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Existence and Uniqueness for Second Order Boundary Value Problems

By

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1. Introduction

The main purpose of the present paper is to get the solvability of the following boundary value problem (BVP for short)

$$(E) \qquad (\rho(t)x'(t))' + f(t, x(t), x(\sigma(t)), x'(t), x'(g(t))) = 0, \qquad t \in [a, b]$$

$$(BC) \qquad \begin{aligned} x(t) &= \phi_1(t), && t \le a \\ x(t) + \gamma x'(b) &= \phi_2(t), && t \ge b, \gamma \ge 0 \end{aligned}$$

Here, $I = [a, b]$, $f: I \times (\boldsymbol{R}^n)^4 \to \boldsymbol{R}^n$ is a continuous function, ρ is a real valued continuous and positive function defined on I, σ and g are continuous real valued functions defined and continuously differentiable on $E(a)$, $E(b)$ respectively, where we assume that

$$-\infty < r(a) = \min_{t \in I} \{\sigma(t), g(t)\} < a$$

$$b < r(b) = \max_{t \in I} \{\sigma(t), g(t)\} < +\infty$$

and we set $E(a) = [r(a), a]$, $E(b) = [b, r(b)]$ and $J = [r(a), r(b)]$. Also we assume that the set $\{t \in I : g(t) = a \text{ or } g(t) = b\}$ is finite.

By a solution of the BVP (E)–(BC) we mean a function $x \in C(J, \boldsymbol{R}^n) \cap C^1(E(a) \cup E(b), \boldsymbol{R}^n)$ which is piecewice twice differentiable on I, satisfies the equation (E) for $t \in I$ and the boundary conditions (BC) for $t \in E(a) \cup E(b)$.

In order to show that the BVP (E)–(BC) has at least one solution, we use in this paper the "**Leray-Schauder alternative**" which follows immediately from the Topological Transversality Theorem of Granas [1]. This method reduces the problem of the existence of solutions of a BVP to the establishment of suitable a priori bounds for solutions of these problems. For applications of Topological Transversality method for ordinary differential equations we refer the reader to [2, 6], whereas, for differential equations with delay or deviating arguments in [7, 8, 10, 11] and the references therein.

In Section 2 we prove the basic existence theorem by assuming a priori bounds on solutions and their derivatives. In order to apply this basic existence theorem, we establish a priori bounds for solutions and their derivatives in Section 3. The required a priori bounds for the possible solutions are obtained via L^2-estimates and a Nagumo type condition, by using some ideas from [5]. We prove also and uniqueness results. It is noteworthy that our results for the choice $\rho(t) = e^{kt}$, $k \neq 0$, $t \in I$, $\sigma(t) = g(t) = t$, $\gamma = 0$ lead to the results of Mawhin [9], whereas for the choice $\rho(t) = e^{kt}$, $k \neq 0$, $t \in I$, $\sigma(t) = -t$, $\gamma = 0$ and f independent of x' to the results of Gupta [3, 4].

2. The basic existence theorem

Let B be the space

$$B = C(J, \boldsymbol{R}^n) \cap C^1(E(a) \cup E(b), \boldsymbol{R}^n) \cap C^1(I, \boldsymbol{R}^n)$$

with the norm

$$\|x\|_1 = \max \{\max_{t \in J} |x(t)|, \max_{t \in E(a) \cup E(b)} |x'(t)|, \max_{t \in I} |x'(t)|\}, \qquad x \in B.$$

The following Lemma is an immediate consequence of the Topological Transversality Theorem of Granas [1, p. 61] known as **"Leray-Schauder alternative"** [1, p. 61].

Lemma 2.1. *Let X be a convex subset of a normed linear space E and assume $0 \in B$. Let $F: X \to X$ be a completely continuous operator, i.e. it is continuous and the image of any bounded set is included in a compact set, and let*

$$E(F) = \{x \in X : x = \lambda F x \text{ for some } 0 < \lambda < 1\}.$$

Then either $E(F)$ is unbounded or F has a fixed point.

Theorem 2.2. *Let $f: I \times (\boldsymbol{R}^n)^4 \to \boldsymbol{R}^n$ be a continuous function. Assume that there exists a constant K, such that*

$$\|x\|_1 \leq K,$$

for every solution x of the BVP

$$(E_\lambda) \qquad (\rho(t)x'(t))' + \lambda f(t, x(t), x(\sigma(t)), x'(t), x'(g(t))) = 0, \qquad t \in I$$

$$(BC) \qquad \begin{aligned} x(t) &= \phi_1(t), & t &\in E(a) \\ x(t) + \gamma x'(b) &= \phi_2(t), & t &\in E(b),\ \gamma \geq 0 \end{aligned}$$

where $\lambda \in (0, 1)$. Then the BVP (E)–(BC) has at least one solution.

Proof. Define $T: B \to B$ by

$$Tx(t) = \begin{cases} \phi_1(t), & t \in E(a) \\ \omega(t) + \int_a^b G(t, s) f(s, x(s), x(\sigma(s)), x'(s), x'(g(s)))ds, & t \in I \\ \phi_2(t) - (Tx)'(b), & t \in E(b) \end{cases}$$

where

$$\omega(t) = \phi_1(a) + \frac{\phi_2(b) - \phi_1(a)}{\rho(b)h(b) + \gamma} \rho(b)h(t), \qquad t \in I$$

$$h(t) = \int_a^t \frac{ds}{\rho(s)}, \qquad t \in I$$

and G is the Green's function which is given by the formula

$$G(t, s) = -\frac{1}{\rho(b)h(b) + \gamma} \begin{cases} \left[\rho(b) \int_a^b \frac{ds}{\rho(s)} + \gamma\right] h(t), & a \le t \le s \\ \left[\rho(b) \int_t^b \frac{ds}{\rho(s)} + \gamma\right] h(s), & s \le t \le b \end{cases}$$

T is clearly continuous. We shall prove that T is completely continuous. For this purpose we consider a bounded sequence $\{x_\nu\}$ in B, i.e.

$$\| x_\nu \|_1 \le M, \qquad \text{for all } \nu,$$

where M is a positive constant. Then we have

$$\| Tx_\nu \|_1 \le M_0,$$

where

$$M_0 = \max \{\Theta K_1 + A_1, \Theta K_2 + A_2\},$$

K_1, K_2 constants with

$$\int_a^b |G(t, s)| ds \le K_1, \quad \int_a^b |G_t(t, s)| ds \le K_2, \quad t \in I,$$

$$\Theta = \max \{|f(t, u, u_1, v, v_1)| : t \in I, |u|, |u_1|, |v|, |v_1| \le M\},$$

and A_1, A_2 constants with

$$\sup\left\{\left|\phi_1(a) + \frac{\phi_2(b) - \phi_1(a)}{\rho(b)h(b) + \gamma}\rho(b)h(t)\right| : t \in I\right\} \leq A_1$$

$$\sup\left\{\left|\frac{\phi_2(b) - \phi_1(a)}{\rho(b)h(b) + \gamma}\right|\left|\frac{h(b)}{\rho(t)}\right| : t \in I\right\} \leq A_2.$$

Next we shall prove that the sequences $\{Tx_v\}$ and $\{(Tx_v)'\}$ are equicontinuous. Indeed, for any t_1, t_2 in J and arbitrary v we have

$$|Tx_v(t_1) - Tx_v(t_2)| = \left|\int_{t_1}^{t_2}(Tx_v)'(s)ds\right| \leq \hat{K}|t_1 - t_2|$$

where

$$\hat{K} = \{K_2\Theta + A_2, \max_{t\in E(a)}|\phi_1'(t)|, \max_{t\in E(b)}|\phi_2'(t)|\},$$

which proves that $\{Tx_v\}$ is equicontinuous. On the other hand for any t_1, t_2 in I and for arbitrary v we have

$$|(Tx_v)'(t_1) - (Tx_v)'(t_2)| = \left|\int_{t_1}^{t_2}(Tx_v)''(s)ds\right| \leq \Theta|t_1 - t_2|.$$

This relation and the fact that ϕ_1, ϕ_2 are continuously differentiable functions imply, obviously, that $\{(Tx_v)'\}$ is an equicontinuous sequence.

Thus the mapping T is completely continuous. Finally, we observe by hypothesis that the set $E(T) = \{x \in B : x = \lambda Tx \text{ for some } \lambda \in (0, 1)\}$ is bounded. Hence, by Lemma 2.1 the operator T has a fixed point $x \in B$. This means that the BVP (E)–(BC) has as least one solution. The proof of the theorem is now complete.

3. Applications

In order to apply Theorem 2.2 we must impose conditions on f which imply the existence of the needed a priori bounds. In the next theorems we assume that $\phi_1(a) = \phi_2(b) = 0$. With this restriction is no loss in generality, since an appropriate change of variables reduces the problem with $\phi_1(a) \neq 0 \neq \phi_2(b)$ to this case.

Theorem 3.1. *Let* $f : I \times (\boldsymbol{R}^n)^4 \to \boldsymbol{R}^n$ *be a continuous function, and* $\sigma, g : I \to \boldsymbol{R}$ *are such that*

$$|\sigma'(t)| \geq \frac{1}{c_1} \quad \textit{and} \quad |g'(t)| \geq \frac{1}{c_2}, \qquad t \in I$$

for some constants $c_1 > 0$ *and* $c_2 > 0$.

Assume that:

(H_1) *There exist nonnegative constants A, B, C, D and G with*

$$(3.1)\qquad 4(A + B\sqrt{c_1})\frac{(b-a)^2}{\pi^2} + [4(C + D\sqrt{c_2}) + 2B\sqrt{2c_1(b-a)(r(b)-b)}]\frac{b-a}{\pi} < \rho_0$$

where $\rho_0 = \min\{\rho(t) : t \in I\}$ *such that*

$$(3.2)\qquad \langle u, f(t, u, u_1, v, v_1)\rangle \le A|u|^2 + B|u||u_1| + C|u||v| + D|u||v_1| + G|u|$$

(H_2) *There exist a continuous function* $h : \mathbf{R}^+ \to \mathbf{R}^+$ *and a constant N such that*

$$(3.3)\qquad \int_{\frac{R^2M^2}{b-a}}^{N} \frac{ds}{h(s)} \ge 2R^2M^2$$

where $R = \max\{\rho(t) : t \in I\}$,

$$(3.4)\qquad M = \frac{2[B\sqrt{c_1}(\|\phi_1\| + \sqrt{2}\|\phi_2\|) + D\sqrt{c_2}(\|\phi_1'\| + \|\phi_2'\|)]\frac{b-a}{\pi} + \frac{2G(b-a)\sqrt{b-a}}{\pi}}{\rho_0 - 4(A + B\sqrt{c_1})\frac{(b-a)^2}{\pi^2} - [4(C + D\sqrt{c_2}) + 2B\sqrt{2c_1(b-a)(r(b)-b)}]\frac{b-a}{\pi}}$$

and

$$(3.5)\qquad |\langle v, f(t, u, u_1, v, v_1)\rangle| \le h(|\rho(t)v|^2)\rho(t)|v|^2$$

for all $t \in I$ *and* $|u|_0 \le \sqrt{b-a}M$.

(*Here* $\|u\|^2 = \int_a^b |u(t)|^2 dt$).

Then the BVP (E)-(BC), *with* $\phi_1(a) = \phi_2(b) = 0$, *has at least one solution.*

Proof. We need only to establish the a priori bounds for the BVP (E_λ)-(BC). Let x be a solution of (E_λ)-(BC). By taking the inner product of the equation (E_λ) with $x(t)$, integrating by parts over I and use of the boundary conditions and (3.2), we get

$$\int_a^b \rho(t)|x'(t)|^2 dt \le \rho(b)\langle x'(b), x(b)\rangle + A\int_a^b |x(t)|^2 dt + B\int_a^b |x(t)||x(\sigma(t))| dt$$
$$+ C\int_a^b |x(t)||x'(t)| dt + D\int_a^b |x(t)||x'(g(t))| dt + G\int_a^b |x(t)| dt$$

which implies, by Cauchy-Schwarz inequality and $\gamma > 0$ (if $\gamma = 0$ then $x(b) = 0$)

$$\begin{aligned}\rho_0 \|x\|^2 \leq & -\frac{1}{\gamma}\rho(b)|x(b)|^2 + A\|x\|^2 + B\|x\|\left(\int_a^b |x(\sigma(t))|^2 dt\right)^{\frac{1}{2}} \\ & + C\|x\|\,\|x'\| + D\|x\|\left(\int_a^b |x'(g(t))|^2 dt\right)^{\frac{1}{2}} + G\|x\|\sqrt{b-a}\end{aligned} \tag{3.6}$$

But

$$\begin{aligned}\int_a^b |x(\sigma(t))|^2 dt &\leq \left|\int_a^b \frac{1}{\sigma'(t)}|x(\sigma(t))|^2 d(\sigma(t))\right| \\ &\leq \int_{\sigma(I)} |x(t)|^2 dt \\ &= c_1\left(\int_a^b |x(t)|^2 dt + \int_{E(a)} |x(t)|^2 dt + \int_{E(b)} |x(t)|^2 dt\right) \\ &= c_1\left[\|x\|^2 + \|\phi_1\|^2 + \int_{E(b)} |\phi_2(t) - \gamma x'(b)|^2 dt\right] \\ &\leq c_1[\|x\|^2 + \|\phi_1\|^2 + 2\|\phi_2\|^2 + 2\gamma^2|x'(b)|^2(r(b)-b)] \\ &= c_1[\|x\|^2 + \|\phi_1\|^2 + 2\|\phi_2\|^2 + 2|x(b)^2(r(b)-b)] \\ &\text{(since } x(b) = -\gamma x'(b)) \\ &\leq c_1[\|x\|^2 + \|\phi_1\|^2 + 2\|\phi_2\|^2 + 2\|x'\|^2(r(b)-b)(b-a)]\end{aligned} \tag{3.7}$$

(by the Wirtinger's inequality $|x|_0 = \sup\{|x(t)| : t \in I\} \leq \sqrt{b-a}\,\|x'\|$)
and likewise

$$\int_a^b |x'(g(t))|^2 dt \leq c_2[\|x'\| + \|\phi_1'\|^2 + \|\phi_2'\|^2] \tag{3.8}$$

Subtitute (3.7) and (3.8) into (3.6) to obtain

$$\begin{aligned}\rho_0\|x'\|^2 \leq & A\|x\|^2 \\ & + B\sqrt{c_1}\|x\|\{\|x\| + \|\phi_1\| + \sqrt{2}\|\phi_2\| + \sqrt{2(r(b)-b)(b-a)}\|x'\|\} \\ & + C\|x\|\,\|x'\| + D\sqrt{c_2}\|x\|\{\|x'\| + \|\phi_1'\| + \|\phi_2'\|\} + G\|x\|\sqrt{b-a}\end{aligned}$$

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Next by applying Wirtinger's inequality $\|x\|^2 \leq \dfrac{4(b-a)^2}{\pi^2}\|x'\|^2$ we get,

$$\rho_0 \| x' \|^2 \leq 4(A + B\sqrt{c_1}) \frac{(b-a)^2}{\pi^2} \| x' \|^2 + 2[B\sqrt{c_1}(\| \phi_1 \| + \sqrt{2} \| \phi_2 \|)$$

$$+ D\sqrt{c_2}(\| \phi_1' \| + \| \phi_2' \|) \frac{b-a}{\pi} \| x' \| + 4(C + D\sqrt{c_2}) \frac{b-a}{\pi} \| x' \|^2$$

$$+ B\sqrt{2c_1(r(b) - b)(b-a)} \frac{2(b-a)}{\pi} \| x' \|^2 + \frac{2G(b-a)\sqrt{b-a}}{\pi} \| x' \|$$

Therefore we deduce

$$\left\{ \rho_0 - 4(A + B\sqrt{c_1}) \frac{(b-a)^2}{\pi^2} - [4(C + D\sqrt{c_2}) + 2B\sqrt{2c_1(r(b) - b)(b-a)}] \frac{(b-a)}{\pi} \right\} \| x' \|$$

$$\leq 2[B\sqrt{c_1}(\| \phi_1 \| + \sqrt{2} \| \phi_2 \|) + D\sqrt{c_2}(\| \phi_1' \| + \| \phi_2' \|)] \frac{b-a}{\pi} + \frac{2G(b-a)\sqrt{b-a}}{\pi}$$

which implies, by (3.1) and (3.4)

(3.8) $$\| x' \| \leq M.$$

By the Wirtinger's inequality $|x|_0 = \sup \{ |x(t)| : t \in I \} \leq \sqrt{b-a} \| x' \|$ we have

$$|x|_0 \leq \sqrt{b-a} M = M_1.$$

Also, (3.8) implies, by the mean value theorem, that there exists $t_0 \in I$ such that

$$(b-a) |x'(t_0)|^2 \leq M^2$$

or

(3.9) $$\rho^2(t_0) |x'(t_0)|^2 \leq \frac{R^2 M^2}{b-a}$$

Now, taking the inner product of (E_λ) with $x'(t)$ we have, by (3.5)

$$\left| \frac{d}{dt} | \rho(t) x(t) |^2 \right| \leq 2h(| \rho(t) x'(t) |^2) \rho^2(t) | x'(t) |^2$$

or

(3.10) $$\left| \frac{d}{dt} \int_a^{|\rho(t) x'(t)|^2} \frac{ds}{h(s)} \right| \leq 2 | \rho(t) x'(t) |^2$$

Integrating (3.10) and using (3.9) we get

$$\int_a^{|\rho(t)x'(t)|^2} \frac{ds}{h(s)} \leq \int_a^{|\rho(t_0)x'(t_0)|^2} \frac{ds}{h(s)} + 2\int_a^b |\rho^2(t)x'(t)|^2 dt$$

$$\leq \int_a^{|\rho(t_0)x'(t_0)|^2} \frac{ds}{h(s)} + 2R^2M^2$$

$$\leq \int_a^{\frac{R^2M^2}{b-a}} \frac{ds}{h(s)} + \int_{\frac{R^2M^2}{b-a}}^N \frac{ds}{h(s)} = \int_a^N \frac{ds}{h(s)}$$

Hence

$$|\rho(t)x'(t)|^2 \leq N$$

or

$$\rho_0^2 |x'(t)|^2 \leq |\rho(t)x'(t)|^2 \leq N$$

which implies

$$|x'|_0 \leq \frac{1}{\rho_0}\sqrt{N} = M_2, \qquad t \in I.$$

Consequently the required a priori bounds are established and the proof of the theorem is complete.

The next theorem concerns uniqueness results for the BVP (E)-(BC) in the case when $\gamma = 0$. We remark that in this case the corresponding Wirtinger's inequality becomes

$$\|x\|^2 \leq \frac{(b-a)^2}{\pi^2} \|x'\|^2$$

and relation (3.1)

$$(3.1)' \qquad (A + B\sqrt{c_1})\frac{(b-a)^2}{\pi^2} + (C + D\sqrt{c_2})\frac{b-a}{\pi} < \rho_0$$

Theorem 3.2. *Let $f: I \times (\boldsymbol{R}^n)^4 \to \boldsymbol{R}^n$ be a continuous function, and $\sigma, g: I \to \boldsymbol{R}$ are such that*

$$|\sigma'(t)| \geq \frac{1}{c_1} \quad \text{and} \quad |g'(t)| \geq \frac{1}{c_2}, \qquad t \in I$$

for some constants $c_1 > 0$ and $c_2 > 0$.

Assume that:

(H_3) *There exist nonnegative constants A, B, C and D satisfying* (3.1)′ *and such that*

(3.11)

$$\langle u - x, f(t, u, u_1, v, v_1) - f(t, x, x_1, y, y_1)\rangle$$
$$\leq A|u - x|^2 + B|u - x||u_1 - x_1| + C|u - x||v - y| + D|u - x||v_1 - y_1|$$

Then the BVP (E)-(BC) with $\phi_1(a) = \phi_2(b) = 0$ *and* $\gamma = 0$ *has at most one solution.*

Proof. Let u, x be two solutions of the BVP (E)-(BC). Then we get

$$0 = -\int_a^b \langle(\rho(t)u'(t))' - (\rho(t)x'(t))', u(t) - x(t)\rangle\, dt$$
$$- \int_a^b \langle f(t, u(t), u(\sigma(t)), y'(t), u'(g(t))) - f(t, x(t), x(\sigma(t)), x'(t), x'(g(t)))\rangle\, dt$$
$$\geq \rho_0 \int_a^b |u'(t) - x'(t)|^2 dt - A\int_a^b |u(t) - x(t)|^2 dt$$
$$- B\int_a^b |u(t) - x(t)||u(\sigma(t)) - x(\sigma(t))|\, dt$$
$$- C\int_a^b |u(t) - x(t)||u'(t) - x'(t)|\, dt - D\int_a^b |u(t) - x(t)||u'(g(t)) - x'(g(t))|\, dt$$
$$\geq \left[\rho_0 - (A + B\sqrt{c_1})\frac{(b-a)^2}{\pi^2} - (C + D\sqrt{c_2})\frac{b-a}{\pi}\right]\|u' - x'\|^2$$
$$\geq \left[\rho_0 - (A + B\sqrt{c_1})\frac{(b-a)^2}{\pi^2} - (C + D\sqrt{c_2})\frac{b-a}{\pi}\right]\frac{\pi}{(b-a)}\|u - x\|^2$$

Therefore by (3.1)′ we conclude that $u(t) = x(t)$ for every $t \in I$, which proves the theorem.

Corollary 3.3. *Let* $f: I \times (\boldsymbol{R}^n)^4 \to \boldsymbol{R}^n$ *be a continuous function, and* $\sigma, g: I \to \boldsymbol{R}$ *are such that*

$$|\sigma'(t)| \geq \frac{1}{c_1} \quad \text{and} \quad |g'(t)| \geq \frac{1}{c_2}, \qquad t \in I$$

for some constants $c_1 > 0$ *and* $c_2 > 0$.

Assume that conditions (H_1) *and* (H_3) *hold. Then the BVP (E)-(BC) with* $\phi_1(a) = \phi_2(b) = 0$ *and* $\gamma = 0$ *has a unique solution.*

Proof. By (3.11) with $x = x_1 = y = y_1 = 0$, we obtain

$$\langle u, f(t, u, u_1, v, v_1)\rangle$$
$$\leq A|u|^2 + B|u||u_1| + C|u||v| + D|u||v_1| + |u||f(t, 0, 0, 0, 0)|.$$

That is, condition (3.2) with $G = \max\{|f(t, 0, 0, 0, 0)|, t \in I\}$ holds. This complete the proof.

Remark 3.4. If $\rho(t) = e^{kt}$, $k \neq 0$, $t \in I_1 = [0, \pi]$ equation (E) gives

$$(E_1) \qquad x''(t) + kx'(t) + h(t, x(t), x(\sigma(t)), x'(t), x'(g(t))) = 0$$

where

$$h(t, x(t), x(\sigma(t)), x'(t), x'(g(t))) = e^{kt} f(t, x(t), x(\sigma(t)), x'(t), x(g(t))).$$

Moreover if $\sigma(t) = g(t) = t$ our Theorems 3.1 and 3.2 immediately imply Theorems 1 and 2 respectively of Mawhin [9]. Also if $\sigma(t) = -t$, $t \in I_2 = [-1, 1]$, i.e. if we have boundary value problems involving reflection of the arguments, and f is independent of x', our Theorems 3.1 and 3.2 imply immediately the results of Gupta [3] and [4] for constant matrix A.

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