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Contiguity Relations for the Lauricella Functions

By

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1. Introduction

Following Gelfand and his collaborators we consider an $\mathscr{A}$-hypergeometric system ($\mathscr{A}$-HGS) M_α with parameter α (cf. [G], [GGZ], [GZK1], [GZK2], [GKZ]). In [Sai1] we defined b-functions for $\mathscr{A}$-HGS's, and computed them when the systems were normal in the sense to be defined later in §2. In this paper we explain a method for deducing explicit contiguity relations of hypergeometric functions (HGF's) from the computation of b-functions.

Let us recall contiguity relations of the Gauss HGF

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!} x^n$$

where

$$(a, n) = \begin{cases} 1, & \text{for } n = 0 \\ a(a+1)\cdots(a+n-1), & \text{for } n \geq 1. \end{cases}$$

We call a, b, c parameters. The Gauss HGF's $F(a \pm 1)$, $F(b \pm 1)$ and $F(c \pm 1)$ are said to be contiguous to F. Here $F(a+1)$ stands for $F(a+1, b; c; x)$, etc. Since we have $(x(d/dx) + a)(a, n)x^n = a(a+1, n)x^n$, we can easily find the following relations:

$$\left(x\frac{d}{dx} + a\right)F = aF(a+1),$$

$$\left(x\frac{d}{dx} + b\right)F = bF(b+1),$$

$$\left(x\frac{d}{dx} + c - 1\right)F = (c-1)F(c-1),$$

which we call obvious contiguity relations. We also have nonobvious contiguity relations

$$\left[x(1-x)\frac{d}{dx}-bx+c-a\right]F=(c-a)F(a-1),$$

$$\left[x(1-x)\frac{d}{dx}-ax+c-b\right]F=(c-b)F(b-1),$$

$$c\left[(1-x)\frac{d}{dx}+c-a-b\right]F=(c-a)(c-b)F(c+1),$$

which are slightly more difficult to find. By these two kinds of relations we obtain

$$\left[x(1-x)\frac{d}{dx}-bx+c-a\right]\left(x\frac{d}{dx}+a\right)F=a(c-a-1)F,$$

$$\left[x(1-x)\frac{d}{dx}-ax+c-b\right]\left(x\frac{d}{dx}+b\right)F=b(c-b-1)F,$$

$$\left[(1-x)\frac{d}{dx}+c-a-b-1\right]\left(x\frac{d}{dx}+c-1\right)F=(c-a-1)(c-b-1)F.$$

In [Sai1], we have shown that the factors appearing in the right hand sides $a(c-a-1)$, $b(c-b-1)$ and $(c-a-1)(c-b-1)$ are understood to be b-functions for an $\mathscr{A}$-HGS. In this paper we show a process of obtaining nonobvious contiguity relations for HGF's whose associated $\mathscr{A}$-HGS's are normal. As examples we exhibit the process for the Barnes HGF ${}_pF_{p-1}$ and the Lauricella function F_C. Contiguity relations of HGF's have been studied by several people. The contiguity relations of the Lauricella function F_D have been obtained by Miller (cf. [M]). Sasaki has obtained the contiguity relations for HGF's whose associated $\mathscr{A}$-HGS's are attached to Grassmannian manifolds (cf. [Sas]); these $\mathscr{A}$-HGS's are normal (cf. [Sai2]). Takayama has shown an algorithm to obtain contiguity relations for HGF's, and presented the explicit contiguity relation of Appell's F_4 with respect to one of its parameters (cf. [T]).

Our process starts with obvious contiguity relations of a HGF of M variables with n parameters, and then we associate an $\mathscr{A}$-HGS on an affine space of dimension $M+n$ to the original HGF (cf. [KKM]). If this $\mathscr{A}$-HGS is normal, then we know its b-functions (cf. [Sai1]). Since we see that obvious contiguity relations composed by nonobvious ones yield b-functions, knowing b-functions is enough in principle for knowing nonobvious contiguity relations. In fact, when we rewrite b-functions in terms of differential operators and expand them with respect to each corresponding variable, we can find differential operators which give nonobvious contiguity relations. Finally we restrict them on the original space to obtain nonobvious contiguity relations for the original HGF.

In §2, we review the notion of $\mathscr{A}$-HGS's and the results in [Sai1]. In §3, we treat the Barnes HGF ${}_pF_{p-1}$, which is a very simple example, in order to get used to the process. In §4, we treat the Lauricella function F_C; this case is not so simple. In §5, we present the nonobvious contiguity relations for the Lauricella function F_A and the ones for the Lauricella function F_B without proof.

Many other functions, such as all 14 two-variable complete series in the Horn's list (cf. [E]), satisfy the required conditions to follow the process above and thus their nonobvious contiguity relations can be calculated by this method.

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2. Review on $\mathscr{A}$-HGS's

In this section we review the notion of $\mathscr{A}$-HGS's and the results in [Sai1]. Suppose we are given a set $\mathscr{A} = \{\chi_j = \sum_{i=1}^n \chi_{ij} e_i \in \boldsymbol{Z}^n = \bigoplus_{i=1}^n \boldsymbol{Z} e_i \mid j = 1, \dots, N\}$ satisfying two conditions:

(1) The vectors $\chi_1, \dots, \chi_N$ generate the lattice $\boldsymbol{Z}^n$.

(2) All the vectors χ_j lie on some affine hyperplane $\sum_{i=1}^n c_i x_i = 1$ in $\boldsymbol{R}^n$, where $c_i \in \boldsymbol{Z}$.

We denote by L the subgroup in $\boldsymbol{Z}^N$ consisting of those $a = (a_j)_{j=1}^N$ satisfying $\sum_{j=1}^N a_j \chi_j = 0$. Let $W = W_V = \boldsymbol{C}[u_1, \dots, u_N, D_1, \dots, D_N]$ denote the Weyl algebra on $V = \boldsymbol{C}^N$ where $(u_1, \dots, u_N)$ is a coordinate system on V and $D_j = \partial / \partial u_j$ for $j = 1, \dots, N$. We put $\Box_a = \prod_{a_j > 0} D_j^{a_j} - \prod_{a_j < 0} D_j^{-a_j}$. For a parameter $\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{C}^n$ we define an $\mathscr{A}$-HGS M_α on V by

$$M_\alpha := W / (\sum_{i=1}^n W(\sum_{j=1}^N \chi_{ij} \theta_j - \alpha_i) + \sum_{a \in L} W \Box_a).$$

Here $\theta_j = u_j D_j$ and $\sum_{a \in L} W \Box_a$ denotes the left W-submodule of W consisting of all sums $\sum_{a \in L} w_a \Box_a$ with $w_a \in W$ where only finitely many w_a are not zero. We denote by Q the Newton polyhedron, i.e., Q is the convex hull in $\boldsymbol{R}^n$ of the points $\chi_1, \dots, \chi_N$ and by $\mathscr{F}$ the set of faces of Q of codimension one. Let $\{s_1, \dots, s_n\}$ be the dual basis of $\{e_1, \dots, e_n\}$. For $\Gamma \in \mathscr{F}$, we denote by φ_Γ the linear form defining the hyperplane spanned by Γ such that the coefficients of φ_Γ are integers, that their greatest common divisor is one, and that $\varphi_\Gamma(\chi_j) \geq 0$ for all $1 \leq j \leq N$. We denote by $W[s]$ the noncommutative ring $\boldsymbol{C}[s_1, \dots, s_n] \otimes_{\boldsymbol{C}} W$ where each s_i is a central element, by $M[s]$ the quotient of $W[s]$ divided by the left ideal generated by $\sum_{j=1}^N \chi_{ij} \theta_j - s_i (i = 1, \dots, n)$ and $\Box_a (a \in L)$. We denote by I_j for $1 \leq j \leq N$ the left ideal of $W[s]$ generated by $\sum_{j=1}^{N} \chi_{ij}\theta_j - s_i (i = 1,\dots,n)$, $\Box_a (a \in L)$ and D_j, and by B_j the kernel of the natural morphism $\boldsymbol{C}[s] = \boldsymbol{C}[s_1,\dots,s_n] \to W[s]/I_j$ which sends 1 to the element represented by 1. Any nonzero polynomial in B_j is called a b-function with respect to D_j. We remark that b-functions in this sense are not b-functions in a usual sense.

Proposition 2.1 [Sai1]. *For a polynomial $b(s) \in B_j$ there exists an operator $Q \in W$ such that $b(\alpha) = QD_j$ in M_α for all $\alpha \in \boldsymbol{C}^n$.*

Corollary 2.2 [Sai1]. *Suppose that there exists a polynomial $b(s) \in B_j$ such that $b(\alpha) \neq 0$. Then the morphism multiplying D_j from the right induces an isomorphism $M_{\alpha - \chi_j} \xrightarrow{\sim} M_\alpha$.*

An $\mathscr{A}$-HGS M_α is said to be normal when the following condition is satisfied:

$$(\boldsymbol{R}_{\geq 0}\chi_1 + \cdots + \boldsymbol{R}_{\geq 0}\chi_N) \cap \boldsymbol{Z}^n = \boldsymbol{Z}_{\geq 0}\chi_1 + \cdots + \boldsymbol{Z}_{\geq 0}\chi_N.$$

The main result in [Sai1] is the following:

Theorem 2.3 [Sai1]. *Let M_α be a normal $\mathscr{A}$-HGS. Then, for $1 \leq j \leq N$, the ideal B_j of $\boldsymbol{C}[s]$ is generated by one element*

$$b_j(s) = \prod_{\Gamma \in \mathscr{F},\, \varphi_\Gamma(\chi_j) > 0} \ \prod_{k=0}^{\varphi_\Gamma(\chi_j) - 1} (\varphi_\Gamma(s) - k).$$

3. The Barnes HGF ${}_pF_{p-1}$

In this section we obtain nonobvious contiguity relations of the Barnes HGF ${}_pF_{p-1}$ by our method. The simplicity of this example helps us to understand the method. The Barnes HGF ${}_pF_{p-1}$ around the origin of $\boldsymbol{C}$ is defined to be

$${}_pF_{p-1}(a_1,\dots,a_p;\, b_1,\dots,b_{p-1};\, x) = \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{p} (a_i, n)}{\prod_{i=1}^{p-1} (b_i, n)} \frac{x^n}{n!}.$$

In order to define an $\mathscr{A}$-HGS associated with ${}_pF_{p-1}$, we use the idea of the canonical system (cf. [KKM]). We have the following obvious differential contiguity relations for ${}_pF_{p-1}$:

$$(\vartheta + a_i)\,{}_pF_{p-1} = a_i\,{}_pF_{p-1}(a_i + 1) \qquad (1 \leq i \leq p),$$

$$(\vartheta + b_i - 1)\,{}_pF_{p-1} = (b_i - 1)\,{}_pF_{p-1}(b_i - 1) \qquad (1 \leq i \leq p - 1),$$

$$\frac{\partial}{\partial x}\,{}_pF_{p-1} = \frac{\prod_{i=1}^{p} a_i}{\prod_{i=1}^{p-1} b_i}\,{}_pF_{p-1}(a_1 + 1,\dots,a_p + 1;\, b_1 + 1,\dots,b_{p-1} + 1)$$

where $\vartheta = x \cdot \partial/\partial x$. We call the differential operators appearing the left hand sides of the above contiguity relations obvious contiguity operators. The set of parameter shifts by the obvious contiguity operators will be a set $\mathscr{A}$. In order to consider the obvious contiguity operators as vector fields, we introduce additional variables $v_{a_1},\dots,v_{a_p}$, $v_{b_1},\dots,v_{b_{p-1}}$ and a new function ${}_p\tilde{F}_{p-1} = {}_pF_{p-1} \cdot v_{a_1}^{a_1} \cdots v_{a_p}^{a_p} \cdot v_{b_1}^{b_1-1} \cdots v_{b_{p-1}}^{b_{p-1}-1}$. We then define the following operators:

$$E^{a_i} := v_{a_i}(\vartheta + \vartheta_{a_i}) \qquad (1 \le i \le p),$$

$$E_{b_i} := v_{b_i}^{-1}(\vartheta + \vartheta_{b_i}) \qquad (1 \le i \le p-1),$$

$$E^{a_1 \dots a_p \cdot b_1 \dots b_{p-1}} := v_{a_1} \cdots v_{a_p} \cdot v_{b_1} \cdots v_{b_{p-1}} \frac{\partial}{\partial x}$$

where $\vartheta_{a_i} = v_{a_i} \cdot \partial/\partial v_{a_i} (1 \le i \le p)$ and $\vartheta_{b_i} = v_{b_i} \cdot \partial/\partial v_{b_i} (1 \le i \le p-1)$.

Then the function ${}_p\tilde{F}_{p-1}$ satisfies

$$\begin{aligned}
&\vartheta_{a_i}\, {}_p\tilde{F}_{p-1} = a_i\, {}_p\tilde{F}_{p-1} && (1 \le i \le p),\\
&\vartheta_{b_i}\, {}_p\tilde{F}_{p-1} = (b_i - 1)\, {}_p\tilde{F}_{p-1} && (1 \le i \le p-1),\\
&E^{a_i}\, {}_p\tilde{F}_{p-1} = a_i\, {}_p\tilde{F}_{p-1}(a_i + 1) && (1 \le i \le p),\\
&E_{b_i}\, {}_p\tilde{F}_{p-1} = (b_i - 1)\, {}_p\tilde{F}_{p-1}(b_i - 1) && (1 \le i \le p-1),\\
&E^{a_1 \dots a_p \cdot b_1 \dots b_{p-1}}\, {}_p\tilde{F}_{p-1}\\
&\quad = \frac{a_1 \cdots a_p}{b_1 \cdots b_{p-1}}\, {}_p\tilde{F}_{p-1}(a_1 + 1,\dots,a_p + 1;\, b_1 + 1,\dots,b_{p-1} + 1).
\end{aligned}$$

Hence the function ${}_p\tilde{F}_{p-1}$ is a solution of the system of differential equations

$$\begin{aligned}
(\vartheta_{a_i} - a_i)\Phi &= 0 \qquad (1 \le i \le p),\\
(\vartheta_{b_i} - b_i + 1)\Phi &= 0 \qquad (1 \le i \le p-1),\\
(E^{a_1} \cdots E^{a_p} - E_{b_1} \cdots E_{b_{p-1}} E^{a_1 \dots a_p \cdot b_1 \dots b_{p-1}})\Phi &= 0.
\end{aligned}$$

Next we change variables from $v_{a_1} \cdots v_{a_p}$, $v_{b_1},\dots,v_{b_{p-1}}$, x to $u_i = -v_{a_i}^{-1} (1 \le i \le p)$, $u_{p+i} = v_{b_i} (1 \le i \le p-1)$, $u_{2p} = v_{a_1}^{-1} \cdots v_{a_p}^{-1} v_{b_1}^{-1} \cdots v_{b_{p-1}}^{-1} x$ so that $E^{a_i} (1 \le i \le p)$, $E_{b_i} (1 \le i \le p-1)$, $E^{a_1 \dots a_p \cdot b_1 \dots b_{p-1}}$ are transformed into $D_i (1 \le i \le p)$, $D_{p+i} (1 \le i \le p-1)$, D_{2p} respectively where $D_j = \partial/\partial u_j (1 \le j \le 2p)$. Hence the function ${}_p\tilde{F}_{p-1}(u_1,\dots,u_{2p})$ is a solution of the system of differential equations

$$\begin{aligned}
(\theta_i + \theta_{2p} + a_i)\Phi &= 0 \qquad (1 \le i \le p),\\
(\theta_{p+i} - \theta_{2p} - b_i + 1)\Phi &= 0 \qquad (1 \le i \le p-1),\\
(D_1 \cdots D_p - D_{p+1} \cdots D_{2p})\Phi &= 0,
\end{aligned}$$

where $\theta_j = u_j D_j (1 \le j \le 2p)$. We consider a complex affine space V of dimension $2p$ on which $(u_1,\dots,u_{2p})$ gives a coordinate system. Since we have $v_{a_i} = -u_i^{-1}(1 \le i \le p)$, $v_{b_i} = u_{p+i}(1 \le i \le p-1)$ and $x = (-1)^p u_1^{-1} \cdots u_p^{-1} u_{p+1} \cdots u_{2p}$, we see that the restriction of ${}_p\widetilde{F}_{p-1}$ to the submanifold X of the complex affine space V defined by $u_i = -1 (1 \le i \le p)$ and $u_{p+i} = 1 (1 \le i \le p-1)$ is ${}_pF_{p-1}$. From the above arguments we take $\{\chi_1 = e_1, \dots, \chi_{2p-1} = e_{2p-1}, \chi_{2p} = \sum_{i=1}^{p} e_i - \sum_{i=1}^{p-1} e_{p+i} \in \boldsymbol{Z}^{2p-1} = \bigoplus_{i=1}^{2p-1} \boldsymbol{Z} e_i\}$ as a set $\mathscr{A}$, and define an $\mathscr{A}$-HGS M_α and a $W[s]$-module $M[s]$. Let $\{s_1, \dots, s_{2p-1}\}$ be the dual basis of $\{e_1, \dots, e_{2p-1}\}$. As in [Sai2], we can calculate the set $\mathscr{F}$; the set of associated linear forms is

$$\{\varphi_\Gamma | \Gamma \in \mathscr{F}\} = \{s_i, s_i + s_{p+j} | 1 \le i \le p, 1 \le j \le p-1\}.$$

At the same time as the calculation of the set $\mathscr{F}$, we see that the vectors $\chi_j (1 \le j \le 2p)$ satisfy the normality condition. It is easy to see that the ideal generated by all $\Box_l (l \in L)$ is actually generated by only $D_1 \cdots D_p - D_{p+1} \cdots D_{2p}$.

By the definition of b-functions, nonobvious contiguity operators can be obtained by factorizing b-functions in the module $M[s]$ by obvious contiguity operators. Clearly the operators $D_k (1 \le k \le 2p)$ are obvious contiguity operators. By Proposition 2.1, there exist operators $Q_k \in W[s]$ $(1 \le k \le 2p)$ such that $b_k(s) \equiv Q_k D_k$ in the module $M[s]$. We first find such operators $Q_k (1 \le k \le 2p)$, and next restrict them on X.

By Theorem 2.3, we have in the module $M[s]$

$$b_k(s) = s_k \prod_{i=1}^{p-1} (s_k + s_{p+i})$$

$$\equiv \prod_{i=1}^{p} (\theta_k + \theta_{p+i}) \qquad (1 \le k \le p), \tag{1.k}$$

$$b_{p+k}(s) = \prod_{i=1}^{p} (s_i + s_{p+k})$$

$$\equiv \prod_{i=1}^{p} (\theta_i + \theta_{p+k}) \qquad (1 \le k \le p-1), \tag{1.p+k}$$

$$b_{2p}(s) = \prod_{i=1}^{p} s_i \equiv \prod_{i=1}^{p} (\theta_i + \theta_{2p}), \tag{1.2p}$$

because we have

$$\theta_i + \theta_{2p} \equiv s_i \qquad (1 \le i \le p),$$

$$\theta_i - \theta_{2p} \equiv s_{p+i} \qquad (1 \le i \le p-1).$$

By expanding (1.k) on θ_k for each k with $1 \le k \le 2p$, we see that for k with $1 \le k \le p$

$$b_k(s) \equiv \sum_{j=0}^{p-1} (\sum_{I\subset[1,p],|I|=j} \prod_{i\in I} \theta_{p+i}) \theta_k^{p-j} + \prod_{i=1}^{p} \theta_{p+i},$$

$$b_{p+k}(s) \equiv \sum_{j=0}^{p-1} (\sum_{I\subset[1,p],|I|=j} \prod_{i\in I} \theta_{i}) \theta_{p+k}^{p-j} + \prod_{i=1}^{p} \theta_{i}.$$

Since $D_1 \cdots D_p - D_{p+1} \cdots D_{2p} \equiv 0$, we have $\prod_{i=1}^{p} \theta_i \equiv \prod_{i=1}^{p} u_i \prod_{i=1}^{p} D_{p+i}$ and $\prod_{i=1}^{p} \theta_{p+i} \equiv \prod_{i=1}^{p} u_{p+i} \prod_{i=1}^{p} D_i$. Hence we put

$$Q_k := \sum_{j=0}^{p-1} (\sum_{I\subset[1,p],|I|=j} \prod_{i\in I} \theta_{p+i}) \theta_k^{p-j-1} u_k + \prod_{i=1}^{p} u_{p+i} \prod_{\substack{1\le i\le p\\ i\ne k}} D_i,$$

$$Q_{p+k} := \sum_{j=0}^{p-1} (\sum_{I\subset[1,p],|I|=j} \prod_{i\in I} \theta_{i}) \theta_{p+k}^{p-j-1} u_{p+k} + \prod_{i=1}^{p} u_{i} \prod_{\substack{1\le i\le p\\ i\ne k}} D_{p+i}$$

for k with $1 \le k \le p$. We have thus proved the following theorem.

Theorem 3.1. *In the module* $M[s]$, *we have* $Q_k D_k \equiv b_k(s)$ *for all* k *with* $1 \le k \le 2p$.

Although we have obtained nonobvious contiguity operators $Q_k (1 \le k \le 2p)$ for a $W[s]$-module $M[s]$, we need to rewrite them in order to restrict them on the submanifold X. We define

$$Q_k(s) := \sum_{j=0}^{p-1} (\sum_{I\subset[1,p],|I|=j} \prod_{i\in I} (\theta_{2p} + s_{p+i})) (s_k - \theta_{2p})^{p-j-1} u_k$$

$$+ \prod_{i=1}^{p} u_{p+i} \prod_{\substack{1\le i\le p\\ i\ne k}} D_i \qquad (1 \le k \le p),$$

$$Q_{p+k}(s) := \sum_{j=0}^{p-1} (\sum_{I\subset[1,p],|I|=j} \prod_{i\in I} (s_i - \theta_{2p})) (s_{p+k} + \theta_{2p})^{p-j-1} u_{p+k}$$

$$+ \prod_{i=1}^{p} u_i \prod_{\substack{1\le i\le p\\ i\ne k}} D_{p+i} \qquad (1 \le k \le p-1),$$

$$Q_{2p}(s) := \sum_{j=0}^{p-1} (\sum_{I\subset[1,p],|I|=j} \prod_{i\in I} (s_i - \theta_{2p}) \theta_{2p}^{p-j-1} u_{2p}$$

$$+ \prod_{i=1}^{p} u_i \prod_{i=1}^{p-1} D_{p+i}.$$

Proposition 3.2. *In the module* $M[s]$, *we have* $Q_k(s) D_k \equiv b_k(s)$ *for all* k *with* $1 \le k \le 2p$.

Proof. Since $s_k \equiv \theta_k + \theta_{2p}(1 \leq k \leq p)$ and $s_{p+k} \equiv \theta_{p+k} - \theta_{2p}(1 \leq k \leq p)$, the proposition is obvious by Theorem 3.1.

For $\beta = \sum_{i=1}^{2p-1} \beta_i e_i \in \boldsymbol{C} \otimes_{\boldsymbol{Z}} (\bigoplus_{i=1}^{2p-1} \boldsymbol{Z} e_i)$, we define operators ${}_XQ_k(\beta)$ $(1 \leq k \leq 2p)$ in $W_X = \boldsymbol{C}[u_{2p}, D_{2p}]$ by

$$
\begin{aligned}
{}_XQ_k(\beta) := &-\sum_{j=0}^{p-1} \Big(\sum_{I \subset [1,p], |I| = j} \prod_{i \in I} (\theta_{2p} + \beta_{p+i}) \Big) (\beta_k - \theta_{2p})^{p-j-1} \\
&+ u_{2p} \prod_{\substack{1 \leq i \leq p \\ i \neq k}} (\theta_{2p} - \beta_i) \qquad (1 \leq k \leq p), \\
{}_XQ_{p+k}(\beta) := &\sum_{j=0}^{p-1} \Big(\sum_{I \subset [1,p], |I| = j} \prod_{i \in I} (\beta_i - \theta_{2p}) \Big) (\beta_{p+k} + \theta_{2p})^{p-j-1} \\
&+ (-1)^p D_{2p} \prod_{\substack{1 \leq i \leq p \\ i \neq k}} (\beta_{p+i} + \theta_{2p}) \qquad (1 \leq k \leq p-1), \\
{}_XQ_{2p}(\beta) := &\sum_{j=0}^{p-1} \Big(\sum_{I \subset [1,p], |I| = j} \prod_{i \in I} (\beta_i - \theta_{2p}) \Big) \theta_{2p}^{p-j-1} u_{2p} \\
&+ (-1)^p \prod_{i=1}^{p-1} (\beta_{p+i} + \theta_{2p}).
\end{aligned}
$$

Let $\alpha = -\sum_{i=1}^{p} a_i e_i + \sum_{i=1}^{p-1} (b_{p+i} - 1) e_{p+i} \in \boldsymbol{C} \otimes_{\boldsymbol{Z}} (\bigoplus_{i=1}^{2p-1} \boldsymbol{Z} e_i)$.

Theorem 3.3. *We have contiguity relations*

$$
\begin{aligned}
b_k(\alpha)\,{}_pF_{p-1} &= a_k\, {}_XQ_k(\alpha - \chi_k)\, {}_pF_{p-1}(a_k + 1) \qquad (1 \leq k \leq p), \\
b_{p+k}(\alpha)\,{}_pF_{p-1} &= (b_k - 1)\, {}_XQ_{p+k}(\alpha - \chi_{p+k})\, {}_pF_{p-1}(b_k - 1) \qquad (1 \leq k \leq p-1), \\
b_{2p}(\alpha)\,{}_pF_{p-1} &= \frac{a_1 \cdots a_p}{b_1 \cdots b_{p-1}}\, {}_XQ_{2p}(\alpha - \chi_{2p}) \\
&\qquad {}_pF_{p-1}(a_1 + 1, \ldots, a_p + 1; b_1 + 1, \ldots, b_{p-1} + 1)
\end{aligned}
$$

where

$$
\begin{aligned}
b_k(\alpha) &= -a_k \prod_{i=1}^{p-1} (b_i - 1 - a_k) \qquad (1 \leq k \leq p), \\
b_{p+k}(\alpha) &= \prod_{i=1}^{p} (b_k - 1 - a_i) \qquad (1 \leq k \leq p-1), \\
b_{2p}(\alpha) &= (-1)^p \prod_{i=1}^{p} a_i.
\end{aligned}
$$

Proof. *For all* k *with* $1 \leq k \leq p$, *we have* $D_k\, {}_p\tilde{F}_{p-1}(\alpha) = a_k\, {}_p\tilde{F}_{p-1}(\alpha - \chi_k)$.

Hence we have for all k with $1 \leq k \leq p$

$$\begin{aligned} a_k Q_k(\alpha - \chi_k)_p\tilde{F}_{p-1}(\alpha - \chi_k) &= a_k Q_k(s)_p\tilde{F}_{p-1}(\alpha - \chi_k) \\ &= Q_k(s) D_{k\,p}\tilde{F}_{p-1}(\alpha) \\ &= b_k(s)_p\tilde{F}_{p-1}(\alpha) \\ &= b_k(\alpha)_p\tilde{F}_{p-1}(\alpha). \end{aligned}$$

Restricting the above equations on X, we obtain for all k with $1 \leq k \leq p$

$$_XQ_k(\alpha - \chi_k) a_{k\,p}F_{p-1}(a_k + 1) = b_k(\alpha)_pF_{p-1}.$$

We can prove the other contiguity relations similarly.

4. The Lauricella function F_C

The Lauricella function F_C around the origin of $\boldsymbol{C}^M$ is defined to be

$$\begin{aligned} &F_C(a, b; c_1, \ldots, c_M; x_1, \ldots, x_M) \\ &= \sum_{n_1, \ldots, n_M \geq 0} \frac{(a, \sum_{j=1}^M n_j)(b, \sum_{j=1}^M n_j)}{\prod_{j=1}^M (c_j, n_j) \cdot n_j!} \prod_{j=1}^M x_j^{n_j} \end{aligned}$$

where $\boldsymbol{n} = (n_1, \ldots, n_M)$ is a multi-index. We have the following obvious differential contiguity relations for F_C:

$$\begin{aligned} &(\vartheta_1 + \cdots + \vartheta_M + a) F_C = a F_C(a + 1), \\ &(\vartheta_1 + \cdots + \vartheta_M + b) F_C = b F_C(b + 1), \\ &\qquad (\vartheta_j + c_j - 1) F_C = (c_j - 1) F_C(c_j - 1) \qquad (1 \leq j \leq M), \\ &\qquad\qquad \frac{\partial}{\partial x_j} F_C = \frac{ab}{c_j} F_C(a + 1, b + 1, c_j + 1) \qquad (1 \leq j \leq M) \end{aligned}$$

where $\vartheta_j = x_j \cdot \partial/\partial x_j$ for $1 \leq j \leq M$. As in §3, we introduce additional variables $v_a, v_b, v_{c_1}, \ldots, v_{c_M}$ and a new function $\tilde{F}_C = F_C \cdot v_a^a \cdot v_b^b \cdot v_{c_1}^{c_1 - 1} \cdots v_{c_M}^{c_M - 1}$, and then change variables from $x_1, \ldots, x_M, v_a, v_b, v_{c_1}, \ldots, v_{c_M}$ to $u_j = v_a^{-1} v_b^{-1} v_{c_j}^{-1} x_j$ $(1 \leq j \leq M)$, $u_{M+1} = -v_a^{-1}$, $u_{-j} = v_{c_j} (1 \leq j \leq M)$, $u_{-(M+1)} = -v_b^{-1}$. Then the function $\tilde{F}_C(u_1, \ldots, u_{M+1}, u_{-1}, \ldots, u_{-(M+1)})$ is a solution of the system of differential equations

$$\begin{aligned} &(\theta_j - \theta_{-j} + c_j - 1)\Phi = 0 \qquad (1 \leq j \leq M), \\ &(\theta_{M+1} - \theta_{-(M+1)} + a - b)\Phi = 0, \\ &\Big(\sum_{j=1}^{M+1} (\theta_j + \theta_{-j}) + a + b + M - \sum_{j=1}^{M} c_j\Big)\Phi = 0, \end{aligned}$$

$$(D_jD_{-j} - D_{M+1}D_{-(M+1)})\Phi = 0 \qquad (1 \le j \le M),$$

where $D_j = \dfrac{\partial}{\partial u_j}$ and $\theta_j = u_jD_j$ $(1 \le |j| \le M + 1)$. We consider a complex affine space V of dimension $2M + 2$ on which $(u_{\pm 1},\dots,u_{\pm(M+1)})$ gives a coordinate system. Since we have $v_a = -u_{M+1}^{-1}$, $v_b = -u_{-(M+1)}^{-1}$, $v_{c_j} = u_{-j}$ $(1 \le j \le M)$ and $x_j = (u_ju_{-j})/(u_{M+1}u_{-(M+1)})$ $(1 \le j \le M)$, we see that the restriction of $\tilde{F}_C$ to the submanifold X of the complex affine space V defined by $u_{-j} = 1$ $(1 \le j \le M)$ and $u_{M+1} = u_{-(M+1)} = -1$ is F_C. Since $\tilde{F}_C$ is a solution of the system above, we take $\{\chi_j = e_j + e_{M+2}, \chi_{-j} = -e_j + e_{M+2} \in \boldsymbol{Z}^{M+2} = \bigoplus_{i=1}^{M+2}\boldsymbol{Z}e_i \mid 1 \le j \le M + 1\}$ as a set $\mathscr{A}$, and consider a $W[s]$-module $M[s]$ associated with the set $\mathscr{A}$. Let $\{s_1,\dots,s_{M+2}\}$ be the dual basis of $\{e_1,\dots,e_{M+2}\}$. As in [Sai2], we can calculate the set $\mathscr{F}$; the set of associated linear forms is

$$\{\varphi_\Gamma \mid \Gamma \in \mathscr{F}\} = \left\{\frac{1}{2}(s_{M+2} + \sum_{j\in J} s_j - \sum_{j\in J'} s_j) \mid J \subset [1, M + 1]\right\}$$

where J' is the complement of J in $[1, M + 1]$. At the same time as the calculation of the set $\mathscr{F}$, we see that the vectors $\chi_{\pm j}$ $(1 \le j \le M + 1)$ satisfy the normality condition. Again as in [Sai2], we can check the ideal generated by all $\Box_l (l \in L)$ is actually generated by only $D_jD_{-j} - D_{M+1}D_{-(M+1)}$ $(1 \le j \le M)$.

For each k with $1 \le k \le M + 1$, we fix a permutation σ_k of the set $[1, M + 1]$ such that $\sigma_k(1) = k$. When $1 \le k \le M$, we assume $\sigma_k(M + 1) = M + 1$. For all k and r with $1 \le k, r \le M + 1$, we define $I_r^k := [1, M + 1] - \{\sigma_k(1),\dots,\sigma_k(r)\}$. For $J \subset I_r^k$, we denote by J' the complement of J in I_r^k.

By Theorem 2.3, we have in the module $M[s]$

$$\begin{aligned} b_k(s) &= 2^{-2M} \prod_{J\subset[1,M+1], J\ni k} (s_{M+2} + \sum_{j\in J} s_j - \sum_{j\in J'} s_j) \\ &\equiv \prod_{J\subset I_1^k} (\theta_k + \sum_{j\in J} \theta_j + \sum_{j\in J'} \theta_{-j}), \end{aligned} \tag{2k}$$

$$\begin{aligned} b_{-k}(s) &= 2^{-2M} \prod_{J\subset[1,M+1], J\ni k} (s_{M+2} + \sum_{j\in J'} s_j - \sum_{j\in J} s_j) \\ &\equiv \prod_{J\subset I_1^k} (\theta_{-k} + \sum_{j\in J} \theta_j + \sum_{j\in J'} \theta_{-j}), \end{aligned} \tag{3k}$$

for all k with $1 \le k \le M + 1$, because we have

$$\theta_k - \theta_{-k} \equiv s_k \qquad (1 \leq k \leq M+1),$$
$$\sum_{j=1}^{M+1} (\theta_j + \theta_{-j}) \equiv s_{M+2}.$$

By expanding (2k) ((3k) respectively) on θ_k (θ_{-k} respectively) for k with $1 \leq k \leq M+1$, we see that $b_k(s) \equiv (\cdots)D_k + d_1^k$ ($b_{-k}(s) \equiv (\cdots)D_{-k} + d_1^k$ respectively) where we define for k and r with $1 \leq k \leq M+1$ and $1 \leq r \leq M$

$$d_r^k := \prod_{J \subset I_r^k} (\sum_{j \in J} \theta_j + \sum_{j \in J'} \theta_{-j}).$$

Hence we need to find $Q'_k \in W[s]$ (Q'_{-k} respectively) $(1 \leq k \leq M+1)$ such that $d_1^k \equiv Q'_k D_k$ ($d_1^k \equiv Q'_{-k} D_{-k}$ respectively) for the factorization of $b_k(s)$ ($b_{-k}(s)$ respectively) by D_k (D_{-k} respectively). In order to describe d_r^k, we define for k and r with $1 \leq k \leq M+1$ and $1 \leq r \leq M$

$$d_r^k := \prod_{J \subset I_r^k} (\sum_{j \in J} \theta_j + \sum_{j \in J'} \theta_{-j}),$$

and for k and r with $1 \leq k \leq M+1$ and $1 \leq r \leq M-1$

$$e_r^k := \prod_{J \subset I_{r+1}^k} (\theta_{\sigma_k(r+1)} + \theta_{-\sigma_k(r+1)} + \sum_{j \in J} \theta_j + \sum_{j \in J'} \theta_{-j}).$$

Then we have for k and r with $1 \leq k \leq M+1$ and $1 \leq r \leq M-1$

$$\begin{aligned} d_r^k &= \prod_{J \subset I_{r+1}^k} (\theta_{\sigma_k(r+1)} + \sum_{j \in J} \theta_j + \sum_{j \in J'} \theta_{-j}) \\ &\qquad \times (\theta_{-\sigma_k(r+1)} + \sum_{j \in J} \theta_j + \sum_{j \in J'} \theta_{-j}) \\ &= \prod_{J \subset I_{r+1}^k} \{\theta_{\sigma_k(r+1)} \theta_{-\sigma_k(r+1)} + (\theta_{\sigma_k(r+1)} + \theta_{-\sigma_k(r+1)} + \sum_{j \in J} \theta_j + \sum_{j \in J'} \theta_{-j}) \\ &\qquad \times (\sum_{j \in J} \theta_j + \sum_{j \in J'} \theta_{-j})\} \\ &= A_r^k + e_r^k d_{r+1}^k \end{aligned}$$

where

$$A_r^k := B_r^k \theta_{\sigma_k(r+1)} \theta_{-\sigma_k(r+1)} \equiv B_r^k u_{\sigma_k(r+1)} u_{-\sigma_k(r+1)} D_k D_{-k},$$

$$B_r^k := \sum_{p=0}^{2^{M-r}-1} \{ \sum_{\substack{p \text{ distinct} \\ J_1, \ldots, J_p \subset I_{r+1}^k}} \prod_{t=1}^{p} (\theta_{\sigma_k(r+1)} + \theta_{-k(r+1)} + \sum_{j \in J_t} \theta_j + \sum_{j \in J'_t} \theta_{-j})$$

$$\times \Big(\sum_{j\in J_t}\theta_j+\sum_{j\in J'_t}\theta_{-j}\Big)\Big\}\,(\theta_{\sigma_k(r+1)}\theta_{-\sigma_k(r+1)})^{2^{M-r-p-1}}.$$

We assume the sums for zero distinct subset to be one. We thus obtain for k with $1\le k\le M+1$

$$d_1^k=\sum_{r=1}^{M-1}\Big(\prod_{i=1}^{r-1}e_i^k\Big)A_r^k+\Big(\prod_{i=1}^{M-1}e_i^k\Big)d_M^k.$$

Here we assume the products $\prod_{i=1}^{r-1}e_i^k$ $(1\le k\le M)$ for $r=1$ to be one. Since $d_M^k=\theta_{\sigma_k(M+1)}\theta_{-\sigma_k(M+1)}\equiv u_{\sigma_k(M+1)}u_{-\sigma_k(M+1)}D_kD_{-k}$, we have for k with $1\le k\le M+1$

$$\begin{aligned}d_1^k\equiv\Big\{&\sum_{r=1}^{M-1}\Big(\prod_{i=1}^{r-1}e_i^k\Big)B_r^ku_{\sigma_k(r+1)}u_{-\sigma_k(r+1)}\\&+\Big(\prod_{i=1}^{M-1}e_i^k\Big)u_{\sigma_k(M+1)}u_{-\sigma_k(M+1)}\Big\}D_kD_{-k}.\end{aligned}$$

Therefore if we define $Q_k, Q_{-k}\in W[s]$ $(1\le k\le M+1)$ by

$$\begin{aligned}Q_k:=&\sum_{p=0}^{2^M-1}\Big\{\sum_{\substack{p\text{ distinct}\\ J_1,\dots,J_p\subset I_1^k}}\prod_{t=1}^{p}\Big(\sum_{j\in J_t}\theta_j+\sum_{j\in J'_t}\theta_{-j}\Big)\Big\}\theta_k^{2^M-p-1}u_k\\&+\Big\{\sum_{r=1}^{M-1}\Big(\prod_{i=1}^{r-1}e_i^k\Big)B_r^ku_{\sigma_k(r+1)}u_{-\sigma_k(r+1)}\\&+\Big(\prod_{i=1}^{M-1}e_i^k\Big)u_{\sigma_k(M+1)}u_{-\sigma_k(M+1)}\Big\}D_{-k},\end{aligned}$$

$$\begin{aligned}Q_{-k}:=&\sum_{p=0}^{2^M-1}\Big\{\sum_{\substack{p\text{ distinct}\\ J_1,\dots,J_p\subset I_1^k}}\prod_{t=1}^{p}\Big(\sum_{j\in J_t}\theta_j+\sum_{j\in J'_t}\theta_{-j}\Big)\Big\}\theta_{-k}^{2^M-p-1}u_{-k}\\&+\Big\{\sum_{r=1}^{M-1}\Big(\prod_{i=1}^{r-1}e_i^k\Big)B_r^ku_{\sigma_k(r+1)}u_{-\sigma_k(r+1)}\\&+\Big(\prod_{i=1}^{M-1}e_i^k\Big)u_{\sigma_k(M+1)}u_{-\sigma_k(M+1)}\Big\}D_k,\end{aligned}$$

then we have the following theorem.

Theorem 4.1. *In the module* $M[s]$, *we have* $Q_kD_k\equiv b_k(s)$ *for all* k *with* $1\le|k|\le M+1$.

We replace e_r^k, B_r^k, etc. by operators in the subalgebra $W_X[s]:=\boldsymbol{C}[s]\otimes_{\boldsymbol{C}}\boldsymbol{C}[u_1,\dots,u_M,D_1,\dots,D_M]$ in order to restrict them on X. We define for k and r with $1\le k\le M$ and $1\le r\le M-1$

$$\begin{aligned} e_r^k(s) := \prod_{J \subset I_{r+1}^k} & (\theta_{\sigma_k(r+1)} - \sum_{j=1}^{r} \theta_{\sigma_k(j)} \\ & + \frac{1}{2}(s_{M+2} + \sum_{j=1}^{r} s_{\sigma_k(j)} + \sum_{j \in J} s_j - s_{\sigma_k(r+1)} - \sum_{j \in J'} s_j)), \end{aligned}$$

and for r with $1 \leq r \leq M-1$

$$e_r^{M+1}(s) := \prod_{J \subset I_{r+1}^{M+1}} (2\theta_{\sigma_{M+1}(r+1)} + \sum_{j \subset I_{r+1}^{M+1}} \theta_j - s_{\sigma_{M+1}(r+1)} - \sum_{j \in J'} s_j),$$

and for k and r with $1 \leq k \leq M$ and $1 \leq r \leq M-1$

$$\begin{aligned} B_r^k(s) := \sum_{p=0}^{2^{M-r-1}} \Bigg\{ \sum_{\substack{p \text{ distinct} \\ J_1, \ldots, J_p \subset I_{r+1}^k}} \prod_{t=1}^{p} \Bigg(& - \sum_{j=1}^{r+1} \theta_{\sigma_k(j)} + \frac{1}{2}(s_{M+2} \\ & + \sum_{j \in J_t} s_j + \sum_{j=1}^{r+1} s_{\sigma_k(j)} - \sum_{j \in J_t'} s_j) \Bigg) \\ & \times \Bigg(\theta_{\sigma_k(r+1)} - \sum_{j=1}^{r} \theta_{\sigma_k(j)} + \frac{1}{2}(s_{M+2} \\ & + \sum_{j \in J_t} s_j + \sum_{j=1}^{r} s_{\sigma_k(j)} - s_{\sigma_k(r+1)} - \sum_{j \in J_t'} s_j) \Bigg) \Bigg\} \\ & \times ((\theta_{\sigma_k(r+1)} - s_{\sigma_k(r+1)}) \theta_{\sigma_k(r+1)})^{2^{M-r-p-1}}, \end{aligned}$$

and for r with $1 \leq r \leq M-1$

$$\begin{aligned} B_r^{M+1}(s) := \sum_{p=0}^{2^{M-r-1}} \{ \sum_{\substack{p \text{ distinct} \\ J_1, \ldots, J_p \subset I_{r+1}^{M+1}}} \prod_{t=1}^{p} & (- \sum_{j \in J_t'} s_j + \sum_{j \in I_{r+1}^{M+1}} \theta_j) \\ & \times (- s_{\sigma_{M+1}(r+1)} + 2\theta_{\sigma_{M+1}(r+1)} - \sum_{j \in J_t'} s_j + \sum_{j \in I_{r+1}^{M+1}} \theta_j) \} \\ & \times ((\theta_{\sigma_{M+1}(r+1)} - s_{\sigma_{M+1}(r+1)}) \theta_{\sigma_{M+1}(r+1)})^{2^{M-r-p-1}}. \end{aligned}$$

We define for k with $1 \leq k \leq M$

$$\begin{aligned} R_k(s) := \sum_{p=0}^{2^{M-1}} \Bigg\{ \sum_{\substack{p \text{ distinct} \\ J_1, \ldots, J_p \subset I_1^k}} \prod_{t=1}^{p} \Bigg(& - \theta_k + \frac{1}{2}(s_{M+2} + s_k \\ & + \sum_{j \in J_t} s_j - \sum_{j \in J_t'} s_j) \Bigg) \Bigg\} \theta_k^{2^{M-p-1}}, \end{aligned}$$

$$R_{-k}(s) := \sum_{p=0}^{2^{M-1}} \Bigg\{ \sum_{\substack{p \text{ distinct} \\ J_1, \ldots, J_p \subset I_1^k}} \prod_{t=1}^{p} \Bigg(- \theta_k + \frac{1}{2}(s_{M+2} + s_k$$

$$+\sum_{j\in J_t} s_j - \sum_{j\in J_t'} s_j\Big)\Big\}(\theta_k - s_k)^{2^M-p-1},$$

$$R_{M+1}(s):=\sum_{p=0}^{2^M-1}\Big\{\sum_{\substack{p\text{ distinct}\\ J_1,\dots,J_p\subset[1,M]}}\prod_{t=1}^{p}\Big(-\sum_{j\in J_t'} s_j+\sum_{j=1}^{M}\theta_j\Big)\Big\}$$

$$\times\Big(-\sum_{j=1}^{M}\theta_j+\frac{1}{2}\sum_{j=1}^{M+2}s_j\Big)^{2^M-p-1},$$

and

$$R_{-(M+1)}(s):=\sum_{p=0}^{2^M-1}\Big\{\sum_{\substack{p\text{ distinct}\\ J_1,\dots,J_p\subset[1,M]}}\prod_{t=1}^{p}\Big(-\sum_{j\in J_t'} s_j+\sum_{j=1}^{M}\theta_j\Big)\Big\}$$

$$\times\Big(-\sum_{j=1}^{M}\theta_j - s_{M+1}+\frac{1}{2}\sum_{j=1}^{M+2}s_j\Big)^{2^M-p-1}.$$

We define for k with $1\le k\le M+1$

$$Q_k(s):=R_k(s)u_k+\Big\{\sum_{r=1}^{M-1}\Big(\prod_{i=1}^{r-1}e_i^k(s)\Big)B_r^k(s)u_{\sigma_k(r+1)}u_{-\sigma_k(r+1)}$$

$$+\Big(\prod_{i=1}^{M-1}e_i^k(s)\Big)u_{\sigma_k(M+1)}u_{-\sigma_k(M+1)}\Big\}D_{-k},$$

$$Q_{-k}(s):=R_{-k}(s)u_{-k}+\Big\{\sum_{r=1}^{M-1}\Big(\prod_{i=1}^{r-1}e_i^k(s)\Big)B_r^k(s)u_{\sigma_k(r+1)}u_{-\sigma_k(r+1)}$$

$$+\Big(\prod_{i=1}^{M-1}e_i^k(s)\Big)u_{\sigma_k(M+1)}u_{-\sigma_k(M+1)}\Big\}D_k.$$

Proposition 4.2. (1) *All operators* $R_k(s)$, $R_{-k}(s)$, $e_r^k(s)$ *and* $B_r^k(s)$ $(1\le k\le M+1,\ 1\le r\le M-1)$ *belong to the algebra* $W_X[s]$.
(2) *In the module* $M[s]$, *we have* $Q_k(s)D_k\equiv b_k(s)$ *for all* k *with* $1\le|k|\le M+1$.

Proof. Since we have $\theta_k-\theta_{-k}\equiv s_k$ $(1\le k\le M+1)$ and $\sum_{j=1}^{M+1}(\theta_j+\theta_{-j})\equiv s_{M+2}$, the proposition is obvious by Theorem 4.1.

For $\beta\in\boldsymbol{C}\otimes_{\boldsymbol{Z}}(\otimes_{i=1}^{M+2}\boldsymbol{Z}e_i)$, we define operators $R_k(\beta)$, $R_{-k}(\beta)$, $B_r^k(\beta)$ and $e_r^k(\beta)$ $(1\le k\le M+1,\ 1\le r\le M-1)$ in $W_X=\boldsymbol{C}[u_1,\dots,u_M,D_1,\dots,D_M]$ by plugging the values $s_j=s_j(\beta)$ $(1\le j\le M+2)$ in $R_k(s)$, $R_{-k}(s)$, $B_r^k(s)$ and $e_r^k(s)$ respectively. We define operators ${}_XQ_k(\beta)\in W_X$ $(1\le|k|\le M+1)$ for $\beta=\sum_{i=1}^{M+2}\beta_i e_i$ by

$$\begin{aligned}{}_XQ_k(\beta) &:= R_k(\beta)u_k + \{ \sum_{r=1}^{M-1} \prod_{i=1}^{r-1} e_i^k(\beta) B_r^k(\beta) u_{\sigma_k(r+1)} \\ &\qquad + \prod_{i=1}^{M-1} e_i^k(\beta) \} (\theta_k - \beta_k) \qquad (1 \leq k \leq M),\end{aligned}$$

$$\begin{aligned}{}_XQ_{-k}(\beta) &:= R_{-k}(\beta) + \{ \sum_{r=1}^{M-1} \prod_{i=1}^{r-1} e_i^k(\beta) B_r^k(\beta) u_{\sigma_k(r+1)} \\ &\qquad + \prod_{i=1}^{M-1} e_i^k(\beta) \} D_k \qquad (1 \leq k \leq M),\end{aligned}$$

$$\begin{aligned}{}_XQ_{M+1}(\beta) &:= -R_{M+1}(\beta) - \{ \sum_{r=1}^{M-1} \prod_{i=1}^{r-1} e_i^{M+1}(\beta) B_r^{M+1}(\beta) u_{\sigma_{M+1}(r+1)} \\ &\quad + \prod_{i=1}^{M-1} e_i^{M+1}(\beta) u_{\sigma_{M+1}(M+1)} \} \left(\sum_{j=1}^{M} \theta_j - \frac{1}{2} (\beta_{M+2} - \beta_{M+1} + \sum_{i=1}^{M} \beta_i) \right),\end{aligned}$$

$$\begin{aligned}{}_XQ_{-(M+1)}(\beta) &:= -R_{-(M+1)}(\beta) - \{ \sum_{r=1}^{M-1} \prod_{i=1}^{r-1} e_i^{M+1}(\beta) B_r^{M+1}(\beta) u_{\sigma_{M+1}(r+1)} \\ &\quad + \prod_{i=1}^{M-1} e_i^{M+1}(\beta) u_{\sigma_{M+1}(M+1)} \} (- \sum_{j=1}^{M} \theta_j + \frac{1}{2} \sum_{i=1}^{M+2} \beta_i).\end{aligned}$$

Let $\alpha = \sum_{j=1}^{M} (-c_j + 1) e_j + (b - a) e_{M+1} + (\sum_{j=1}^{M} c_j - a - b - M) e_{M+2} \in \boldsymbol{C} \otimes_{\boldsymbol{Z}} (\bigoplus_{i=1}^{M+2} \boldsymbol{Z} e_i)$.

Theorem 4.3. *We have contiguity relations*

$$\begin{aligned}b_k(\alpha) F_C &= \frac{ab}{c_k} {}_XQ_k(\alpha - \chi_k) F_C(a+1, b+1, c_k+1) \qquad (1 \leq k \leq M), \\ b_{M+1}(\alpha) F_C &= a {}_XQ_{M+1}(\alpha - \chi_{M+1}) F_C(a+1), \\ b_{-k}(\alpha) F_C &= (c_k - 1) {}_XQ_{-k}(\alpha - \chi_{-k}) F_C(c_k - 1) \qquad (1 \leq k \leq M), \\ b_{-(M+1)}(\alpha) F_C &= b {}_XQ_{-(M+1)}(\alpha - \chi_{-(M+1)}) F_C(b+1)\end{aligned}$$

where

$$\begin{aligned}b_k(\alpha) = \prod_{J \subset [1,M],\, J \ni k} &[(\sum_{j \in J'} (c_j - 1) - a) \\ &\times (\sum_{j \in J'} (c_j - 1) - b)] \qquad (1 \leq k \leq M),\end{aligned}$$

$$b_{M+1}(\alpha) = \prod_{J \subset [1,M]} (\sum_{j \in J'} (c_j - 1) - a),$$

$$b_{-k}(\alpha) = \prod_{J \subset [1,M], J \ni k} [(\sum_{j \in J} (c_j - 1) - a)$$

$$\times (\sum_{j \in J} (c_j - 1) - b)] \qquad (1 \le k \le M),$$

$$b_{-(M+1)}(\alpha) = \prod_{J \subset [1,M]} (\sum_{j \in J} (c_j - 1) - b).$$

Proof. For k with $1 \le k \le M$, we have $D_k \tilde{F}_C(\alpha) = (ab/c_k)\tilde{F}_C(\alpha - \chi_k)$. Hence we have for k with $1 \le k \le M$

$$\begin{aligned}\frac{ab}{c_k} Q_k(\alpha - \chi_k)\tilde{F}_C(\alpha - \chi_k) &= \frac{ab}{c_k} Q_k(s)\tilde{F}_C(\alpha - \chi_k)\\ &= Q_k(s)D_k\tilde{F}_C(\alpha)\\ &= b_k(s)\tilde{F}_C(\alpha)\\ &= b_k(\alpha)\tilde{F}_C(\alpha).\end{aligned}$$

Restricting the aboe equations on X, we obtain for all k with $1 \le k \le M$

$$\frac{ab}{c_k} {}_xQ_k(\alpha - \chi_k)F_C(a+1, b+1, c_k+1) = b_k(\alpha)F_C.$$

We can prove the other contiguity relations similarly.

5. The Lauricella functions F_A and F_B

The associated $\mathscr{A}$-HGS's to the Lauricella functions F_A and F_B are the same with different parameters. Since the calculation of obtaining nonobvious contiguity relations for F_A or F_B is very similar to the one for F_C, we present only the associated $\mathscr{A}$-HGS to F_A, the set $\{\varphi_\Gamma | \Gamma \in \mathscr{F}\}$, the nonobvious contiguity relations for F_A, and the ones for F_B.

The Lauricella function F_A around the origin of $\mathbf{C}^M$ is defined to be

$$\begin{aligned}&F_A(a, b_1, \dots, b_M; c_1, \dots, c_M; x_1, \dots, x_M)\\ &\quad = \sum_{n_1, \dots, n_M \ge 0} \frac{(a, \sum_{j=1}^M n_j)\prod_{j=1}^M (b_j, n_j)}{\prod_{j=1}^M (c_j, n_j) \cdot n_j!} \prod_{j=1}^M x_j^{n_j}.\end{aligned}$$

The associated $\mathscr{A}$-HGS to F_A of M variables is

$$(\theta_0 + \sum_{j=1}^M \theta_{2M+j} + a)\Phi = 0,$$

$$(\theta_j + \theta_{2M+j} + b_j)\Phi = 0 \qquad (1 \le j \le M),$$

$$\begin{aligned}(\theta_{M+j} - \theta_{2M+j} - c_j + 1)\Phi &= 0 \qquad (1 \le j \le M),\\ (D_0 D_j - D_{M+j} D_{2M+j})\Phi &= 0 \qquad (1 \le j \le M).\end{aligned}$$

We then have

$$\{\varphi_\Gamma \mid \Gamma \in \mathscr{F}\} = \left\{\begin{matrix} s_j,\ s_j + s_{M+j} & (1 \le j \le M) \\ s_0 + \sum\limits_{j\in J} s_{M+j} & (J \subset [1, M]) \end{matrix}\right\}.$$

By Theorem 2.3, we have

$$\begin{aligned}
b_0(s) &= \prod_{J\subset[1,M]} (s_0 + \sum_{j\in J} s_{M+j}),\\
b_k(s) &= s_k(s_k + s_{M+k}) && (1 \le k \le M),\\
b_{M+k}(s) &= (s_k + s_{M+k}) \prod_{J\subset[1,M],\, J\ni k} (s_0 + \sum_{j\in J} s_{M+j}) && (1 \le k \le M),\\
b_{2M+k}(s) &= s_k \prod_{J\subset[1,M],\, k\notin J} (s_0 + \sum_{j\in J} s_{M+j}) && (1 \le k \le M).
\end{aligned}$$

For each k with $1 \le k \le M$, we fix a permutation σ_k such that $\sigma_k(1) = \mathrm{k}$. We put $\sigma_0(i) = i (1 \le i \le M)$. For all k and r with $0 \le k \le M$ and $1 \le r \le M$, we define $I_r^k := [1, M] - \{\sigma_k(1), \dots, \sigma_k(r)\}$, and we set $I_0^0 := [1, M]$. For $J \subset I_r^k$ we denote by J' the complement of J in I_r^k.

We define for $\beta = \sum_{i=0}^{2M} \beta_i e_i \in \bigoplus_{i=0}^{2M} Ce_i$

$$e_{A,r}^k(\beta) := \prod_{J\subset I_{r+1}^k} (2\vartheta_{\sigma_k(r+1)} + \sum_{j\in I_{r+1}^k} \vartheta_j + \beta_{M+\sigma_k(r+1)} + \sum_{j\in J} \beta_{M+j}),$$

$$\begin{aligned}
B_{A,r}^k(\beta) := \sum_{p=0}^{2^{M-r-1}-1} \{ \sum_{\substack{p \text{ distinct}\\ J_1,\dots,J_p \subset I_{r+1}^k}} \prod_{t=1}^{p} (\beta_{M+\sigma_k(r+1)} + 2\vartheta_{\sigma_k(r+1)} + \sum_{j\in J_t} \beta_{M+j}\\
+ \sum_{j\in I_{r+1}^k} \vartheta_j)(\sum_{j\in J_t} \beta_{M+j} + \sum_{j\in I_{r+1}^k} \vartheta_j)\}\\
\times ((\vartheta_{\sigma_k(r+1)} + \beta_{M+\sigma_k(r+1)}) \cdot \vartheta_{\sigma_k(r+1)})^{2^{M-r-1}-p},
\end{aligned}$$

$$\begin{aligned}
C_0(\beta) := -\sum_{p=0}^{2^M-1} \sum_{\substack{p \text{ distinct}\\ J_1,\dots,J_p \subset [1,M]}} \prod_{t=1}^{p} (\sum_{j\in J_t} \beta_{M+j} + \sum_{j=1}^{M} \vartheta_j)\\
\times (\beta_0 - \sum_{j=1}^{M} \vartheta_j)^{2^M-1-p},
\end{aligned}$$

and

$$\begin{aligned} Q_0^A(\beta) := C_0(\beta) &+ \sum_{r=0}^{M-2}\prod_{i=0}^{r-1} e_i^0(\beta)B_r^0(\beta)x_{r+1}(\vartheta_{r+1}-\beta_{r+1}) \\ &+ \prod_{i=0}^{M-2} e_i^0(\beta)x_M(\vartheta_M-\beta_M). \end{aligned}$$

Here the product $\prod_{i=0}^{r-1} e_i^0(\beta)$ for $r=0$ is assumed to be one. For k with $1 \le k \le M$ we define

$$Q_k^A(\beta) := -(\vartheta_k + \beta_{M+k} - \beta_k) + x_k\Big(\sum_{j=1}^{M}\vartheta_j - \beta_0\Big).$$

We define for k with $1 \le k \le M$

$$f_0^k(\beta) := \prod_{J\subset I_1^k}\Big(\beta_0 + \beta_{M+k} + \sum_{j\in J}\beta_{M+j}\Big),$$

$$g_0^k(\beta) := \prod_{J\subset I_1^k}\Big(\beta_0 + \sum_{j\in J}\beta_{M+j}\Big),$$

$$\begin{aligned} f_0'^k(\beta) := &\sum_{p=0}^{2^{M-1}-1}\ \sum_{\substack{p\text{ distinct}\\ J_1,\dots,J_p\subset I_1^k}}\ \prod_{t=1}^{p}\Big(\beta_0 + \sum_{j\in J_t}\beta_{M+j} - \vartheta_k\Big) \\ &\times(\beta_k+\vartheta_k)^{2^{M-1}-p-1}, \end{aligned}$$

$$g_0'^k(\beta) := \sum_{p=0}^{2^{M-1}-1}\ \sum_{\substack{p\text{ distinct}\\ J_1,\dots,J_p\subset I_1^k}}\ \prod_{t=1}^{p}\Big(\beta_0 + \sum_{j\in J_t}\beta_{M+j} - \vartheta_k\Big)\vartheta_k^{2^{M-1}-p-1},$$

$$\begin{aligned} B_0^k(\beta) := &\sum_{p=0}^{2^{M-1}-1}\ \sum_{\substack{p\text{ distinct}\\ J_1,\dots,J_p\subset I_1^k}}\ \prod_{t=1}^{p}\Big(\sum_{j\in J_t}\beta_{M+j} + \sum_{j\neq k}\vartheta_j\Big) \\ &\times\Big(\beta_0 - \sum_{j=1}^{M}\vartheta_j\Big)^{2^{M-1}-p-1}. \end{aligned}$$

We define for k with $1 \le k \le M$

$$\begin{aligned} Q_{M+k}^A(\beta) := &\,(f_0^k(\beta) + f_0'^k(\beta)(\beta_k-\vartheta_k)) + B_0^k(\beta)\frac{\partial}{\partial x_k} \\ &+ \sum_{r=1}^{M-2}\prod_{i=1}^{r-1} e_i^k(\beta)B_r^k(\beta)x_{\sigma_k(r+1)}(\beta_{\sigma_k(r+1)} - \vartheta_{\sigma_k(r+1)})\frac{\partial}{\partial x_k} \\ &+ \prod_{i=1}^{M-2} e_i^k(\beta)x_{\sigma_k(M)}(\beta_{\sigma_k(M)} - \vartheta_{\sigma_k(M)})\frac{\partial}{\partial x_k}, \end{aligned}$$

$$Q_{2M+k}^A(\beta) := (g_0^k(\beta) + g_0'^k(\beta)(\beta_k-\vartheta_k))x_k + B_0^k(\beta)(\vartheta_k+\beta_{M+k})$$

$$
\begin{aligned}
&+ \sum_{r=1}^{M-2} \prod_{i=1}^{r-1} e_i^k(\beta) B_r^k(\beta) x_{\sigma_k(r+1)} \\
&\quad \times (- \vartheta_{\sigma_k(r+1)} + \beta_{\sigma_k(r+1)})(\vartheta_k + \beta_{M+k}) \\
&+ \prod_{i=1}^{M-2} e_i^k(\beta) x_{\sigma_k(M)}(- \vartheta_{\sigma_k(M)} + \beta_{\sigma_k(M)})(\vartheta_M + \beta_{2M}).
\end{aligned}
$$

Here the products $\prod_{i=1}^{r-1} e_i^k(\beta)$ $(1 \le k \le M)$ for $r = 1$ are assumed to be one.

Let $\alpha_A = - ae_0 - \sum_{j=1}^M b_j e_j + \sum_{j=1}^M (c_j - 1) e_{M+j}$.

Theorem 5.1 *For all k with $1 \le k \le M$, we have contiguity relations*

$$
\begin{aligned}
&a Q_0^A(\alpha_A - \chi_0) F_A(a + 1) = \prod_{J \subset [1,M]} (- a + \sum_{j \in J} (c_j - 1)) F_A, \\
&b_k Q_k^A(\alpha_A - \chi_k) F_A(b_k + 1) = b_k(b_k - c_k + 1) F_A, \\
&(c_k - 1) Q_{M+k}^A(\alpha_A - \chi_{M+k}) F_A(c_k - 1) \\
&\qquad\qquad = (c_k - b_k - 1) \prod_{J \subset [1,M], J \ni k} (- a + \sum_{j \in J} (c_j - 1)) F_A, \\
&\frac{ab_k}{c_k} Q_{2M+k}^A(\alpha_A - \chi_{2M+k}) F_A(a + 1, b_k + 1, c_k + 1) \\
&\qquad\qquad = - b_k \prod_{J \subset [1,M] - \{k\}} (- a + \sum_{j \in J} (c_j - 1)) F_A.
\end{aligned}
$$

The Lauricella function F_B around the origin of $\boldsymbol{C}^M$ is defined to be

$$
\begin{aligned}
&F_B(a_1, \dots, a_M, b_1, \dots, b_M; c; x_1, \dots, x_M) \\
&\quad = \sum_{n_1, \dots, n_M \ge 0} \frac{\prod_{j=1}^M (a_j, n_j) \prod_{j=1}^M (b_j, n_j)}{(c, \sum_{j=1}^M n_j) \prod_{j=1}^M n_j!} \prod_{j=1}^M x_j^{n_j}.
\end{aligned}
$$

For k and r and j with $1 \le k \le M$, $1 \le r \le M - 1$, $1 \le j \le M$ or $k = 0$, $0 \le r \le M - 1$, $1 \le j \le M$, we define

$$
\begin{aligned}
e_{rB}^k(\beta) := &\prod_{J \subset I_{r+1}^k} (2(\beta_{\sigma_k(r+1)} - \vartheta_{\sigma_k(r+1)}) + \sum_{j \in I_{r+1}^k} (\beta_j - \vartheta_j) \\
&+ \beta_{M+\sigma_k(r+1)} + \sum_{j \in J} \beta_{M+j}), \\
B_{rB}^k(\beta) := &\sum_{p=0}^{2^{M-r-1}-1} \{ \sum_{\substack{p \text{ distinct} \\ J_1, \dots, J_p \subset I_{r+1}^k}} \prod_{t=1}^{p} (\beta_{M+\sigma_k(r+1)} + 2(\beta_{\sigma_k(r+1)} - \vartheta_{\sigma_k(r+1)})
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j\in J_t}\beta_{M+j}+\sum_{j\in I_{r+1}^k}(\beta_j-\vartheta_j))\\
&\times(\sum_{j\in J_t}\beta_{M+j}+\sum_{j\in I_{r+1}^k}(\beta_j-\vartheta_j))\}\\
&\times((\beta_{\sigma_k(r+1)}+\beta_{M+\sigma_k(r+1)}-\vartheta_{\sigma_k(r+1)})\\
&\times(\beta_{\sigma_k(r+1)}-\vartheta_{\sigma_k(r+1)}))^{2^{M-r-1}-p},
\end{aligned}
$$

$$
\begin{aligned}
C_{0B}(\beta):=&\sum_{p=0}^{2^M-1}\sum_{\substack{p\text{ distinct}\\ J_1,\dots,J_p\subset[1,M]}}\prod_{t=1}^{p}(\sum_{j\in J_t}\beta_{M+j}+\sum_{j=1}^{M}(\beta_j-\vartheta_j))\\
&\times(\beta_0-\sum_{j=1}^{M}(\beta_j-\vartheta_j))^{2^M-1-p},
\end{aligned}
$$

$$
\begin{aligned}
f_{0B}'^k(\beta):=&\sum_{p=0}^{2^{M-1}-1}\sum_{\substack{p\text{ distinct}\\ J_1,\dots,J_p\subset I_1^k}}\prod_{t=1}^{p}(\beta_0+\sum_{j\in J_t}\beta_{M+j}-\beta_k+\vartheta_k)\\
&\times(2\beta_k-\vartheta_k)^{2^{M-1}-p-1},
\end{aligned}
$$

$$
\begin{aligned}
g_{0B}'^k(\beta):=&\sum_{p=0}^{2^{M-1}-1}\sum_{\substack{p\text{ distinct}\\ J_1,\dots,J_p\subset I_1^k}}\prod_{t=1}^{p}(\beta_0+\sum_{j\in J_t}\beta_{M+j}-\beta_k+\vartheta_k)\\
&\times(\beta_k-\vartheta_k)^{2^{M-1}-p-1},
\end{aligned}
$$

$$
\begin{aligned}
B_{0B}^k(\beta):=&\sum_{p=0}^{2^{M-1}-1}\sum_{\substack{p\text{ distinct}\\ J_1,\dots,J_p\subset I_1^k}}\prod_{t=1}^{p}(\sum_{j\in J_t}\beta_{M+j}+\sum_{j\neq k}(\beta_j-\vartheta_j))\\
&\times(\beta_0-\sum_{j=1}^{M}(\beta_j-\vartheta_j))^{2^{M-1}-p-1}.
\end{aligned}
$$

We define

$$
\begin{aligned}
Q_0^B(\beta):=C_{0B}(\beta)&+\sum_{r=0}^{M-2}\prod_{i=0}^{r-1}e_{iB}^0(\beta)B_{rB}^0(\beta)\frac{\partial}{\partial x_{r+1}}\\
&+\prod_{i=0}^{M-2}e_{iB}^0(\beta)\frac{\partial}{\partial x_M},
\end{aligned}
$$

and for k with $1\leq k\leq M$

$$
Q_k^B(\beta):=(\beta_{M+k}-\vartheta_k)x_k+(\sum_{j=1}^{M}\vartheta_j+\beta_0-\sum_{j=1}^{M}\beta_j),
$$

$$
Q_{M+k}^B(\beta):=-(f_0^k(\beta)+f_{0B}'^k(\beta)\vartheta_k)+B_{0B}^k(\beta)x_k(\vartheta_k-\beta_k)
$$

$$+ \sum_{r=1}^{M-2} \prod_{i=1}^{r-1} e_{iB}^{k}(\beta) B_{rB}^{k}(\beta) x_k(\vartheta_k - \beta_k) \frac{\partial}{\partial x_{\sigma_k(r+1)}}$$

$$+ \prod_{i=1}^{M-2} e_{iB}^{k}(\beta) x_k(\vartheta_k - \beta_k) \frac{\partial}{\partial x_{\sigma_k(M)}},$$

$$Q_{2M+k}^{B}(\beta) := -(g_0^k(\beta) + g_{0B}'^{k}(\beta)\vartheta_k) + B_{0B}^{k}(\beta) x_k(\vartheta_k - \beta_k - \beta_{M+k})$$

$$+ \sum_{r=1}^{M-2} \prod_{i=1}^{r-1} e_{iB}^{k}(\beta) B_{rB}^{k}(\beta) x_k(\vartheta_k - \beta_k - \beta_{M+k}) \frac{\partial}{\partial x_{\sigma_k(r+1)}}$$

$$+ \prod_{i=1}^{M-2} e_{iB}^{k}(\beta) x_k\Big(\vartheta_k + \beta_0 - \sum_{j=1}^{M} \beta_j\Big) \frac{\partial}{\partial x_{\sigma_k(M)}}.$$

Here the products $\prod_{i=0}^{r-1} e_{iB}^{0}(\beta)$ for $r = 0$ and $\prod_{i=1}^{r-1} e_{iB}^{k}(\beta)$ $(1 \le k \le M)$ for $r = 1$ are assumed to be one.

Let $\alpha_B = (c - 1 - \sum_{i=1}^{M} a_i)e_0 - \sum_{i=1}^{M} a_i e_i + \sum_{i=1}^{M} (a_i - b_i) e_{M+i}$.

Theorem 5.2. *For all k with $1 \le k \le M$, We have contiguity relations*

$$(c-1)Q_0^B(\alpha_B - \chi_0)F_B(c-1) = \prod_{J \subset [1,M]} \Big(c - 1 - \sum_{j \in J} b_j - \sum_{j \in J'} a_j\Big) F_B,$$

$$\frac{a_k b_k}{c} Q_k^B(\alpha_B - \chi_k) F_B(a_k + 1, b_k + 1, c + 1) = a_k b_k F_B,$$

$$b_k Q_{M+k}^B(\alpha_B - \chi_{M+k}) F_B(b_k + 1)$$

$$= -b_k \prod_{J \subset [1,M], J \ni k} \Big(c - 1 - \sum_{j \in J} b_j - \sum_{j \in J'} a_j\Big) F_B,$$

$$a_k Q_{2M+k}^B(\alpha_B - \chi_{2M+k}) F_B(a_k + 1)$$

$$= -a_k \prod_{J \subset [1,M], k \notin J} \Big(c - 1 - \sum_{j \in J} b_j - \sum_{j \in J'} a_j\Big) F_B.$$

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