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Non Convex Set-Valued Systems in Banach Spaces[1]

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§ 0. Introduction

In this paper we give sufficient conditions for the solvability of set-valued systems of the form

$$\begin{cases} 0 \in F(x, y) \\ 0 = g(x, y) \end{cases}, \tag{1}$$

where F and g are defined in the following way

$$F(x, y) = y - \bar{F}(x, y)$$

and

$$g(x, y) = x - \bar{g}(x, y),$$

where $x \in X$, $y \in Y$, with X, Y Banach spaces, $\bar{F}$ is an upper semicontinuous, compact multivalued map, $\bar{g}$ is a continuous map; both maps are defined on the closure of a suitable open subset U of the product $X \times Y$ and take values in Y and X respectively. Our approach is a development, in various aspects, of the one introduced in [8] in the case when $\bar{F}$ is a continuous, compact, singlevalued map. Roughly speaking, we solve the first equation in terms of y as a function of "the parameter" x considering the application $x \multimap S(x) = \{y \in Y: 0 \in F(x, y)\}$, and hence we introduce the solution set $S(x)$ in the second one. The fixed points of the composite function $\bar{g}(x, S(x))$ are the solutions of system (1). In recent years upper semicontinuous multivalued maps with non convex values have been studied by several authors, see e.g. [1], [3], [6], [7]. The paper is organized as follows.

In Section 1 we introduce the definition and the properties of UV^{∞} sets and recall a graph approximation theorem for upper semicontinuous multivalued maps with UV^{∞} values, given independently in [1] and [7]. Then we state the definitions of weighted maps, introduced in [4].

In Section 2 we state two existence results for system (1).

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In Section 3 an application to control theory of the obtained results, is given.

§ 1. Definitions, notations and preliminary results

Definition 1.1: Let X and Y be metric spaces. A set-valued map M from X into Y, with nonempty values, is said to be *upper semicontinuous at* $x \in X$ if for any neighborhood V of $M(x)$ there exists a neighborhood U of x such that $M(u) \subset V$ for any $u \in U$. If, for every $x \in X$, M is upper semicontinuous at x, then M is said to be upper semicontinuous (u.s.c.) on X.

If M sends bounded sets into relatively compact sets, then it is said *compact*. M is said *proper* if, for each compact set K of Y, $M^{-1}(K)$ is compact.

We will denote a multivalued map M from X to Y with the symbol $M: X \multimap Y$. It is well known that, if $\overline{M(X)}$ is compact and $M(x)$ is closed for any $x \in X$, then M is u.s.c. if and only if M has closed graph, i.e. $x_n \in X$, $x_n \to x_0$, $y_n \to y_0$, $y_n \in M(x_n)$ implies $y_0 \in M(x_0)$.

The multivalued map $M: X \multimap Y$ is said *lower semicontinuous* (l.s.c.) on X if, for every $x \in X$ and every neighborhood V of every $y \in M(x)$, there exists a neighborhood U of x such that $M(u) \cap V \neq \emptyset$ for all $u \in U$. If $M: X \multimap Y$ is both u.s.c. and l.s.c. on X, then M is said *continuous* on X.

Definition 1.2: Let X be a metric space and let K be a compact subset of X. We say that K is a UV^∞ set if, for any neighborhood U of K, there exists a neighborhood V, $K \subset V \subset U$, such that any two points in V can be joined by a path in U, and there is a base point $y_0 \in K$ such that the inclusion of V into U induces the trivial homomorphism $\pi_n(V, y_0) \to \pi_n(U, y_0)$ for each positive integer n.

Definition 1.3: Let X and Y be metric spaces and let $M: X \multimap Y$ be an upper semicontinuous multivalued mapping. We say that M is a *UV^∞-mapping* if the set $M(x) \subset Y$ is UV^∞ for each $x \in X$.

Definition 1.4: A map $\mu: X \to Y$ is said to be an *ε-graph approximation*, shortly ε-approximation, of the multivalued map $M: X \multimap Y$ if $\mathrm{Gr}\,\mu \subset \varepsilon\,\mathrm{Gr}\,M$, where $\mathrm{Gr}\,M$ denotes the graph of M and $\varepsilon\,\mathrm{Gr}\,M$ denotes the set of points of distance less than ε from the set $\mathrm{Gr}\,M$. Moreover we will say that a map $\mu: X \to Y$ is a *ε-pointwise approximation* for $M: X \multimap Y$ if $\mu(x) \in \varepsilon M(x)$ for all $x \in X$.

Lemma 1.1: ([7]) *If for a compact set K, one of the following conditions holds*

1) *K is a fundamental absolute retract;*
2) *K has trivial shape;*
3) *K is an R_δ − set;*
4) *K is an absolute retract;*
5) *K is contractible;*
then K is a UV^∞ set.

The classes of absolute retracts and absolute neighborhood retracts are denoted by AR and ANR, respectively, (see [5]).

Theorem 1.1: ([1], [7]): *Let X be a compact ANR space, Y a metric space. For any $\varepsilon > 0$ and any UV^∞-mapping $M: X \multimap Y$, there exists a continuous ε-approximation, $\mu: X \to Y$ of M.*

We want to recall here (see [7]) that for an UV^∞-mapping the notion of fixed point index with the usual properties is well defined. The relative definition is given by using Theorem 1.1.

Definition 1.5: Let U be an open subset of a compact ANR space X. Denote by ∂U the boundary of U. Let $\phi: \bar{U} \to E$ be an UV^∞ mapping. Assume that it is fixed points free on ∂U. Then we can define $\mathrm{ind}(X, \phi, U)$ (see [7]). Notice that this index satisfies all the standard properties.

Definition 1.6: Let X and Y be topological Hausdorff spaces. A finite valued upper semicontinuous map $M: X \multimap Y$ will be called a *weighted map* (shortly w-map) if, to each x and $y \in M(x)$ a multiplicity or weight $\mathrm{m}(y, M(x)) \in \boldsymbol{Z}$ is assigned in such a way that the following property holds:
if U is an open set in Y with $\partial U \cap M(x) = \emptyset$, then

$$\sum_{y \in M(x) \cap U} \mathrm{m}(y, M(x)) = \sum_{y' \in M(x') \cap U} \mathrm{m}(y', M(x'))$$

whenever x' is close enough to x, (see [4] and [8]).

Definition 1.7: The number $\mathrm{i}(M(x), U) = \sum_{y \in M(x) \cap U} \mathrm{m}(y, M(x))$ will be called the *index* or *multiplicity* of $M(x)$ in U. If U is a connected set, the number $\mathrm{i}(M(x), U)$ does not depend on $x \in X$. In this case the number $\mathrm{i}(M) = \mathrm{i}(M(x), U)$ will be called the *index of the weighted map M*.

Definition 1.8: Let X be a Banach space and let $\bar{B}(0, r)$ be the closed ball in X of radius r, centered at the origin. We will say that the upper semicontinuous map $M: \bar{B}(0, r) \multimap X$ verifies the *Borsuk–Ulam (B.U.) property on $\partial B(0, r)$* if for all $x \in \partial B(0, r)$ for which $0 \notin M(x)$, $M(x)$ and $M(-x)$ are strictly separated by an hyperplane, i.e. for all $x \in \partial B(0, r)$ there exists a continuous functional $x^* \in X^*$, the dual space of X, such that $x^*(y) > 0$ for all $y \in M(x)$ and $x^*(y) < 0$ for all $y \in M(-x)$.

For sake of simplicity, in the sequel we will write B instead of $B(0, r)$.

Definition 1.9: Let X, Y be metric spaces. Given $S \subset X \times Y$ we denote by

$$S(x) = \{y \in Y: (x, y) \in S\};$$

$$S(A) = \{y \in Y: (x, y) \in S,\ x \in A\};$$

$$S_x = S \cap (\{x\} \times Y) \quad \text{and} \quad S_A = S \cap (A \times Y) \quad \text{for } A \subset X;$$

$$S^F = \{(x, y) \in \bar{U}: y \in \bar{F}(x, y)\};$$

$$\mathscr{D}^F = \{x \in X: S_x^F \cap \partial U = \emptyset\}.$$

If F is defined in all $X \times Y$ then $\mathscr{D}^F = \{x \in X$: x is not a bifurcation point from infinity of the equation $0 \in F(x, y)\}$.

§ 2. An existence result

We want now to prove the existence of solutions for the system

$$\text{(1)} \qquad \begin{cases} y \in \bar{F}(x, y) \\ x = \bar{g}(x, y) \end{cases} \quad \text{or} \quad \begin{cases} 0 \in y - \bar{F}(x, y) = F(x, y) \\ 0 = x - \bar{g}(x, y) = g(x, y). \end{cases}$$

For this we have the following.

Theorem 2.1: *Let X be a Banach space and let K be a compact, convex subset of a Banach space Y. Let U be a relatively open, bounded, convex set in $X \times K$, $\bar{F}: \bar{U} \multimap K$ an UV^∞-mapping and $\bar{g}: \bar{U} \to X$ a continuous compact map. Suppose that there exists $r > 0$ such that $\bar{B} \subset \mathscr{D}^F$ and that* $\operatorname{ind}(K, \bar{F}(0, \cdot), U(0)) \neq 0$. *Let $T: \bar{B} \multimap X$ be the application defined by $T(x) = x - \bar{T}(x)$, where $\bar{T}(x) = \bar{g}(x, S(x))$ and $S(x) = \{y \in K: y \in \bar{F}(x, y)\}$. Let us suppose that for any $x \in \partial B$ such that $0 \notin T(x)$, $T(x)$ and $T(-x)$ are strictly separated by an hyperplane. Then system* (1) *has a solution.*

First, we prove that the map $x \multimap S(x)$ is upper semicontinuous. Observe that, under our assumptions, $S(x)$ is compact and nonempty for every $x \in \mathscr{D}^F$.

Lemma 2.1: *The multivalued map $x \multimap S(x)$ is u.s.c. at every $x \in \mathscr{D}^F$.*

Proof. Let $x_0 \in \mathscr{D}^F$, we shall prove that S is upper semicontinuous at x_0, that is, if V is an open neighborhood of $S(x_0)$, then there exists an open neighborhood N of x_0, $N \subset \mathscr{D}^F$, such that $S(x) \subset V$, $\forall x \in N$. To prove this, let $y \in S(x_0)$ and let us consider neighborhoods of the form $N_{x_0} \times V_y$, with N_{x_0} a neighborhood of x_0 in $\mathscr{D}^F$ and V_y a neighborhood of y in K, such that

$$V_y \subset U(x_0) \cap V \qquad \text{and} \qquad N_{x_0} \times V_y \subset U\,.$$

By the compactness of $S_{x_0} = S \cap (\{x_0\} \times K)$ there exists a finite number, say s, of neighborhoods of the previous form covering S_{x_0}. Let

$$N_0 = \bigcap_{i=1}^{s} N_i \qquad \text{and} \qquad V' = \bigcup_{i=1}^{s} V_i\,.$$

Clearly for each neighborhood N of x_0, with $N \subset N_0$, we have that $N \times V' \subset U$ and $V' \subset V$. Let us prove that there exists a neighborhood N of x_0 such that $S(x) \subset V'$ for all $x \in N$. Suppose not, then there exists a bounded sequence $\{(x_n, y_n)\}$ with $x_n \to x_0$, $y_n \in \bar{F}(x_n, y_n)$ and $y_n \notin V'$. As $\bar{F}$ is upper semicontinuous with compact values and K is compact, we may assume (by passing to a subsequence, if necessary) that $y_n \to y_0$. Then $y_0 \in \bar{F}(x_0, y_0)$, that is $y_0 \in S(x_0)$, contradicting $S(x_0) \subset V'$. □

Lemma 2.2: *Let $X = \boldsymbol{R}^n$, K as in Theorem 2.1 and let $\bar{B} \subset \mathscr{D}^F$. Then for each neighborhood W of S_B^F there exists $\varepsilon > 0$ such that if $\bar{f}\colon \bar{U}_B \to K$ is an ε-approximation of $\bar{F}_{\backslash \bar{U}_B}$, then $S_B^f \subset W$.*

Proof. S_B^F is a closed set. In fact, let $\{(x_n, y_n)\} \subset S_B^F$ and $(x_n, y_n) \to (x_0, y_0)$. As $\{(x_n, y_n)\} \subset S_B^F$, we have that $y_n \in \bar{F}(x_n, y_n)$ for any $n \in \boldsymbol{N}$. As $\bar{F}$ is upper semicontinuous with compact values, then $y_0 \in \bar{F}(x_0, y_0)$, that is $(x_0, y_0) \in S_B^F$. Then S_B^F as a closed subset of $\bar{B} \times K$ is a compact set. Let W be a neighborhood of S_B^F, V an ε_1-neighborhood of S_B^F, $V \subset W$, with $\partial V \cap \partial W = \emptyset$. Let $\varepsilon_2 = \mathrm{d}(\partial V, \partial W)$ and $A = \bar{U}_B \backslash V$. Since A is a compact set, we have that

$$\inf_{(\tilde{x}, \tilde{y}) \in A} \{|s - \tilde{y}|,\, s \in \bar{F}(\tilde{x}, \tilde{y})\} = \varepsilon_3 > 0\,.$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and let $\bar{f}\colon \bar{U} \to Y$ be an ε-approximation of $\bar{F}_{\backslash \bar{U}_B}$. Let $(x, y) \in \bar{U}_B \setminus W$ and let $y = \bar{f}(x, y)$, so (x, y) is in S_B^f, then there exists $(\bar{x}, \bar{y}) \in \bar{U}_B$ such that

$$|(x, y) - (\bar{x}, \bar{y})| + |\bar{f}(x, y) - z| < \varepsilon \qquad \text{for some } z \in \bar{F}(\bar{x}, \bar{y})\,.$$

As $|(x, y) - (\bar{x}, \bar{y})| < \varepsilon$ it follows that $(\bar{x}, \bar{y}) \notin S_B^F$. Since $y = \bar{f}(x, y)$ we get that $|\bar{y} - z| < \varepsilon$, then $(\bar{x}, \bar{y}) \notin A$. Thus $(\bar{x}, \bar{y}) \in V \backslash S_B^F$. This is an absurd, since $(x, y) \in \bar{U}_B \backslash W$ and $|(x, y) - (\bar{x}, \bar{y})| < \varepsilon$, and so $(\bar{x}, \bar{y}) \notin V$.

Lemma 2.3: *Let $X = \boldsymbol{R}^n$ and let K be a compact, convex subset of $Y = \boldsymbol{R}^m$. There exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ there exists $\bar{f}\colon U \to Y$, such that $\bar{f}_{\backslash \bar{U}_B}$ is an ε-approximation of $\bar{F}_{\backslash \bar{U}_B}$ and*

a) *S_B^f is a finite subset of $U(x)$, $\forall x \in \bar{B}$;*

b) $\mathrm{ind}(K, \bar{F}(0, \cdot), U(0)) = \mathrm{ind}(K, \bar{f}(0, \cdot), U(0))$.

Proof. Theorem 4.5 in [7] ensures the existence of a positive number, say ε_0, with the property that for any $\varepsilon < \varepsilon_0$ every ε-approximation of $\bar{F}_{\backslash \bar{U}_B}$ has the same index of $\bar{F}_{\backslash \bar{U}_B}$. Fix $\varepsilon < \varepsilon_0$ and let $\tilde{f}\colon \bar{U}_B \to K$ be an $\varepsilon/2$ approximation of $\bar{F}_{\backslash \bar{U}_B}$. Using the same arguments of Lemma 2.4 of [8] we can prove that there exists $\tilde{f}_1\colon \bar{U}_B \to K$, $\varepsilon/2$-pointwise approximation of $\tilde{f}$, that satisfies property a). Then $\tilde{f}_1$ is an ε-approximation of $\bar{F}_{\backslash \bar{U}_B}$ that satisfies property a). The map $\bar{f}$ is then any continuous extension of $\bar{U}$ of $\tilde{f}_1$, and b) follows immediately from the choice of ε. □

Lemma 2.4: *Let X be a Banach space and K a convex, compact subset of any Banach space Y. Let U be an open subset in $X \times K$. Let us suppose that for all $x \in \mathscr{D}^F$ the equation $y \in \bar{F}(x, y)$ has only isolated solutions. Then the application $x \multimap S(x)$ is a w-map and* $\mathrm{i}(S) = \mathrm{ind}(K, \bar{F}(x, \cdot), U(x))$, *for any $x \in \mathscr{D}^F$.*

Proof. We have already seen that the map $x \multimap S(x)$ is an upper semicontinuous map. We want to prove that it is possible to associate an integer $\mathrm{m}(y, S(x))$ to any $y \in S(x)$ with the property of Definition 1.6. If y is an isolated solution for $\bar{F}$, then there exists a neighborhood Ω of y such that $\Omega \cap S(x) = \{y\}$. Let us define $\mathrm{m}(y, S(x)) = \mathrm{ind}(K, \bar{F}(x, \cdot), \Omega)$. Using the excision property of the fixed point index, $\mathrm{m}(y, S(x))$ does not depend on the choice of Ω.

Let W be an open set in K such that $S(x) \cap \partial W = \emptyset$. As S is upper semicontinuous there exists a ball $\bar{B}(x, r)$ such that $\forall x' \in \bar{B}(x, r)$ we get $S(x') \cap \partial W = \emptyset$.

Let us consider now the following homotopy:
$H\colon [0, 1] \times W \multimap K$ defined by

$$H(t, y) = \bar{F}(tx + (1 - t)x', y)\,.$$

As $tx + (1 - t)x' \in \bar{B}(x, r)$ for all $t \in [0, 1]$, H is an admissible homotopy between $\bar{F}(x, \cdot)_{\backslash W}$ and $\bar{F}(x', \cdot)_{\backslash W}$. From this fact, using the additivity property of the index, we get

$$\sum_{y \in S(x) \cap W} \mathrm{m}(y, S(x)) = \mathrm{ind}(K, \bar{F}(x, \cdot), W) = \mathrm{ind}(K, \bar{F}(x', \cdot), W)$$

$$= \sum_{y \in S(x') \cap W} \mathrm{m}(y, S(x'))\,.$$ □

The following lemmata, whose proof can be found in [8], also hold.

Lemma 2.5: *Let B be an open ball in $\boldsymbol{R}^n$ and let $M\colon B \multimap \boldsymbol{R}^n$ be a w-map with* $\mathrm{i}(M) \neq 0$. *If M verifies the B.U. condition on ∂B, then there exists $x \in \bar{B}$ such that $0 \in M(x)$.*

Lemma 2.6: *Let $M = I - \bar{M}\colon \bar{B} \multimap X$ be a compact vector field satisfying* *the B.U. property. Then there exists* $\bar{\varepsilon} > 0$ *such that every* ε*-approximation of* M, $0 < \varepsilon < \bar{\varepsilon}$, *satisfies the B.U. property.*

We can now give the proof of Theorem 2.1

Proof of Theorem 2.1. Observe that the system (1) has solutions if and only if $0 \in T(x)$, for some $x \in \bar{B}$. Assume to the contrary that $0 \notin T(x)$ for all $x \in \bar{B}$. Then there exists $\varepsilon_1 > 0$ such that $0 \notin \varepsilon_1 T(\varepsilon_1 x)$ for all $x \in \bar{B}$. In fact if not, then there exist sequences $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, $\{x_n\} \subset \bar{B}$ such that $0 \in \varepsilon_n T(\varepsilon_n x_n)$. It follows that there exists $\{x'_n\} \subset \bar{B}$ and $\{y_n\}$ such that $y_n \in T(x'_n)$, $|y_n| < \varepsilon_n$, $|x'_n - x_n| < \varepsilon_n$. As $y_n \to 0$ and T is an upper semicontinuous compact vector field, there exists a subsequence $\{x'_{n_k}\} \subset \{x'_n\}$ such that $x'_{n_k} \to x \in \bar{B}$, and $0 \in T(x)$, absurd. On the other hand from Lemma 2.6 there exists ε_2 such that every ε_2-approximation T' of T, $T': \bar{B} \multimap X$ verifies the B.U. property. Let $\delta = \min\{\varepsilon_1, \varepsilon_2\}$ and let $V \subset U$ defined by $V = \{(x, y) \in U: (x, g(x, y)) \in \delta \operatorname{Gr} T\}$. Clearly V is an open set being the inverse image of the open set $\delta \operatorname{Gr} T$, trought the continuous map (I, g), where I stands for the identity map.

We divide now the proof in three parts.

First part: $X = \boldsymbol{R}^n$, $K \subset Y = \boldsymbol{R}^m$.

Let ε^* be that one given in Lemma 2.2, i.e. every ε^* approximation $\bar{f}$ of $\bar{F}$ has the property that $S^f_B \subset V$. Let ε_0 given by Lemma 2.3 and let $\varepsilon' = \min\{\varepsilon^*, \varepsilon_0, \delta\}$. By Lemmata 2.3 and 2.4 there exists a continuous map $\bar{f}$: $\bar{V} \to \boldsymbol{R}^n$ which is an ε'-approximation for $\bar{F}$ on $\bar{V}_B$ and such that the set-valued map $S': \bar{B} \multimap \boldsymbol{R}^m$ defined by $x \multimap S'(x) = S^f(x)$ is a w-map such that $S'_B \subset V$.

The index of the map is given by

$$\mathrm{i}(S') = \operatorname{ind}(K, \bar{F}(0, \cdot), V(0)) = \operatorname{ind}(K, \bar{F}(0, \cdot), U(0)) \neq 0 .$$

The set valued map $T'(x) = g(x, S'(x))$ is a w-map with index $\mathrm{i}(T') = \mathrm{i}(S') \neq 0$ (see [4]). As $S'_B \subset V$ we have that $\operatorname{Gr} T' \subset \delta \operatorname{Gr} T$; but T' verify the B.U. property, then there exists $x \in B$ such that $0 \in T'(x)$. Then $0 \in \delta T(\delta x)$, which contradicts the fact that $0 \notin \varepsilon_1 T(\varepsilon_1 x)$, as $\delta \leq \varepsilon_1$.

Second part: $X = \boldsymbol{R}^n$, $K \subset Y$ *Banach space.*

As $\bar{U}_B$ is a compact ANR there exists a $\varepsilon'/2$-approximation $\tilde{f}$ of $\bar{F}$ on $\bar{U}_B$. On the other hand there exists a $\varepsilon'/2$ pointwise approximation $\bar{f}$ of $\tilde{f}$ whose range is contained in a finite dimensional convex set K_1.

By Lemma 2.2 and the properties of the index, we get

$$0 \neq \operatorname{ind}(K, \bar{F}(0, \cdot), V(0)) = \operatorname{ind}(K, \bar{f}(0, \cdot), V(0)) = \operatorname{ind}(K_1, \bar{f}(0, \cdot)_{\backslash V_1}, V(0) \cap K_1),$$

with $V_1 = V \cap (X \times K_1)$ and $S^f_B \subset V_1$. Then $S^f_B \neq \emptyset$, with $f = I - \bar{f}$. Let $g_1 = g_{\backslash V_1}$ and $\bar{f}_1 = \bar{f}_{\backslash V_1}$. These two maps satisfy the hypotheses of Theorem 2.1, then, for the first part of the proof, the set valued map $T''(x) = g_1(x, S^{f_1}(x))$

has a zero in $\bar{B}$. As $S_{\bar{B}}^{f_1}(x) \subset V$ we have that Gr $T'' \subset \delta$ Gr T, contradicting the fact that $0 \notin \varepsilon_1 T(\varepsilon_1 x)$.

Third part: *X*, *Y Banach spaces.*

Let $\bar{g}_2: \bar{U}_B \to X$ be an ε-pointwise approximation of $\bar{g}$ on $\bar{U}_B$ with finite dimensional range. Let $X_1 \subset X$ be the subspace containing the range of $\bar{g}_2$ and let $g_2 = I - \bar{g}_{2\backslash(X_1 \times K)\cap\bar{U}_B}$ and $\bar{F}_2 = \bar{F}_{\backslash(X_1 \times K)\cap\bar{U}_B}$. Let $T': \bar{B}' = \bar{B} \cap X_1 \multimap X_1$ defined by $T'(x) = g_2(x, S(x))$. T' is an ε-approximation of $T_{\backslash\bar{B}'}$ then, by Lemma 2.6, for $\varepsilon > 0$ sufficiently small, T' satisfies the B.U. property on $\partial B'$. By the second part of the proof T' has then a zero on $\bar{B}' \subset \bar{B}$, which is absurd. □

With a similar proof to that one of Theorem 1.4 of [8] we can prove the following

Theorem 2.2: *Let X and K as in Theorem* 2.1. *Let U be an open, bounded set in $X \times K$ and let $\bar{F}: \bar{U} \multimap K$ be an UV^∞-mapping. Consider a compact, convex set $Q \subset X$ such that for every $x \in Q$ we have $y \notin \bar{F}(x, y)$ on $\partial U(x)$. Assume that for some (and hence for all) $x \in Q$ we have* $\mathrm{ind}(K, \bar{F}(x, \cdot), U(x)) \neq 0$. *If $\bar{g}: S_Q \to X$ is any continuous map such that $\bar{g}(x, S(x)) \subset Q$ for any $x \in Q$, then there exists a solution $(x, y) \in U$ of* (1) *with $x \in Q$ and $y \in S(x)$.*

§ 3. Application

Consider the following nonlinear boundary value control problem

$$\text{(C)} \qquad \begin{cases} \ddot{x} + x = h(t, x, \dot{x}, u)\,, & t \in [0, \pi] \\ x(0) = x(\pi)\,, \quad \dot{x}(0) = \dot{x}(\pi)\,, & \\ u(t) \in U(t, x(t))\,, & \text{a.a. } t \in [0, \pi]\,. \end{cases}$$

Assume the following conditions

h_1) the function $h: [0, \pi] \times \boldsymbol{R}^{3n} \to \boldsymbol{R}^n$, given by $(t, p, q, r) \to h(t, p, q, r)$, is t-measurable and (p, q, r)-continuous;

c_0) the multivalued map $U: [0, \pi] \times \boldsymbol{R}^n \multimap \boldsymbol{R}^n$, $(t, p) \multimap U(t, p)$ is t-measurable and p-continuous for a.a. $t \in [0, \pi]$ with nonempty and compact values;

c_1) $0 \in U(t, p)$ for a.a. $t \in [0, \pi]$ and any $p \in \boldsymbol{R}^n$. Moreover, the set $U(t, p)$ is star-shaped with respect to the origin for any $(t, p) \in [0, \pi] \times \boldsymbol{R}^n$;

c_2) there exists $\rho > 0$ such that $|U(t, p)| < \rho$ for a.a. $t \in [0, \pi]$ and any $p \in \boldsymbol{R}^n$.

In order to formulate our conditions on the control law $t \to u(t)$, $u(t) = (u_1(t), \ldots, u_n(t)) \in U(t, x(t))$, we need some preliminaries. First, we recall that every essentially bounded function $v \in L^\infty([0, \pi], \boldsymbol{R})$ can be regarded, except for a set of measure zero, as the uniform limit of a sequence of simple functions $\{\phi_n\}_{n \in \boldsymbol{N}}$, where, for any fixed $n \in \boldsymbol{N}$,

$$\phi_n = \sum_{j=1}^{n \cdot 2^n} a_j \cdot \chi_{E_j}\,, \qquad a_j = \operatorname*{ess\,inf}_{t \in E_j} u(t)$$

and the sets E_j, $1 \le j \le n \cdot 2^n$, are given by

$$E_j = \{t \in [0, \pi]: j \cdot 2^{-n} \le v(t) \le (j+1) \cdot 2^{-n}\},$$

(see [2]). Obviously if v is a simple function, $\phi_n = v$ for sufficiently large values of n. (The convention $\chi_\emptyset = 0$ is assumed).

From the covering theorem of Vitali, we have that for all $1 \le j \le n \cdot 2^n$ there exists a finite or countable family of nondegenerate closed intervals having no internal point in common $\{I_j^k\}_{k=1}^{K(j)}$ such that, except for a set of measure zero, we get

$$E_j = \bigcup_{k=1}^{K(j)} I_j^k .$$

Let $(\alpha_j^i, \beta_j^i)_{i=1}^{D(j)}$ be the connected components of the set $\bigcup_{k=1}^{K(j)} I_j^k$ where $D(j)$ is finite or infinite, and let D_n be the set: $D_n = \{D(1), D(2), \dots, D(j), \dots, D(n \cdot 2^n)\}$.

Let R be any positive constant. Let us denote by U_R the closed ball $\bar{B}(0, R) \subset L^\infty([0, \pi], \boldsymbol{R})$. For all $v \in U_R$ let us consider now the sequence $\{\phi_n\}_{n \in N}$ where

$$\phi_n = \sum_{j=1}^{n \cdot 2^n} a_j \cdot \chi_{E_j} = \sum_{j=1}^{n \cdot 2^n} \sum_{i \in D(j)} a_j^i \cdot \chi_{(\alpha_j^i, \beta_j^i)}, \qquad a_j^i = a_j \ \forall i \in D(j) .$$

The last equality holds except for a set of measure zero. Let $d_n = \sum_{j=1}^{n \cdot 2^n} D(j)$ and define $d(v) = \limsup_{n \to +\infty} d_n$. For $u \in L^\infty([0, \pi], \boldsymbol{R}^n)$ we define $d(u) = \max_i d(u_i(t))$. Let $\eta: \boldsymbol{R} \cup \{\infty\} \to \boldsymbol{R} \cup \{\infty\}$ a continuous, non decreasing, non negative function with $\eta(\infty) = \infty$.

Assume the following condition on the controls u

c_3) There exists $N \in \boldsymbol{R}_+$ such that $\eta(d(u)) \le N$ for all $u \in L^\infty([0, \pi], \boldsymbol{R}^n)$ such that $u(t) \in U(t, x(t))$ for any $x \in AC([0, \pi], \boldsymbol{R}^n)$.

The meaning of the function $\eta(d(\cdot))$ is to indicate that we pay a cost each time that we allow the control function to be nonconstant or jump. Therefore we confine ourselves to consider only piecewise constant controls with a finite number of switches.

We can write the problem

$$\text{(CI)} \qquad \begin{cases} \ddot{x} + x = h(t, x, \dot{x}, u) \\ x(0) = x(\pi), \qquad \dot{x}(0) = \dot{x}(\pi), \end{cases}$$

in the form

$$\text{(CV)} \quad \begin{cases} \dot{z} = Az + K(t, z, u) \\ z(0) = z(\pi) \end{cases}, \qquad \text{where } A = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \qquad z = \begin{bmatrix} x \\ x_1 \end{bmatrix} \quad \text{and}$$

$$K(t, z, u) = \begin{bmatrix} 0 \\ h(t, x, x_1, u) \end{bmatrix}.$$

We put $|z| = |x| + |x_1|$, $X = L^1([0, \pi], \mathbf{R}^n)$, $Y = L^\infty([0, \pi], \mathbf{R}^n)$, $Z = AC([0, \pi], \mathbf{R}^{2n})$, and $V = L^1([0, \pi], \mathbf{R}^{2n})$.

Consider the linear operator $L: D(L) \subset Z \to V$ defined by $D(L) = \{z \in Z: z(0) = z(\pi)\}$ and

$$(Lz)(t) = \dot{z}(t) - Az(t) \qquad \text{for a.a. } t \in [0, \pi].$$

The operator L admits an inverse $\mathscr{G}: V \to D(L)$ defined by $z = \mathscr{G}w$ where

$$\begin{cases} \dot{z} - Az = w \\ z(0) = z(\pi) \end{cases}$$

For a fixed $u \in Y$, let $\mathscr{K}(\cdot, u): V \to V$ be the Nemitskii operator generated by K, that is

$$\mathscr{K}(z, u) = K(t, z(t), u(t)) \qquad \text{for a.a. } t \in [0, \pi].$$

Therefore problem (CV) can be rewritten, for a given $u \in Y$, in the operator form

$$z = \bar{g}(z, u),$$

where $\bar{g}: V \times Y \to V$ is given by

$$\bar{g} = \mathscr{G} \circ \mathscr{K}$$

It is clear that $\bar{g}$ is a compact operator on $V \times K$, for any compact set $K \subset Y$, via the compact imbedding of $D(L)$ into V and the continuity of the operators $\mathscr{G}$ and $\mathscr{K}$. Let us define now the multivalued Nemitskii operator $\bar{F}: V \multimap Y$ as follows

$$\bar{F}(z) := \{u \in Y: u(t) \in U(t, x(t)) \qquad \text{for a.a. } t \in [0, \pi]; \eta(d(u)) \le N\},$$

where the vector $x = x(t)$ represents the n-first components of z; obviously, we can also denote the set $\bar{F}(z)$ by $\bar{F}(x)$. Finally, we can write the problem (C) in the form

$$\begin{cases} u \in \bar{F}(z) \\ z = \bar{g}(z, u) \end{cases} \tag{3.1}$$

In order to apply Theorem 2.1, we have to prove the following

Proposition 3.1:
i) $0 \in \bar{F}(x)$ *for all* $x \in X$;
ii) $\bar{F}(x)$ *is star shaped with respect to the origin for all* $x \in X$;
iii) $\operatorname{Im} \bar{F}$ *is a compact subset of* Y;
iv) $\bar{F}$ *is u.s.c. for all* $x \in X$.

Proof. i) follows immediately from hypothesis c_1).

ii) Assume $u \in \bar{F}(x)$; we want to show that $\lambda u \in \bar{F}(x)$ for all $\lambda \in [0, 1]$. Since $u(t) \in U(t, x(t))$ a.e. on $[0, \pi]$ and $\mu(u) \leq N$, then from c_1) it follows that $\lambda u(t) \in U(t, x(t))$ and $\mu(\lambda u) \leq N$ for every $\lambda \in [0, 1]$ and for a.a. $t \in [0, \pi]$. Hence $\lambda u \in \bar{F}(x)$ for every $\lambda \in [0, 1]$.

iii) From c_2) it follows that the $\operatorname{Im} \bar{F}$, i.e. $\bar{F}(X)$, is a bounded set of Y, say it is contained in a ball of radius $M > 0$. Using c_3) and the arguments of [9] we can prove that it is also compact.

iv) Let $\{x_n\}$ be a sequence in X, $x_n \to x$; let $\{u_n\}$ be a sequence such that $u_n \to u$ and $u_n \in \bar{F}(x_n)$, i.e. $u_n(t) \in U(t, x_n(t))$ a.e. on $[0, \pi]$ and $\mu(u_n) \leq N$. Hence, passing to a subsequence if necessary, $u_n(t) \to u(t)$ for a.a. $t \in [0, \pi]$ and so from c_0) it follows that $u(t) \in U(t, x(t))$ a.e. on $[0, \pi]$ and $\mu(u) \leq N$. So $\bar{F}$ has closed graph and the statement is proved. □

Moreover, assume that the following conditions are satisfied.

h_2) $|K(t, z, u)| \leq a|z| + b|u|$ a.e. in $[0, \pi]$ and for all $u \in \boldsymbol{R}^n$ and $z \in R^{2n}$, where $0 < a < \frac{1}{\pi}$ and $b > 0$;

h_3) let $i_K(t, z, u) = \langle z, Az + K(t, z, u)\rangle = \langle z, K(t, z, u)\rangle$, where $\langle \alpha, \beta\rangle$ denotes the inner product of α, $\beta \in \boldsymbol{R}^{2n}$.

Assume that

$$\int_0^\pi \liminf_{|z| \to \infty} \frac{i_K(t, z, u)}{|z|^2} > 0 \qquad \text{for every } u \in \boldsymbol{R}^n \text{ such that } |u| \leq M\,.$$

Here M is the constant of Proposition 3.1–(ii). We have the following result.

Theorem 3.1: *If assumptions* h_1), h_2), h_3), c_0), c_1), c_2), c_3) *are satisfied, then problem* (C) *has solutions.*

Proof. Hypotheses h_2) and h_3) ensure the existence of a constant $R > 0$ such that, if z is a solution of (CV) corresponding to a control $u \in Y$ with $|u|_Y \leq M$, then $\max_{t \in [0, \pi]} |z(t)| < R$.

In fact, assume the contrary. Then there exists a sequence $\{z_n\}$ of π-periodic solutions of (CV), with corresponding $u_n \in \bar{F}(z_n)$ such that $\max_{t \in [0, \pi]} |z_n(t)|^2 \to \infty$ as $n \to \infty$. Consider $\frac{d}{dt} \frac{|z(t)|^2}{2} = \langle z_n(t), \dot{z}_n(t)\rangle = \langle z_n(t), Az_n(t) + K(t, z_n(t), u_n(t)\rangle$ a.e. on $[0, \pi]$. Dividing by $1 + |z_n(t)|^2$ and integrating over $[\tau, t]$, $\tau \in \boldsymbol{R}$, $t \in [\tau, \tau + 1]$, taking in account of h_2) and the boundedness of $\{u_n\}$, we obtain

$$\frac{1}{2}(\ln(1 + |z_n(t)|^2) - \ln(1 + |z_n(\tau)|^2)) = \int_\tau^t \frac{\langle z_n(s), Az_n(s) + K(s, z_n(s), u_n(s))\rangle}{1 + |z_n(s)|^2} ds$$

$$\leq C\,;$$

since z_n is a periodic function we get

$$\ln \max_{t\in[0,\pi]} (1+|z_n(t)|^2) \le 2C + \ln \min_{t\in[0,\pi]} (1+|z_n(t)|^2)$$

so that $\min_{t\in[0,\pi]} (1+|z_n(t)|^2) \to \infty$ for $n\to\infty$. Hence $|z_n(t)| \neq 0$ for large n and we have $\displaystyle\int_0^\pi \frac{\langle z_n(s), Az_n(s) + K(s, z_n(s), u_n(s))\rangle}{|z_n(s)|^2} ds = \ln\frac{|z_n(\pi)|}{|z_n(0)|} = 0$ for large n. Using h_2) we obtain

$$0 = \liminf_{n\to\infty} \int_0^\pi \frac{\langle z_n(s), Az_n(s) + K(s, z_n(s), u_n(s))\rangle}{|z_n(s)|^2} ds \ge \int_0^\pi \frac{i_K(s, z_n(s), u_n(s))}{|z_n(s)|^2} ds\,,$$

contradicting h_3).

Let B an open ball in V containing all the solutions of problem (CV) corresponding to a control $u\in Y$ with $|u|_Y \le M$. Since $\operatorname{Im}\bar{F}$ is a compact subset of Y, then $\overline{\text{co}}\operatorname{Im}\bar{F}$ is a convex and compact subset of Y. Take $K\subset Y$ an arbitrary compact and convex subset such that $K \supset \overline{\text{co}}\operatorname{Im}\bar{F}$, $\partial K \cap \overline{\text{co}}\operatorname{Im}\bar{F} = \emptyset$. Let $W = B\times K$. Clearly all the possible solutions of (3.1) are in W and $\bar{B}\subset \mathscr{D}^F$. Finally, consider the following homotopy: $H: [0,1]\times K \multimap K$ defined by $H(\lambda, u) = \lambda\bar{F}(u, 0)$, where $\bar{F}: K\times V \multimap K$ is given by $\bar{F}(u, z) = \bar{F}(z) \subset K$ for any $u\in K$. We have that $u\notin H(\lambda, u)$ for any $u\in \partial K$ and any $\lambda\in[0,1]$, since $0\in \bar{F}(u,z)$ for any $(u,z)\in Y\times V$. Therefore, $\operatorname{ind}(K, \bar{F}(\cdot, 0), U(0)) \neq 0$, where $U(0)$ is any relatively open set in K such that $U(0)\supset \bar{F}(0)$. □

We end with the following result.

Lemma 3.1: *For sufficiently large $R > 0$, the map $T(z) = z - \bar{g}(z, S(z))$, with $S(z) = \bar{F}(z)$, satisfies the B.U. condition on $\partial B = \partial B(0, R) \subset V$.*

Proof. The proof is based on the following result stated in [8].

—A map $T: \mathscr{B}(0, R) = \multimap V$ satisfies the B.U. condition if and only if the multivalued map $\tilde{T}(z) = Q(\overline{\text{co}}\, T(z)) - \overline{\text{co}}(T(-z))$ has no zeros on $\partial\mathscr{B}$.—

Here $Q(A) = \{\lambda x: \lambda\in[0,1], x\in A\}$ and $\overline{\text{co}}(A)$ is the closure of the convex hull of A. Consider now $v\in \bar{g}(z, S(z))$, for $z\in\partial\mathscr{B}(0,R)$. We have that

$$\begin{cases} \dot{v} - Av = K(t, z(t), u(t)) \\ v(0) = v(\pi) \end{cases}$$

for some $u\in S(z)$. Therefore $v(t) = \displaystyle\int_0^\pi G(t,s)K(s, z(s), u(s))ds$

$$\text{where}\qquad G(t,s) = \begin{cases} [(e^{-A} - I)^{-1} + I]e^{A(t-s)} & 0\le s\le t\le \pi\,, \\ (e^{-A} - I)^{-1} e^{A(t-s)} & 0\le t\le s\le \pi\,. \end{cases}$$

We get $|v(t)| \leq \int_0^\pi |G(t, s)||K(s, z(s), u(s))| ds$. Using the fact that the logarithm norm $\mu(A)$ of the matrix A is zero we obtain

$$|v(t)| \leq \int_0^\pi |K(s, z(s), u(s))| ds \leq a\pi R + b\pi M .$$

By h_2), for sufficiently large $R > 0$ we have that $|v|_{L^1} < R$. Assume now that there exists $z \in \partial B(0, R)$, $\lambda \in [0, 1]$, $y_1 \in \overline{co}(T(z))$ and $y_2 \in \overline{co}(T(-z))$ such that

$$\lambda z - \lambda y_1 + z + y_2 = 0 .$$

Thus

$$0 = |(1 + \lambda)z - (\lambda y_1 - y_2)|_{L^1} \geq (1 + \lambda)|z|_{L^1} - \lambda|y_1|_{L^1} + |y_2|_{L^1} > 0 ,$$

since $|y_1|_{L^1}$, $|y_2|_{L^1} < R$. Therefore, using the above result of [8], we can conclude that $T: B(0, R) \multimap V$ satisfies the B.U. condition. □

Remark 3.1 Let $x \in X$ and let $f: \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ defined as follows

$$f(x(t), u(t)) = \begin{cases} 1 & \text{if } u(t) \in U(t, x(t)) \\ +\infty & \text{if } u(t) \notin U(t, x(t)) . \end{cases}$$

If $u \in Y$, we have that $\int_0^\pi f(x(t), u(t)) dt = 1$ if and only if $u(t) \in U(t, x(t))$ a.e. on $[0, \pi]$. Hence we can define the map $\overline{F}: X \multimap Y$ as follows

$$\overline{F}(x) = \left\{ u \in Y: \int_0^\pi f(x(t), u(t)) dt \leq M, \mu(u) \leq N \right\} .$$

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