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The First Eigenvalues of Some Abstract Elliptic Operators

By

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Dedicated to Professor Hiroki Tanabe on the occasion of his 60th birthday

1 Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega$, and consider the following well-known Poincaré's inequality:

$$\text{(P)} \qquad |u|_{L^p} \leq C|\nabla u|_{L^p} \quad u \in W_0^{1,p}(\Omega), \quad 1 < p < \infty.$$

Since the injection from $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact, it is easy to find an element $u \neq 0$ in $W_0^{1,p}(\Omega)$ which attains the best possible constant for (P), that is to say

$$R(u) = \sup \{R(v);\, v \in W_0^{1,p}(\Omega),\, v \neq 0\} =: \frac{1}{\lambda_1}, \qquad R(v) = |v|_{L^p}/|\nabla v|_{L^p}.$$

Then it can be shown that u must satisfy the equation:

$$(\text{E})_\lambda \qquad \begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \mathscr{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

with $\lambda = \lambda_1$, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

For the case $p = 2$, as a matter of course, λ_1 is **the first eigenvalue** of $-\Delta$ with zero Dirichlet boundary condition and it is well known that the associated eigenfunctions form a one dimensional linear subspace of $H_0^1(\Omega)$, i.e., λ_1 is **simple**. As for the case $p \neq 2$, this kind of result was first obtained by Ôtani [7] for the case $N = 1$, where it is shown that all the eigenvalues forms a countable set and they are all simple.

For higher dimension $N \geq 2$, Sakaguchi [11] showed that the first eigenvalue λ_1 is simple, provided that $\partial\Omega$ is connected. His method of proof relies on **a maximum principle** for $-\Delta_p$.

The main purpose of this paper is to present a method based on **"variational principle"** to show the following properties without assuming that $\partial\Omega$

is connected.

(I) The first eigenvalue λ_1 is simple;

(II) $(E)_\lambda$ has a positive solution if and only if $\lambda = \lambda_1$, (i.e., other eigenfunctions must change their sign.)

Anane [1] also proved these results. However, her method of proof essentially depends on the peculiarity of the operator Δ_p and it seems that this does not work for other types of operators. Our method of proof is quite different from those of [1], [11] and [12] and is based on the variational method. Therefore, as far as the simplicity of the first eigenvalue is concerned, the regularity of solutions required here is only $C(\Omega) \cap W_0^{1,p}(\Omega)$, much less than the usual regularity (say C^1 $(\bar{\Omega})$). Our results are formulated in an abstract form and can cover the eigenvalue problems for other different types of operators. Especially, we would like to emphasize that our abstract framework can be applied also for proving the uniqueness of positive solutions of some subprincipal elliptic equations, i.e., the homogeneous degree of the perturbed term is less than that of the principal term.

2 Main results

In order to formulate our results in an abstract form, we need the notion of subdifferential, a generalized notion of Fréchet derivative. Let V be a real Banach space with dual space V^* and let $\langle \cdot, \cdot \rangle$ be the duality between V^* and V. We denote by $\Phi(V)$ the family of all proper lower semicontinuous convex functions φ from V into $(-\infty, +\infty]$, where "proper" means that the effective domain $D(\varphi) = \{x \in V | \varphi(x) < +\infty\}$ of φ is not empty. The subdifferential $\partial\varphi$ of φ at u is defined by

$$\partial\varphi(u) = \{h \in V^*;\ \varphi(v) - \varphi(u) \geq \langle h, v - u \rangle \quad \forall v \in D(\varphi)\}$$

with domain $D(\partial\varphi) = \{u \in V;\ \partial\varphi(u) \neq \phi\}$. In general, $\partial\varphi$ is a multivalued operator. However, for the sake of simplicity, we here always assume that $\partial\varphi$ is single valued. If φ is convex and Fréchet differentiable, then the notion of Fréchet derivative coincides with the notion of subdifferential. (For fundamental facts, we refer to [2] and [10].)

Let us consider the following abstract eigenvalue problem:

$$(AE)_\lambda \qquad Au = \lambda Bu.$$

We impose the following conditions on A and B.

(A1) (i) $A = \partial f^1$ and $B = \partial f^2$ with $f^1, f^2 \in \Phi(V)$,

(ii) $D(f^1) = D(\partial f^2) = V$,
(iii) V is a function space defined on a domain $\Omega \subset \boldsymbol{R}^N$ such that $V \subset L^1_{\text{loc}}(\Omega)$.

(A2) (i) $R(|v|) \geq R(v) := f^2(v)/f^1(v)\ \forall v \in V$,
(ii) $f^1(v) \geq 0\ \forall v \in V$ and $f^1(v) = 0$ if and only if $v = 0$.
(iii) $\exists u \in V$ s.t. $u \neq 0$ and $R(u) = \sup\{R(v);\ v \in V,\ v \neq 0\} > 0$.

(A3) $\exists \alpha > 1$ s.t. $f^i(tv) = t^\alpha f^i(v)\ \forall v \in V^+ = \{w \in V;\ w(x) \geq 0$ a.e. $x \in \Omega\}$, $\forall t > 0$, $i = 1, 2$.

(A4) (i) $f^1(u \vee w) + f^1(u \wedge w) \leq f^1(u) + f^1(w)$, $\forall u, \forall w \in V^+$,
(ii) $f^2(u \vee w) + f^2(u \wedge w) \geq f^2(u) + f^2(w)$, $\forall u, \forall w \in V^+$,
where $(u \vee w)(x) = \max(u(x), w(x))$ and $(u \wedge w)(x) = \min(u(x), w(x))$.

Furthermore we assume

(A0) Every non-negative nontrivial solution u of $(\text{AE})_\lambda$ belongs to $C(\Omega) \cap L^\infty(\Omega)$ and satisfies $u(x) > 0$ for all $x \in \Omega$.

Then our main result can be stated as follows.

Theorem I. *Assume* (A0)-(A4). *Put* $\lambda_1 = 1/\sup\{R(v);\ v \in V,\ v \neq 0\}$. *Then we have*

(i) $(\text{AE})_\lambda$ *has no nontrivial solution for* $\lambda \in (0, \lambda_1)$,
(ii) λ_1 *is simple, i.e.,* $(\text{AE})_{\lambda_1}$ *has a positive solution and the set of all solutions of* $(\text{AE})_{\lambda_1}$ *is a one dimensional linear subspace of* V.

Theorem II. *Assume* (A0)-(A4) *and*

(A5) f^1 *is strictly convex,*

(A6) $B(u) \leq B(v)$ *if* $0 \leq u \leq v$.

Furthermore, we assume

(A0)′ *Every positive solution* u *of* $(\text{AE})_\lambda$ *satisfies* $u \in C^1(\bar{\Omega})$ *and* $\partial u/\partial n(x) < 0$ *on* $\partial\Omega$.

Then $(\text{AE})_\lambda$ *has a positive solution if and only if* $\lambda = \lambda_1$.

Remark 1.

(i) Take $V = W_0^{1,p}(\Omega)$, $f^1(v) = |\nabla v|^p_{L^p}/p$ and $f^2(v) = |v|^p_{L^p}/p$, then $(\text{E})_\lambda$ can be reduced to $(\text{AE})_\lambda$, and all conditions (A0)-(A4) above and (A5), (A6), (A0)′ in Theorem II are satisfied.

(ii) If $Au = \operatorname{div} \vec{a}(x, u; u) + a_0(x, u; u)$ and $Bu = b(x, u)$, then some sufficient conditions for (A0) can be given in terms of $\vec{a}$, a_0 and b. Since these are somewhat complicated, but general enough, we do not go into the details here. (See Ladyzhenskaya and Ural'tseva [5], Trudinger [14] and Tolksdorf [13].)

(iii) Conditions (A1)-(A4) are not enough to assure the simplicity of λ_1. In fact, it is easy to give a trivial counterexample by taking $f^1 = f^2 = f$ for a suitable f. In this case, $\lambda_1 = 1$ and the set of all eigenvectors becomes V. It is also possible to give a nontrivial counterexample: Let $V = \boldsymbol{R}^2$, and put $f^1(x) = x_1^4 + x_2^4$ and $f^2(x) = x_1^2 x_2^2$ for all $x = (x_1, x_2)$. Then it is easy to show that (A2)-(A4) are fulfilled and that $\lambda_1 = 2$ and the set of all eigenvectors is the set $\{x = (x_1, x_2); x_1 = x_2$ or $x_1 = -x_2\}$, which is not a linear subspace of V.

Remark 2.

To assure condition (A0)$'$, one must prove a Hopf-type maximum principle, and generally this is not so easy. In this sense, this is rather restrictive. However, in most cases, we can exclude this condition by applying some approximation procedure. (Say for the case $B(u) = |u|^{p-2}u$, set $B_\varepsilon(u) = b_\varepsilon(x)|u|^{p-2}u$ with $b_\varepsilon(x) = 1$ if $\operatorname{dis}(x, \partial\Omega) \geq \varepsilon$, $b_\varepsilon(x) = 0$ if $\operatorname{dis}(x, \partial\Omega) < \varepsilon$ and prove the corresponding first eigenvalue λ_1^ε converges to λ_1 as ε tends to zero. Then it suffices to repeat the same argument as in the proof of Theorem II for a sufficiently small ε.) The details will be discussed in a forthcoming paper.

3 Proofs of theorems

Proof of Theorem I.

(i) First of all, we see that (A3) implies

$$\langle \partial f^i(v), v \rangle = \alpha f^i(v), \qquad \forall v \in V,\ i = 1, 2.$$

Indeed, noting

$$\langle \partial f^i(v), v \rangle = \frac{1}{t-1} \langle \partial f^i(v), tv - v \rangle$$

$$\leq \frac{1}{t-1} \{ f^i(tv) - f^i(v) \} = \frac{t^\alpha - 1}{t-1} f^i(v) \qquad (\text{for } t > 1),$$

and letting $t \to 1 \pm 0$, we get the relation above. Hence, if u is a solution of $(\mathrm{AE})_\lambda$ with $\lambda \in (0, \lambda_1)$, then multiplication of $(\mathrm{AE})_\lambda$ by u gives

(1) $$\alpha f^1(u) = \lambda \alpha f^2(u), \quad \text{i.e.,} \quad R(u) = 1/\lambda.$$

This is a contradiction, since $1/\lambda = R(u) > 1/\lambda_1 = \sup R(v)$.
(ii) Set $J_\lambda(v) = f^1(v) - \lambda f^2(v)$, then

(2) $$J_\lambda(v) \underset{(=)}{<} 0 \quad \text{if and only if} \quad R(v) \underset{(=)}{>} 1/\lambda.$$

Therefore it follows from (iii) of (A2) that there exists $u \in V$ such that $u \neq 0$ and

(3) $$\min\{J_{\lambda_1}(v);\ v \in V,\ v \neq 0\} = 0 = J_{\lambda_1}(u).$$

Hence we get

$$f^1(v) - f^1(u) \geq \lambda_1 f^2(v) - \lambda_1 f^2(u) \geq \lambda_1 \langle \partial f^2(u), v - u \rangle \qquad \text{for all } v \in V.$$

Consequently, $\partial f^1(u) = \lambda_1 \partial f^2(u)$, i.e., u becomes a nontrivial solution of $(AE)_{\lambda_1}$. Conversely, if u is a solution of $(AE)_{\lambda_1}$, then, by (1), $J_{\lambda_1}(u) = 0$. Thus we find that

(4) $$u \text{ is a solution of } (AE)_{\lambda_1} \text{ if and only if } J_{\lambda_1}(u) = 0.$$

Furthermore, by (2), (3) and (i) of (A2), $J_{\lambda_1}(u) = 0$ implies $J_{\lambda_1}(|u|) = 0$. Thus

(5) If u is a solution of $(AE)_{\lambda_1}$, then $|u|$ is also a solution of $(AE)_{\lambda_1}$.

Then, by (A0), $|u|$ has no zero in Ω. Consequently, every nontrivial solution u of $(AE)_{\lambda_1}$ is positive or negative in Ω.

Let u, v be two positive solutions of $(AE)_{\lambda_1}$ and put

$$M(t, x) = \max(u(x), tv(x)) \quad \text{and} \quad m(t, x) = \min(u(x), tv(x)).$$

Then, by (A4), we get

$$0 \leq J_{\lambda_1}(M) + J_{\lambda_1}(m) \leq J_{\lambda_1}(u) + J_{\lambda_1}(tv) = J_{\lambda_1}(u) + t^\alpha J_{\lambda_1}(v) = 0,$$

whence follows $J_{\lambda_1}(M) = J_{\lambda_1}(m) = 0$. Hence, by (4), M and m turn out to be solutions of $(AE)_{\lambda_1}$ for all $t \geq 0$.

Here we note that since every solution u of $(AE)_\lambda$ belongs to $C(\Omega) \cap W_0^{1,p}(\Omega)$, u is absolutely continuous in each variable (on segments in Ω) for almost all values of the other variables, and its partial and generalized derivatives coincide almost everywhere, (see [6]). Moreover, by virtue of the facts that $v(x) > 0$ in Ω and $u, v \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, u/v also belongs to $C(\Omega) \cap W_{loc}^{1,p}(\Omega)$.

For a.e. $x_0 \in \Omega$, set $t_0 = u(x_0)/v(x_0) > 0$. Then, for any unit vector e, we have

$$u(x_0 + he) - u(x_0) \leq M(t_0, x_0 + he) - M(t_0, x_0),$$

$$t_0 v(x_0+he) - t_0 v(x_0) \le M(t_0, x_0+he) - M(t_0, x_0).$$

Dividing these inequalities by $h>0$ and $h<0$ and letting t tend to ± 0, we get

$$\nabla_x u(x_0) = \nabla_x M(t_0, x_0) = t_0 \nabla_x v(x_0).$$

Hence

$$\nabla_x \left(\frac{u}{v}(x_0) \right) = (\nabla_x u(x_0) v(x_0) - u(x_0) \nabla_x v(x_0))/v(x_0)^2 = 0.$$

Thus we see that $u(x)/v(x) = \text{Const.}$ in Ω. (QED)

Proof of Theorem II.
By using convex analysis, we can prove the following lemma.

Lemma 3. *Assume* (i) *of* (A4) *and* (A5). *Then $Au \le Av$ implies $u \le v$.*

Proof of Lemma 3.
Put $[u-v]^+(x) = \max(u(x)-v(x), 0)$, then it is easy to see that $[u-v]^+ = u \vee v - v = u - u \wedge v$. Then taking the duality between $Av - Au$ and $[u-v]^+$, we find, by (i) of (A4),

$$\begin{aligned} 0 &\le \langle Av - Au, [u-v]^+ \rangle \\ &= \langle Av, u \vee v - v \rangle + \langle Au, u \wedge v - u \rangle \\ &\le f^1(u \vee v) - f^1(v) + f^1(u \wedge v) - f^1(u) \le 0, \end{aligned}$$

which implies $\langle Av, u \vee v - v \rangle = f^1(u \vee v) - f^1(v)$.

Hence we obtain

$$\begin{aligned} t\{f^1(u \vee v) - f^1(v)\} &= \langle Av, t(u \vee v) + (1-t)v - v \rangle \\ &\le f^1(t(u \vee v) + (1-t)v) - f^1(v) \\ &\le t f^1(u \vee v) + (1-t) f^1(v) - f^1(v) \\ &= t\{f^1(u \vee v) - f^1(v)\}, \end{aligned}$$

whence follows

$$f^1(t(u \vee v) + (1-t)v) = t f^1(u \vee v) + (1-t) f^1(v).$$

Then the strict convexity of f^1 says that $u \vee v = v$, i.e. $u \le v$. (QED)

Suppose now that $(\mathrm{AE})_\lambda$ with $\lambda > \lambda_1$ has a positive solution v, and let u be a positive solution of $(\mathrm{AE})_{\lambda_1}$. By virtue of (A0)′ and the fact that tv is also a solution of $(\mathrm{AE})_\lambda$, we may assume without loss of generality that $u \le v$. Then, by (A.6), we get

$$Au = \lambda_1 Bu \le \lambda_1 Bv = \lambda B(\eta v) = A(\eta v) \quad \text{with} \quad \eta = (\lambda_1/\lambda)^{1/(\alpha-1)} < 1,$$

where we used the fact that B is a homogeneous operator of order $\alpha - 1$. Hence it follows from Lemma 3 that $u \leq \eta v$. Now, repeating this argument n times, we obtain $0 \leq u \leq \eta^n v$. Therefore, by letting n tend to infinity, we finally deduce $u \equiv 0$. This is a contradiction. (QED)

4 Applications

4.1 The Eigenvalue problems

As is mentioned above, our abstract framework can cover some problems more complicated than $(E)_\lambda$. For example, let $V = W_0^{1,p}(\Omega)$ and put

$$f^1(u) = \frac{1}{p} \int_\Omega \{(a_1(x)|u|^2 + a_2(x)|\nabla u|^2)^{p/2} + a^3(x)|u|^p\}\, dx,$$

$$f^2(u) = \frac{1}{p} \int_\Omega b(x)|u|^p dx,$$

where a_i $(i = 1, 2, 3)$, $b \in L^\infty(\Omega)$; $a_1(x)$, $a_3(x)$, $b(x) \geq 0$, $a_2(x) \geq \rho > 0$ a.e. $x \in \Omega$.

Then

$$\begin{aligned} Au = \partial f^1(u) &= \operatorname{div}((a_1|u|^2 + a_2|\nabla u|^2)^{(p-2)/2} a_2 \nabla u) \\ &\quad + (a_1|u|^2 + a_2|\nabla u|^2)^{(p-2)/2} a_1 u + a_3|u|^{p-2}u \\ Bu = \partial f^2(u) &= b|u|^{p-2}u. \end{aligned}$$

It is obvious that (i) of (A2), (A3) with $\alpha = p$, (A5) and (A6) hold. Furthermore, (ii) and (iii) of (A2) are assured by Poincaré's inequality and Rellich's compactness theorem respectively.

By virtue of Stampacchia's theorem (see Kinderlehrer and Stampacchia [4]), $u \vee w$ and $u \wedge w$ also belong to $V^+ = (W_0^{1,p}(\Omega))^+$ for all $u, w \in V^+$ and the equalities in (i) and (ii) of (A4) hold. Applying the same Moser's method as in [8], we can derive the $L^\infty(\Omega)$ bound for solutions of $(AE)_\lambda$. Hence, it follows from the results by Ladyzhenskaya and Ural'tseva [5] and Trudinger [14] that every non-negative solution u of $(AE)_\lambda$ belongs to $C^\alpha(\bar{\Omega})$ and $u(x) > 0$ for all $x \in \Omega$. Thus conditions (A0)-(A6) are all fulfilled for this case, so the assertion of Theorem I is valid.

For the case where $a_1 \equiv 0$, the much stronger condition (A0)′ can be verified by Proposition 3.7 of [13] and Lemma 4 of [9] together with Lemma 3. Then Theorem II is applicable for this case.

4.2 The uniqueness of positive solutions for sub-principal elliptic equations

It should be noted that our abstract framework for eigenvalue problems can cover apparently different type of problems, i.e., the uniqueness problems for positive solutions of some elliptic equations with sub-principal terms. A typical example is stated as follows.

Theorem 4. *Let* $b \in L^\infty(\Omega)$, $b(x) \geq 0$ *a.e.* $x \in \Omega$ *and* $1 < q < p$. *Then*

$$(6) \qquad \begin{cases} -\Delta_p u = b(x)|u|^{q-2}u & \text{in } \mathscr{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega) \setminus \{0\},\ u(x) \geq 0 & \text{a.e. } x \in \Omega \end{cases}$$

has a unique solution.

Proof of Theorem 4.
The existence part is easy, (see e.g. [8]). Let $V = W_0^{1,p}(\Omega)$, and put $f^1(v) = \frac{1}{p}\int_\Omega |\nabla v|^p dx$ and $f^2(v) = \frac{1}{p}(\int_\Omega b(x)|v|^q dx)^{p/q}$. Then $(\mathrm{AE})_\lambda$ becomes

$$(7) \qquad -\Delta_p u = \lambda |b^{1/q}u|_{L^q}^{p-q} b(x)|u|^{q-2}u,$$

and all assumptions in Theorems I and II are satisfied. (f^2 satisfies (ii) of (A4) if and only if $q \leq p$.)

Let u and v be different solutions of (6). Then u and v satisfy (7) with $\lambda = |b^{1/q}u|_{L^q}^{q-p}$ and $\lambda = |b^{1/q}v|_{L^q}^{q-p}$ respectively. Then Theorems I and II say that

$$\lambda_1 = |b^{1/q}u|_{L^q}^{q-p} = |b^{1/q}v|_{L^q}^{q-p} \quad \text{and} \quad u = tv \text{ for some } t > 0,$$

whence follows $u = v$. (QED)

Remark 5.
A same type of result as Theorem 4 is already obtained by Diaz and Saa [3]. But their result does not cover the case where meas $\{x \in \Omega;\ b(x) = 0\} > 0$.

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