z)^{-1}f_2; X\|\\ &\leq |z|^{-1}\|f_1; D(B)\| + m_3(1+|z|)^{-\beta-1}\|f_1; D(B)\| + m_1(1+|z|)^{-\beta}\|f_2; X\|\\ &\leq m_4(1+|z|)^{-\beta}\|(f_1, f_2); D(B)\times X\|.\end{aligned}$$

Analogously, one easily verifies that

$$\|-CP(z)^{-1}Af_1 + zCP(z)^{-1}f_2; X\| \leq m_5(1+|z|)^{-\beta}\|(f_1, f_2); D(B)\times X\|.$$

This is not only formal but ensured by (H). Therefore the operator pencil $zM + L$ has just the properties required in Proposition 2. Further,

$$\|M(zM+L)^{-1}; L(D(B)\times X)\| \leq m(1+|z|)^{-\beta}, \qquad z\in\textstyle\sum_\eta.$$

Before establishing some consequences of this result we must define precisely what we mean by a solution to (Q). To this end, we introduce:

Definition 4. A function $u\colon (0, T] \to Y$ is a classical solution of (Q) if $u(t)\in D(A)$, $0 < t \leq T$, $u\in C^1((0, T]; Y)$, $u'(t)\in D(B)$, $Cu'(\cdot)\in C^1((0, T]; X)$, $Bu'(\cdot)\in C((0, T]; X)$, and

$$\text{(1)}\qquad \frac{d}{dt}(Cu'(t)) + Bu'(t) + Au(t) = f(t),\ 0 < t \leq T,$$

$$\text{(2)}\qquad \|u(t) - u_0; Y\| \to 0 \text{ as } t \to 0+,$$

$$\text{(3)}\qquad \|Cu'(t) - Cu_1; X\| \to 0 \text{ as } t \to 0+,$$

where $f\in C([0, T]; X)$ and $u_0\in Y$, $u_1\in D(C)$.

Definition 5. A function u: $[0, T] \to Y$ is said to be a strict solution to (Q) if $u(t)\in D(A)$, $0 \leq t \leq T$, $u\in C^1([0, T]; Y)$, $u'(t)\in D(B)$, $Cu'(\cdot)\in C^1([0, T]; X)$, $Bu'(\cdot)\in C([0, T]; X)$, and

$$\text{(4)}\qquad \frac{d}{dt}(Cu'(t)) + Bu'(t) + Au(t) = f(t),\ 0 < t \leq T,$$

$$\text{(5)}\qquad u(0) = u_0,$$

$$\text{(6)}\qquad Cu'(0) = Cu_1,$$

f being a continuous function from $[0, T]$ into X.

We have thus proven

Theorem 1. *Assume* (H) *and* $D(B)\subseteq D(A)$. *If* $u_0, u_1\in D(B)$ *and* $f\in C^\sigma([0, T]; X)$, *with* $2\eta + \beta > 2$, $(2-\eta-\beta)/\eta < \sigma \leq 1$, *then Problem* (1)–(3) *has a unique classical solution.*

Notice that if $\eta = 1$, then β can move in the whole interval (0, 1).

Proof. It is an easy consequence of Proposition 2. #

Theorem 2. *Let us assume* (H) *with* $\eta = 1$, *and let* $\sigma \in (1 - \beta, 1)$. *If* $f \in C^{\sigma}([0, T]; X)$, *and* $u_0, u_1 \in D(B)$ *satisfy the compatibility relation*

$$f(0) - Au_0 - Bu_1 \in C(D(B)),$$

then Problem (4)–(6) *has a unique strict solution u such that*

$$Bu', (Cu')' \in C^{\sigma+\beta-1}([0, T]; X).$$

Proof. It suffices to observe that Proposition 3 applies provided that

$$(0, f(0)) - (-u_1, Au_0 + Bu_1) = (x, Cy),$$

for suitable $x, y \in D(B)$. Hence the conclusion is obvious. #

Example 1. Consider the initial-boundary-value problem

$$(7)\qquad \begin{cases} \dfrac{\partial}{\partial t}\left(m(x)\dfrac{\partial u}{\partial t}\right) - \Delta\dfrac{\partial u}{\partial t} + A(x, D)u = f(t, x), \text{ in } (0, T] \times \Omega, \\[2ex] u = \dfrac{\partial u}{\partial t} = 0, \text{ in } (0, T] \times \partial\Omega, \\[2ex] u(0, x) = u_0(x), \ x \in \Omega, \\[2ex] m(x)\dfrac{\partial u}{\partial t}(t, x) \to m(x)u_1(x) \text{ for } t \to 0+, \ x \in \Omega. \end{cases}$$

Here Ω is a bounded open set in $\boldsymbol{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$, $m(\cdot) \in L^{\infty}(\Omega)$, $m(x) \geq 0$ for all $x \in \Omega$, $A(x, D) = \sum_{|\alpha| \leq 2} a_{\alpha}(x)D^{\alpha}$, a_{α} is a scalar-valued function that is continuous on $\bar{\Omega}$, u_0, u_1 are two given functions, $f(t, x)$ is continuous on $[0, T] \times \bar{\Omega}$.

Such a problem can be easily generalized to embrace more general operator-coefficients. It has been treated very much in the literature and we refer to Carroll-Showalter [3]. Usually, one works in the space of distributions $H^{-1}(\Omega)$, but we showed in Favini-Yagi [6] that also the space $L^2(\Omega)$ is allowed, provided that less restrictive growth conditions for the modified resolvent are permitted.

In fact, if A denotes $A(x, D)$ with Dirichlet boundary conditions in $H^{-1}(\Omega)$, (that is $D(A) = H_0^1(\Omega)$ and A acts from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$), $B = -\Delta$ with $D(B) = D(A)$ and C is the bounded linear operator from $H_0^1(\Omega)$ into $L^2(\Omega)$ of multiplication by $m(x)$, it is proven in [6] that $zC + B$ has a bounded inverse in $H^{-1}(\Omega)$ for all z in a sector of the type Σ and

$$\|C(zC+B)^{-1}; L(H^{-1}(\Omega))\| \leq m|z|^{-1}, \ z\in\Sigma, \ |z| \text{ large},$$

holds. Then Theorem 1 applies with $\beta=\eta=1$.

On the other hand, if $D(B)=H_0^1(\Omega)\cap H^2(\Omega)$, $B=-\Delta$ operates in the Hilbert space $X=L^2(\Omega)$ and C is multiplication by $m(x)$, then

$$\|C(zC+B)^{-1}; L(X)\| \leq m|z|^{-1/2}, \ z\in\Sigma, \ |z| \text{ large}.$$

This time A is viewed as a bounded operator from $H^2(\Omega)$ into X and Theorems 1 and 2 hold with $\sigma\in(1/2, 1)$. In particular, we conclude that if

$$f(0,\cdot)+\Delta u_1-A(\cdot, D)u_0=m(\cdot)w(\cdot)$$

with $w\in H_0^1(\Omega)\cap H^2(\Omega)$, $u_0, u_1\in D(B)$, $f\in C^\sigma([0, T]; L^2(\Omega))$, then problem (7) has a unique strict solution u with the regularity $\Delta\dfrac{\partial u}{\partial t}(\cdot, x)\in C^{\sigma-1/2}([0, T];$ $L^2(\Omega))$ and also $\dfrac{\partial}{\partial t}\left(m(x)\dfrac{\partial u}{\partial t}(\cdot, x)\right)\in C^{\sigma-1/2}([0, T]; L^2(\Omega))$.

Example 2. For sake of simplicity, we confine our discussion to the interval $I=(0, 1)\subset \boldsymbol{R}$, taking into consideration the problem

$$\begin{cases} \dfrac{\partial}{\partial t}\left(m(x)\dfrac{\partial u}{\partial t}\right)-\dfrac{\partial^3 u}{\partial x^2\partial t}+C_0\dfrac{\partial u}{\partial t}+\dfrac{\partial^2 u}{\partial x^2}=f(t, x), \ 0<t\leq T, \ x\in(0, 1), \\ u(t, 0)=u(t, 1)=\dfrac{\partial u}{\partial t}(t, 0)=\dfrac{\partial u}{\partial t}(t, 1)=0, \ t\in(0, T], \\ \lim\limits_{t\to 0+} u(t, x)=u_0(x), \ 0<x<1, \\ \lim\limits_{t\to 0+} m(x)\dfrac{\partial u}{\partial t}(t, z)=m(x)u_1(x), \ 0<x<1. \end{cases}$$

This time we take $X=L^p(0, 1)$, $1<p<\infty$, so that $D(B)=W^{2,p}(0, 1)\cap W_0^{1,p}(0, 1)$, $(Bu)(x)=-u''(x)+C_0u(x)$, $u\in D(B)$, $C_0>0$, $m(x)\geq 0$, $m(\cdot)\in L^\infty(\Omega)$, C is multiplication by $m(\cdot)$ in $L^p(0, 1)$.

By Favini-Yagi [7, Example 5], $zC+B$ has a bounded inverse for all z in a sector Σ containing $\operatorname{Re} z\geq 0$ and

$$\|C(zC+B)^{-1}; L(X)\| \leq m|z|^{-1/p}, \ z\in\Sigma, \ |z| \text{ large}.$$

Therefore the conclusions corresponding to Theorem 2 are deduced with $\beta=1/p$.

Example 3. This is an abstract version of the model equation considered in the first part of Example 1.

Let V, H be two complex Hilbert spaces such that $V \subset H$ densely and continuously.

We denote by $\| \cdot \|$, $| \cdot |$ and $\| \cdot \|_*$ the norms in V, H and V', respectively. Further, $(\cdot, \cdot)$ denotes the inner product in H and $\langle \cdot, \cdot \rangle$ is the pairing between V and V', such that $\langle u, v \rangle = (u, v)$ if $u, v \in V$.

Assume

(8) *A, B are two bounded linear operators from V to V' such that* $\operatorname{Re} \langle Bu, u \rangle \geq$ $\geq a_0 \|u\|^2$, $u \in V$, *where* $a_0 > 0$.

(9) *C is a self-adjoint non negative bounded operator from H into itself.*

For example, A, C could be the operators defined by two sesquilinear forms in V and H, respectively, and B a positive (of parabolic type) operator $\in L(V, V')$.

It is known (see Barbu-Favini [2]) that under (8), (9) the estimate

$$\| B(zC + B)^{-1}; L(V') \| \leq \text{Const.}$$

is verified for all complex numbers z in a sector Σ containing $\operatorname{Re} z \geq 0$. Therefore, hypotheses (8), (9) permit to handle problem (Q) in this interesting case.

Now we want to describe in more details what happens when C has also a bounded inverse in the space H.

Then, using the preceding notation, the operator LM^{-1} is given by

$$LM^{-1} = \begin{bmatrix} 0 & -C^{-1} \\ A & BC^{-1} \end{bmatrix}$$

and its opposite generates a holomorphic semigroup in the space $V \times V'$, with domain $D(LM^{-1}) = V \times D(BC^{-1})$, (that is not necessarily dense in $V \times V'$). To see this, it is enough to observe that $M(zM + L)^{-1}$ coincides with $(z + LM^{-1})^{-1}$ and hence the required estimate for the norm of the resolvent is ensured.

We are now in a position to apply to the problem

$$(10) \qquad \begin{cases} y'(t) + LM^{-1} y(t) = (0, f(t)), & 0 < t \leq T, \\[2ex] y(0) = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \end{cases}$$

what has been established in a large literature on existence and regularity of the solutions to parabolic Cauchy problems. We refer, for example, to

Angement [1].

We then deduce the following two results to be compared to those of hyperbolic type in Lagnese [11, p. 31], where classical solutions are established by semigroup theory instead.

Proposition 4. *Let us assume* (8), (9) *and* C *have a bounded inverse, with* V *invariant under* C *and* C^{-1}.
Then for any $u_0 \in V$, $u_1 \in H$ *and* $f \in L^2(0, T; V')$, *Problem* (10) *has a unique* L^2*-solution* u *such that*

(1) *holds a.e. on* $(0, T)$,
$u, u' \in L^2(0, T; V)$, $(Cu')' \in L^2(0, T; V')$, $Au + Bu' \in L^2(0, T; V')$,
$u(t) \to u_0$ *in* V, $Cu'(t) \to Cu_1$ *in* V' *as* $t \to 0+$.

Proof. It suffices to apply Da Prato-Grisvard [4, p. 338] and observe that the hypotheses on C guarantee $D(BC^{-1}) = D(B)$. #

Proposition 5. *Under the same assumptions as in Proposition* 4, *if* $u_0, u_1 \in V$, $Au_0 + Bu_1 \in H$ *and* $f \in L^2(0, T; H)$, *then Problem* (10) *has a unique* L^2*-solution* u *such that*

(1) *holds a.e. on* $(0, T)$,
$u, u' \in L^2(0, T; V)$, $(Cu')' \in L^2(0, T; H)$, $Au + Bu' \in L^2(0, T; H)$,
$u(t) \to u_0$ *in* V, $Cu'(t) \to Cu_1$ *in* H *as* $t \to 0+$.

Proof. We transform problem (10) into the problem

$$\begin{cases} w'(t) + LM^{-1}w(t) = -LM^{-1}M\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, & 0 < t < T, \\ w(0) = 0, \end{cases}$$

and apply Da Prato-Grisvard [4, p. 337]. In this case, their result reads

$$(0, f(t)) - (-u_1, Au_0 + Bu_1) \in L^2(0, T; [V \times V', V \times V]_{1/2}) = L^2(0, T; V \times H)$$

as a condition entailing the enunciated regularity. #

3. The case $D(A)$ strictly contained into $D(B)$

As it was outlined in the introduction, in order to handle problem (Q) when $D(A)$ is smaller than $D(B)$, first we shall suitably modify the device used for by Krein [10] and Yakubov [15], relative to regular problems.

To this purpose, the first step consists in noting that system (Q) can be transformed into the equivalent problem

$$\text{(R)}\quad \begin{cases} \dfrac{d}{dt}\begin{bmatrix} B & C \\ 0 & C \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = -\begin{bmatrix} A & 0 \\ A & B \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} f(t) \\ f(t) \end{bmatrix}, & 0 < t \leq T, \\ \begin{bmatrix} B & C \\ 0 & C \end{bmatrix}\begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & C \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \end{cases}$$

where the operator-matrices act from $Y \times Y$ into $X \times X$.

Indeed, if $Bu' = Bv$, then $(Bu)' = Bv$ and thus

$$(Bu + Cv)' = Bv - Au - Bv + f = -Au + f.$$

Conversely, if (R) holds, then $(Bu)' = (Bu + Cv - Cv)' = (Bu + Cv)' - (Cv)' = -Au + f - (-Au - Bv + f) = Bv$.

Since B has a bounded inverse, as we always assume, this says that (R) is satisfied. We then want to apply the general Proposition 1 to problem (R) and hence whereas the operator C is univalent, C^{-1} is to be viewed as a multivalued linear operator.

If we introduce, for sake of brevity, $AB^{-1} = U$, $BC^{-1} = V$, we have

$$\begin{bmatrix} A & 0 \\ A & B \end{bmatrix}\begin{bmatrix} B^{-1} & -B^{-1} \\ 0 & C^{-1} \end{bmatrix} = \begin{bmatrix} U & -U \\ U & -U+V \end{bmatrix}.$$

Therefore we must study the resolvent

$$\begin{bmatrix} U+z & -U \\ U & -U+V+z \end{bmatrix}^{-1}, \quad \text{where } z \in \Sigma_\eta.$$

Now, if

$$\mathcal{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} 0 & U \\ 0 & U \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} U & 0 \\ U & V \end{bmatrix},$$

such a resolvent coincides with

$$(11)\qquad (\mathcal{V} + z)^{-1}(\mathcal{I} - \mathcal{U}(\mathcal{V} + z)^{-1})^{-1},$$

provided that the inverses exist as univalent operators.

Consider then the system

$$\begin{cases} (U + z)x = f \\ Ux + (V + z)y = g \end{cases}$$

where $z \in \Sigma_\eta$, $f, g \in X$.

Formally its solution (x, y) is given by

$$x = (U + z)^{-1}f, \; y = -(V + z)^{-1}U(U + z)^{-1}f + (V + z)^{-1}g,$$

so that

$$\mathscr{U}(\mathscr{V}+z)^{-1}=\begin{bmatrix} -U(V+z)^{-1}U(U+z)^{-1} & U(V+z)^{-1} \\ -U(V+z)^{-1}U(U+z)^{-1} & U(V+z)^{-1} \end{bmatrix}.$$

The following assumptions seem then suitable in order to obtain the required properties of the resolvent.

(a)″ $\rho(C, B)$ and $\rho(AB^{-1})$ contain Σ_η and $\Sigma_{\eta'}$, respectively,

(b)″ $\| C(zC+B)^{-1}; L(X)\| \le k(1+|z|)^{-\beta}$, $z \in \Sigma_\eta$,

(c)″ $\| B(zB+A)^{-1}; L(X)\| \le k_1(1+|z|)^{-\alpha}$, $z \in \Sigma_{\eta'}$,

(d)″ $D(V) \subseteq D(U)$ and there is $\delta > 0$ such that
$\| U(V+z)^{-1}; L(X)\| \le k(1+|z|)^{-\delta}$, $z \in \Sigma_\eta$, $|z|$ large.

Here the constants α, β, η, η' satisfy $0 \le \alpha, \beta \le \eta \le \eta' \le 1$. We observe that for the applications we have in mind the condition $\eta \le \eta'$ is not restritive. Moreover, (a)″-(d)″ hold when $D(U) = X$, the situation considered in the preceding section.

If $x, y \in X$, $z \in \Sigma_\eta$, $|z|$ large, since

$$\begin{aligned} &U(U+z)^{-1} = I - zB(zB+A)^{-1}, \\ &\| U(V+z)^{-1}U(U+z)^{-1}x - U(V+z)^{-1}y; X\| \\ &\qquad \le k_2|z|^{-\delta-\alpha+1}\|x; X\| + k_3|z|^{-\delta}\|y; X\|, \end{aligned}$$

so that $\alpha + \delta > 1$ implies that $\|\mathscr{U}(\mathscr{V}+z)^{-1}; L(X \times X)\| \to 0$ as $|z| \to \infty$.

In view of (11), an estimate for the norm of the inverse of

$$z\mathscr{I} + \begin{bmatrix} A & 0 \\ A & B \end{bmatrix}\begin{bmatrix} B^{-1} & -B^{-1} \\ 0 & C^{-1} \end{bmatrix}$$

in $L(X \times X)$ is obtained by a bound for $\|(\mathscr{V}+z)^{-1}; L(X \times X)\|$.

Now, our hypotheses (a)″-(d)″ guarantee that

$$\|(V+z)^{-1}U(U+z)^{-1}; L(X)\| \le k(1+|z|)^{1-(\alpha+\beta)}, \qquad z \in \Sigma_\eta.$$

Therefore, we are allowed to apply Proposition 1. However, on the ground of the preceding discussion, we can furnish a direct proof of the result as follows.

Theorem 3. *Assume* (a)″-(d)″, *with* $0 < \alpha, \beta \le \eta \le \eta' \le 1$, $\alpha + \beta$, $\alpha + \delta > 1$, $2\eta + \alpha + \beta > 3$. *Let* $(3 - \eta - \alpha - \beta)/\eta < \sigma \le 1$. *Then for any* $u_0 \in D(A)$, $u_1 \in D(B)$ *and* $f \in C^\sigma([0, T]; X)$, *Problem* (1)-(3) *has a unique classical solution.*

Proof. Let us first note that

$$\begin{aligned}P(z) &= z^2C + zB + A = (z + A(zC + B)^{-1})(zC + B)\\ &= (z + UB(zC + B)^{-1})(zC + B) = (z + U[I - zC(zC + B)^{-1}])(zC + B).\end{aligned}$$

Here we observe that $(zC + B)(D(A)) \subseteq D(U)$, since $B(D(A)) \subseteq D(AB^{-1})$ and (d)″ implies $C(D(A)) = CB^{-1}B(D(A)) = V^{-1}B(D(A)) \subseteq D(V) \subseteq D(U)$. So that,

$$\begin{aligned}P(z) &= (z + U - zUC(zC + B)^{-1})(zC + B)\\ &= (z + U)(I - zU(z + U)^{-1}(z + V)^{-1})(zC + B).\end{aligned}$$

Next let us note that (d)″ implies that

$$Q(z) = I - zU(z + U)^{-1}(z + V)^{-1}$$

has a bounded inverse. Therefore, $P(z)$ has a bounded inverse

$$P(z)^{-1} = (zC + B)^{-1}Q(z)^{-1}(z + U)^{-1}.$$

As in the proof of Theorem 1, we shall work in the product space $D(B) \times X$ and use the same notation for the operators. In fact, if M and L are defined by

$$M = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}, \quad L = \begin{bmatrix} 0 & -I \\ A & B \end{bmatrix},$$

with $D(M) = D(B) \times D(B)$, $D(L) = D(A) \times D(B)$, then

$$M(zM + L)^{-1} = \begin{bmatrix} z^{-1} + z^{-1}R(z) & P(z)^{-1} \\ CR(z) & zCP(z)^{-1} \end{bmatrix},$$

where $R(z) = zP(z)^{-1}(zC + B) - I$ (which is equal to $-P(z)^{-1}A$ on the domain $D(A)$). This is a consequence of the analogous formula in Section 2, when we notice that now $-z^{-1}P(z)^{-1}A$ is restriction of $z^{-1}\{zP(z)^{-1}(zC + B) - I\}$ to $D(A)$.

An essential thing is then the following resolvent calculation relative to $R(z)$:

$$\begin{aligned}R(z) &= (zC + B)^{-1}Q(z)^{-1}[z(z + U)^{-1}(zC + B) - Q(z)(zC + B)]\\ &= (zC + B)^{-1}Q(z)^{-1}[z(z + U)^{-1}(zC + B) -\\ &\quad - [I - zU(z + U)^{-1}C(zC + B)^{-1}](zC + B)]\\ &= (zC + B)^{-1}Q(z)^{-1}[z(z + U)^{-1}(zC + B) - (zC + B) + zU(z + U)^{-1}C]\\ &= (zC + B)^{-1}Q(z)^{-1}[z(z + U)^{-1}(zC + B) - B - z^2(z + U)^{-1}C]\\ &= (zC + B)^{-1}Q(z)^{-1}[z(z + U)^{-1} - 1]B.\end{aligned}$$

In addition, it follows that with $f = (f_1, f_2) \in D(B) \times X$

$$\begin{aligned}
&\| BR(z)f_1 ; X \| \\
&\quad \le \| B(zC + B)^{-1}; L(X)\| \, \| Q(z)^{-1}; L(X)\| \, \| z(z + U)^{-1} - I; L(X)\| \, \| Bf_1 ; X \| \\
&\quad \le C|z|^{2-(\alpha+\beta)} \| f_1 ; D(B)\|, \\
&\| CR(z)f_1 ; X \| \\
&\quad \le \| C(zC + B)^{-1}; L(X)\| \, \| Q(z)^{-1}; L(X)\| \, \| z(z + U)^{-1} - I; L(X)\| \, \| Bf_1 ; X \| \\
&\quad \le C|z|^{1-(\alpha+\beta)} \| f_1 ; D(B)\|,
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\| BP(z)^{-1}f_2 ; X \| \\
&\quad \le \| B(zC + B)^{-1}; L(X)\| \, \| Q(z)^{-1}; L(X)\| \, \| (z + U)^{-1}; L(X)\| \, \| Bf_2 ; X \| \\
&\quad \le C|z|^{1-(\alpha+\beta)} \| f_2 ; X \|, \\
&\| CP(z)^{-1}f_2 ; X \| \\
&\quad \le \| C(zC + B)^{-1}; L(X)\| \, \| Q(z)^{-1}; L(X)\| \, \| (z + U)^{-1}; L(X)\| \, \| f_2 ; X \| \\
&\quad \le C|z|^{-(\alpha+\beta)} \| f_2 ; X \|.
\end{aligned}$$

From these it is verified that

$$\| M(zM + L)^{-1}; L(D(B) \times X)\| \le C|z|^{1-(\alpha+\beta)}.$$

Hence if $\alpha + \beta > 1$, condition (b)′ is satisfied.

To complete the proof of the theorem it suffices to apply Proposition 1 and to translate the information given into the assumptions and in the previous estimates. #

Theorem 4. *Assume* $0 < \alpha$, $\beta \le 1$, $\alpha + \beta$, $\alpha + \delta > 1$, $\eta = \eta' = 1$, *and let* $2 - \alpha - \beta < \sigma < 1$. *Then for any* $f \in C^{\sigma}([0, T]; X)$ *and* $u_0, u_1 \in D(A)$ *satisfying*

$$f(0) - Au_0 - Bu_1 \in C(D(B)),$$

there is a unique strict solution u *to* (4)–(6) *for which*

$$Au,\ Bu',\ (Cu')' \in C^{\sigma+\alpha+\beta-2}([0, T]; X).$$

Proof. We only need read the condition

$$\begin{bmatrix} f(0) \\ f(0) \end{bmatrix} - \begin{bmatrix} A & 0 \\ A & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & C \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

for certain $x_0 \in D(A)$, $x_1 \in D(B)$. #

Remark 1. In the best posible situation, $\alpha = \beta = \eta = 1$, σ can be taken arbitrarily in (0, 1) and we have the maximal regularity property for the solution.

Notice that (a)″-(d)″ contain the one in section 2.

In the conditions (a)″-(d)″ above, the hypothesis really restrictive is given by (d)″, since it imposes a very strict link between the operators U and V and hence between A, B and C. We refer to Favini-Obrecht [9] for concrete examples where (d)″ is satisfied, but $C = I$.

In the next result we formulate a hypothesis entailing that, under (a)″-(c)″, (d)″ holds too.

Theorem 5. *Let us assume hypotheses* (a)″-(c)″, $X = Y$, *and*

(i) $D(B)$ *is invariant under* C,

(ii) $\|BCu; X\| \leq k\|Bu; X\|$ *for all* $u \in D(B)$,

(iii) $D(AB^{-1}) \supseteq (X, D(B))_{\tau,1}$, $0 < \tau < \beta$,
where $(X, D(B))_{\tau,1}$ *denotes the real interpolation space of Lions-Peetre.*

Then condition (d)″ *holds.*

Proof. First of all, we quote Triebel [14, p. 24] for the mean-method. Assumption (b)″ implies that $C(zC + B)^{-1}$ is bounded from X into itself for all $z \in \Sigma_\eta$ and $\|C(zC + B)^{-1}; L(X)\| \leq k(1 + |z|)^{-\beta}$, $z \in \Sigma_\eta$.

Conditions (i) and (ii) guarantee that

$$\begin{aligned}\|BC(zC + B)^{-1}u; X\| &= \|B(z + BC^{-1})^{-1}u; X\| \\ &\leq k\|B(zC + B)^{-1}u; X\| \leq k'(1 + |z|)^{1-\beta}\|u; X\|,\end{aligned}$$

for all $z \in \Sigma_\eta$.

Therefore, by interpolation

$$\|C(zC + B)^{-1}; L(X, (X, D(B))_{\tau,1})\| \leq c|z|^{\tau-\beta}, \qquad z \in \Sigma_\eta,\ |z| \text{ large}.$$

This implies, by (iii), that

$$\|AB^{-1}C(zC + B)^{-1}; L(X)\| = \|U(z + V)^{-1}; L(X)\| \leq c|z|^{\tau-\beta},$$

and the condition (d)″ follows. #

Remark 2. If B is a positive operator with bounded imaginary powers B^{is}, for real s, $\|B^{is}; L(X)\| \leq \text{Const.}$ if $|s| \leq \rho$, and such that $D(B^{1+\tau}) \subseteq D(A)$, then (iii) is satisfied since, by Triebel [14, p. 103], $D(B^\tau) = [X, D(B)]_\tau$, the complex interpolation space.

Example 4. Let Ω be a bounded domain in $\boldsymbol{R}^n$, $n \geq 1$, with a smooth boundary. Let $1 < p < \infty$ and denote $L^p(\Omega) = X$, $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) = Y$. If Δ is the Laplacian in $\boldsymbol{R}^n$, with $D(\Delta) = Y$, and m is a positive integer, let $D(\Delta^m) = \{u \in D(\Delta^{m-1}); \Delta^{m-1}u \in Y\}$. Let us fix two positive integers k, m such that $k < m$ and $m+1 < 2k$. Define the operators A, B, C in the space X by

$$D(A) = D(\Delta^m), \ Au = (-1)^m \Delta^m u, \ u \in D(A),$$

$$D(B) = D(\Delta^k), \ Bu = (-1)^k \Delta^k u, \ u \in D(B),$$

$$D(C) = Y, \ Cu = (I - \Delta)u, \ u \in Y.$$

It is well known that $-AB^{-1}$ generates an analytic semigroup in X. On the other hand, since

$$(-1)^k \Delta^k (I-\Delta)^{-1} = (-1)^{k-1}\Delta^{k-1} + (-1)^k[-I + (I-\Delta)^{-1}]\Delta^{k-2},$$

$-BC^{-1}$ generates another analytic semigroup in X, so that (a)$''$ $-$ (c)$''$ are verified with $\alpha = \beta = \eta = \eta' = 1$.

In view of [14, p. 103], and the assumption $m+1 < 2k$ we have that

$$\|(BC^{-1}+z)^{-1}; L(X, D((-\Delta)^{(k-1)\theta}))\| \le c|z|^{\theta-1}, \qquad 0 < \theta < 1,$$

and $D(BC^{-1})$ is strictly contained into $D(AB^{-1})$.

Hence assumption (d)$''$ follows. #

Example 5. We confine ourselves to the domain $I = (a, b)$ an open bounded interval of $\boldsymbol{R}$ and to a very simple differential operator on I, but it is not too difficult to think how an extension to general $\Omega \subset \boldsymbol{R}^n$ could work. Let C denote the multiplication operator by $m(\cdot)$ in the space $L^2(I) = X$, where $m(\cdot)$ is a sufficiently smooth and non negative function on $\bar{I}$. Let us introduce the operator K by

$$D(K) = H^2(I) \cap H_0^1(I), \ (Ku)(x) = -u''(x), \ u \in D(K), \ x \in I.$$

We then observe that if $B = K^m$, where m is a positive integer, and $(zC+B)u = f$, with $f \in X$, $u \in D(B)$, $z \in \boldsymbol{C}$, the boundary conditions determined by $D(B)$, that is, $u(a) = u(b) = u''(a) = u''(b) = \cdots = u^{(m-2)}(a) = u^{(m-2)}(b) = 0$, imply that

$$z\int_I m(x)|u(x)|^2 dx + (-1)^m \int_I u^{(2m)}(x)\bar{u}(x)dx = \int_I f(x)\bar{u}(x)dx,$$

and hence

$$z\int_I m(x)|u(x)|^2 dx + \|u^{(m)}; X\|^2 = \int_I f(x)\bar{u}(x)dx,$$

Therefore we deduce that

$$(\operatorname{Re} z)\int_I m(x)|u(x)|^2 dx + \|u^{(m)}; X\|^2 = \operatorname{Re}\int_I f(x)\bar{u}(x)dx,$$

$$|\operatorname{Im} z| \int_I m(x)|u(x)|^2 dx = \left|\operatorname{Im} \int_I f(x)\bar{u}(x)dx\right|,$$

$$(\operatorname{Re} z + |\operatorname{Im} z|) \int_I m(x)|u(x)|^2 dx + \|u^{(m)}; X\|^2 \leq 2\|f; X\| \|u; X\|.$$

If $\operatorname{Re} z + |\operatorname{Im} z| \geq a_0 > 0$, since

$$\|u; X\| \leq k\|u^{(m)}; X\|, \qquad u \in D(B),$$

we deduce

$$\|u; X\| \leq k'\|f; X\|,$$

and

$$|z| \int_I m(x)|u(x)|^2 dx \leq k_1 \|f; X\|^2,$$

that is,

$$|z| \|Cu; X\|^2 \leq k_2 \|(zC + B)u; X\|^2.$$

We now must only recall that B has a bounded inverse, B and C are selfadjoint, so that, taking the adjoint, we arrive to the conclusion that $zC + B$ is onto X and (a)″ – (b)″ hold with $\eta = 1$, $\beta = 1/2$.

Let $A = K^{m+q}$, where $q \in N$, $q < m$, so that $AB^{-1} = K^q$. Since $-K^q$ generates an analytic semigroup in X, assumption (c)″ is satisfied too, with $\alpha = 1$.

In order to apply Theorem 5 we must require something more to the function $m(\cdot)$. Precisely,

$$\text{(13)} \qquad m(\cdot) \in C^{2m}(\bar{I}),\ m^{(2j+1)}(a) = m^{(2j+1)}(b) = 0, \qquad j = 0, 1, \dots, m-1.$$

This hypothesis ensures that $D(B)$ is invariant under C and condition (i) in Theorem 5 is verified. Moreover, since $B = K^m$, $B^{is} = K^{ims}$ is a bounded operator in X for all real number s.

$A = K^{m+q}$ says that $A = B^{1+q/m}$ and thus condition (iii) holds, with $\tau = q/m$.

Therefore also condition (d)″ is satisfied if $2q < m$. This result permits us to handle some boundary-value problems connected with the equation

$$\frac{\partial}{\partial t}\left(m(x)\frac{\partial}{\partial t}u(t, x)\right) + (-1)^m \frac{\partial^{2m+1}}{\partial x^{2m}\partial t}u(t, x) + (-1)^{m+q}\frac{\partial^{2(m+q)}}{\partial x^{2(m+q)}}u(t, x) = f(t, x),$$

where $0 < t \leq T$ and $x \in I$.

Following the approach of Favini-Yagi [7] and using the results of deLaubenfels [5], it should be possible to study the preceding problems in the space $L^p(a, b)$, too, with $1 < p < \infty$.

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