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Nonlinear Eigenvalue Problem for a Modified Capillary Surface Equation

By

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1. Introduction

In this article we discuss the existence of nonzero solutions of the boundary value problem

$$-p\,\mathrm{div}\left(\frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\right)=\lambda f(x,u) \quad \text{in } \Omega \tag{1.1}$$

$$u\geqq 0 \quad \text{in } \Omega \tag{1.2}$$

$$u=0 \quad \text{on } \partial\Omega\,, \tag{1.3}$$

where $p>1$, Ω is a bounded domain in $\boldsymbol{R}^n$ ($n\geqq 2$), ∇u denotes the gradient of u and λ is a positive parameter. In the case $p=1$, the equation (1.1) is the mean curvature equation or the capillary surface equation. When $f(x,u)=u$, the equation describes the free surface of a pendent drop filled with liquid under gravitational field (c.f. R. Finn [8]). For the capillary surface equation, radially symmetric solutions in the case when Ω is a ball or entire space have been investigated precisely. See e.g. [6] [12] [15]. Recently, many authors ([2] [10] [13] etc.) have considered (1.1) for more general function $f(u)$ and investigated radially symmetric solutions. But when Ω is a general domain, only few results concerning the problem (1.1) (1.2) (1.3) are obtained. See Coffman-Ziemer [5]. Hence it seems to be worth considering the problem (1.1) (1.2) (1.3), even though the equation (1.1) is not exactly the capillary surface equation.

Solutions of (1.1) (1.3) correspond to critical points of the functional

$$I_\lambda[u]=\int_\Omega\left(\sqrt{1+|\nabla u|^{2p}}-1\right)dx-\lambda\int_\Omega F(x,u)dx \tag{1.4}$$

defined on the usual Sobolev space $W_0^{1,p}(\Omega)$, where

$$F(x,u)=\int_0^u f(x,\xi)d\xi\,.$$

In order to find out an unstable critical point of (1.4), there are two approaches stated as in [1] [4] [14] etc. The one is to look for critical points of the functional

$$\int_{\Omega} \sqrt{1 + |\nabla u|^{2p}}\, dx$$

on the subset

$$\left\{ u \in W_0^{1,p}(\Omega) \,\middle|\, \int_{\Omega} F(x, u) dx = 1 \right\}.$$

Such a critical point u satisfies the equation

$$-p \operatorname{div}\left(\frac{|\nabla u|^{2p-2} \nabla u}{\sqrt{1 + |\nabla u|^{2p}}} \right) = \mu f(x, u) \qquad \text{in } \Omega,$$

where μ is a Lagrange multiplier. In the case when the principal part of the equation is Laplacian or p-Laplacian and $f(x, u)$ is homogeneous, the method of stretching of μ is valid. See Brezis-Nirenberg [4] and Guedda-Veron [9]. But we do not know the value of μ in this case, and further, since the functional is not homogeneous, it is impossible to stretch the Lagrange multiplier. The other is the method to use the mountain pass lemma for the functional $I_\lambda[u]$. But it is not trivial to show that the functional I_λ satisfies the Palais-Smale condition. Hence in this paper we show the existence of a local minimizer of $I_\lambda[u]$ in Section 2, and next give the proof to obtain an unstable critical point by using the mountain pass lemma without Palais-Smale condition and the monotone operator theory in Section 3.

Finally noting

$$\sqrt{1 + |\nabla u|^{2p}} - 1 \sim \begin{cases} |\nabla u|^p & \text{as } |\nabla u| \to \infty \\ \frac{1}{2}|\nabla u|^{2p} & \text{as } |\nabla u| \to 0, \end{cases}$$

we may consider $I_\lambda[u]$ as a neutral type of the functionals

$$J_\lambda^1[u] = \int_{\Omega} \{|\nabla u|^p - \lambda F(x, u)\} dx \qquad \text{for large } |\nabla u| \tag{1.5}$$

and

$$J_\lambda^2[u] = \int_{\Omega} \{\tfrac{1}{2}|\nabla u|^{2p} - \lambda F(x, u)\} dx \qquad \text{for small } |\nabla u|. \tag{1.6}$$

The Euler equations of J_λ^1 and J_λ^2 are p-Laplace and $2p$-Laplace equations respectively. In Section 4, for simplicity in the case of $f(x, u) = qu^{q-1}$, we remark that the dependence of solutions of (1.1) (1.2) (1.3) on the parameter

λ is very similar to that of solutions of the Euler equations of (1.5) and (1.6) when $|\nabla u|$ is large and small respectively.

Throughout this paper, we assume that the measure of Ω is equal to 1 without loss of generality.

2. Existence of a local minimizer

In this section we find out a local minimizer of the functional

$$I_\lambda[u] = \int_\Omega (\sqrt{1 + |\nabla u|^{2p}} - 1)dx - \lambda \int_\Omega F(x, u)dx$$

defined on the Sobolev space $W_0^{1,p}(\Omega)$ under appropriate conditions on $f(x, u)$. Here

$$F(x, u) = \int_0^u f(x, \xi)d\xi\,.$$

Throughout this paper, we put assumptions on $f(x, \xi)$ as follows:

(A1) f; $\Omega \times \boldsymbol{R} \to \boldsymbol{R}$ is continuous,

(A2) $f(x, \xi) \geqq 0$ on $\Omega \times [0, \infty)$, $f(x, \xi) = 0$ on $\Omega \times (-\infty, 0]$,

(A3) there exists a constant q with $1 \leqq q < np/(n - p)$ and the inequality

$$f(x, \xi) \leqq d_1 \xi^{q-1} + d_2$$

holds on $\Omega \times [0, \infty)$ with some positive constants d_1, d_2.

Under these assumptions (A1) (A2) (A3), it is well known that I_λ is a C^1-function on $W_0^{1,p}(\Omega)$. For the proof, see e.g. [14, Appendix B]. Needless to say, a local minimizer u in $W_0^{1,p}(\Omega)$ of I_λ satisfies (1.1) (i.e. the Euler equation of I_λ) weakly.

Now we have

Theorem 2.1. *In addition to the assumptions* (A1) (A2) (A3), *let us assume* $q < 2p$ *in* (A3) *and the following*:

(A4) *There exists a constant* r *with* $1 < r < 2p$ *and the inequality*

$$f(x, \xi) \geqq d_3 \xi^{r-1} \tag{2.1}$$

holds on $\Omega \times [0, \xi_0)$ *with some constants* $d_3 > 0$ *and* $\xi_0 > 0$.

Then there exists a positive constant λ^* *such that, for any* $0 < \lambda < \lambda^*$, *there exists a non-negative, nonzero local minimizer* u_λ *of* I_λ *in* $W_0^{1,p}(\Omega)$, *i.e. a weak solution of* (1.1) (1.2) (1.3). *Further it satisfies*

$$\|\nabla u_\lambda\|_{L^p(\Omega)} \to 0 \qquad \textit{as } \lambda \to 0\,.$$

Proof. Since the norm

$$\|\nabla u\|_{L^p(\Omega)} = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}$$

is equivalent to the usual Sobolev norm in $W_0^{1,p}(\Omega)$ by Poincare's inequality, we adopt this norm as the one in $W_0^{1,p}(\Omega)$. Note the function $\phi(X) = \sqrt{1 + X^2}$ is convex. Putting $X = |\nabla u|^p$ and using Jensen's inequality, we have

$$\int_\Omega \sqrt{1 + |\nabla u|^{2p}} dx \geqq \sqrt{1 + \left(\int_\Omega |\nabla u|^p dx\right)^2} \tag{2.2}$$

for $u \in W_0^{1,p}(\Omega)$. Note that we assume $\mathrm{meas}(\Omega) = 1$. Further, by the Poincare-Sobolev inequality, we have

$$\int_\Omega |u| dx \leqq M_1 \left(\int_\Omega |\nabla u|^p dx\right)^{1/p} \tag{2.3}$$

and

$$\int_\Omega |u|^q dx \leqq M_2 \left(\int_\Omega |\nabla u|^p dx\right)^{q/p} \tag{2.4}$$

for $u \in W_0^{1,p}(\Omega)$, where M_1 and M_2 are the Poincare-Sobolev constants. From the assumption (A3) and the inequalities (2.3) (2.4), the inequality

$$\begin{aligned} \int_\Omega |F(x, u)| dx &\leqq c_1 \int_\Omega (|u| + |u|^q) dx \\ &\leqq c_2(\rho + \rho^q) \end{aligned} \tag{2.5}$$

holds on the sphere $\|\nabla u\|_{L^p(\Omega)} = \rho$ with some constants c_1, $c_2 > 0$. Hence, from (2.2) and (2.5), we have

$$I_\lambda[u] \geqq \sqrt{1 + \rho^{2p}} - 1 - c_2\lambda(\rho + \rho^q) \tag{2.6}$$

on $\|\nabla u\|_{L^p(\Omega)} = \rho$. Putting $\rho = \rho_\lambda \equiv \lambda^\alpha$ in (2.6), we have

$$I_\lambda[u] \geqq \sqrt{1 + \lambda^{2p\alpha}} - 1 - c_2\lambda(\lambda^\alpha + \lambda^{q\alpha}) \tag{2.7}$$

on $\|\nabla u\|_{L^p(\Omega)} = \rho_\lambda$. Noting $1 \leqq q < 2p$ and taking a constant α such that $0 < \alpha < 1/(2p - 1)$, we have

$$2p\alpha < \alpha + 1 \leqq q\alpha + 1 .$$

Hence there exists a constant $\lambda^* > 0$ such that the right hand of (2.7) is positive for any $0 < \lambda \leqq \lambda^*$. Namely $I_\lambda[u] > 0$ on $\|\nabla u\|_{L^p(\Omega)} = \rho_\lambda$ for any $0 < \lambda \leqq \lambda^*$.

Now let us consider the infimum of $I_\lambda[u]$ over the set $B(\rho_\lambda) \equiv \{u \in W_0^{1,p}(\Omega) \mid \|\nabla u\|_{L^p(\Omega)} \leqq \rho_\lambda\}$. From the inequality (2.6), I_λ is lower bounded on $B(\rho_\lambda)$. Put

$$\inf_{u \in B(\rho_\lambda)} I_\lambda[u] = I^\lambda > -\infty .$$

Let us take a nonzero function $\phi(x) \in C_0^\infty(\Omega) \cap B(\rho_\lambda)$ satisfying $0 \leqq \phi(x) < \xi_0$ in Ω, where ξ_0 is the constant stated in the assumption (A4). Then, from the assumption (A4), the inequality

$$F(x, \varepsilon\phi) = \int_0^{\varepsilon\phi(x)} f(x, \xi) d\xi \geqq c_3 \varepsilon^r \phi(x)^r$$

holds on Ω for any $0 < \varepsilon < 1$ with some constant $c_3 > 0$. Further, it is clear that $\varepsilon\phi$ belongs to $B(\rho_\lambda)$ for $0 < \varepsilon < 1$. Putting $u = \varepsilon\phi$ in $I_\lambda[u]$, we have

$$\begin{aligned} I_\lambda[\varepsilon\phi] &= \int_\Omega (\sqrt{1 + \varepsilon^{2p}|\nabla\phi|^{2p}} - 1) dx - \lambda \int_\Omega F(x, \varepsilon\phi) dx \\ &\leqq \frac{1}{2}\varepsilon^{2p} \int_\Omega |\nabla\phi|^{2p} dx - c_3 \lambda \varepsilon^r \int_\Omega \phi^r \, dx . \end{aligned}$$

Noting $r < 2p$ by the assumption (A4) and taking ε small, we see $I_\lambda[\varepsilon\phi] < 0$. Hence $I^\lambda < 0$. Next let $\{u_n\}$ be a sequence in $B(\rho_\lambda)$ such that

$$I_\lambda[u_n] \to I^\lambda \qquad \text{as } n \to \infty .$$

Since $\|\nabla u_n\|_{L^p(\Omega)} \leqq \rho_\lambda$, i.e. bounded in $W_0^{1,p}(\Omega)$, there exists a subsequence $\{u_{n_k}\}$ which converges weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^q(\Omega)$. Here we used the fact that $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$, since $1 \leqq q < np/(n-p)$. Let $u_\lambda = \lim_{k \to \infty} u_{n_k}$. Since

$$\|\nabla u_\lambda\|_{L^p(\Omega)} \leqq \liminf_{k \to \infty} \|\nabla u_{n_k}\|_{L^p(\Omega)} \leqq \rho_\lambda ,$$

u_λ belongs to $B(\rho_\lambda)$. The functional

$$\int_\Omega (\sqrt{1 + |\nabla u|^{2p}} - 1) dx$$

is weakly lower semicontinuous, since a convex and lower semicontinuous functional defined on a Banach space is weakly lower semicontinuous. For the proof, see e.g. Dacorogna [7, Theorem 1.2 in Chap. 3]. Further, noting that the functional

$$\int_\Omega F(x, u) dx$$

is continuous in $L^q(\Omega)$, we see that the functional I_λ is lower semicontinuous in $W_0^{1,p}(\Omega)$. Thus we have

$$I^\lambda = \lim_{k\to\infty} I_\lambda[u_{n_k}] \geqq I_\lambda[u_\lambda]\,.$$

Hence this limit function u_λ is a minimizer of I_λ in $B(\rho_\lambda)$. On the other hand, as is proved in the former part, $I_\lambda[u] > 0$ on $\|\nabla u\|_{L^p(\Omega)} = \rho_\lambda$ for any $0 < \lambda \leqq \lambda^*$. Since I^λ is negative, u_λ is an interior point of the set $B(\rho_\lambda)$. Namely u_λ is a local minimizer of the functional $I_\lambda[u]$ in $W_0^{1,p}(\Omega)$. Further, since $\rho_\lambda = \lambda^\alpha \to 0$ as $\lambda \to 0$, we have $\|\nabla u_\lambda\|_{L^p(\Omega)} \to 0$ as $\lambda \to 0$. Since $I_\lambda[u_\lambda] < 0$, u_λ is nonzero. The nonnegativity of u_λ is proved in the following lemma.

Lemma 2.1. *If u in $W_0^{1,p}(\Omega)$ satisfies* (1.1) *weakly, that is, the equality*

$$p\int_\Omega \frac{|\nabla u|^{2p-2}}{\sqrt{1+|\nabla u|^{2p}}}\nabla u \nabla\phi\,dx = \lambda\int_\Omega f(x,u)\phi\,dx \tag{2.8}$$

holds for any $\phi \in C_0^\infty(\Omega)$, then $u(x) \geqq 0$ almost everywhere in Ω.

Proof. From the assumption (A3) and the Sobolev imbedding theorem, it is easy to see that $f(x,u)$ belongs to $W_0^{1,p}(\Omega)^*$ ($\equiv$ the adjoint space of $W_0^{1,p}(\Omega)$). Hence the equality (2.8) holds for any $\phi \in W_0^{1,p}(\Omega)$. Since $f(x,u) \geqq 0$ in Ω,

$$p\int_\Omega \frac{|\nabla u|^{2p-2}}{\sqrt{1+|\nabla u|^{2p}}}\nabla u \nabla v\,dx = \lambda\int_\Omega f(x,u)v\,dx \leqq 0$$

holds for any $v(x) \leqq 0$ in $W_0^{1,p}(\Omega)$. Putting $v(x) = \min\{u(x), 0\}$, we have

$$\begin{aligned}
0 &\geqq \int_\Omega \frac{|\nabla u|^{2p-2}}{\sqrt{1+|\nabla u|^{2p}}}\nabla u \nabla v\,dx \\
&= \int_{\Omega\setminus\Omega^-} \frac{|\nabla u|^{2p-2}}{\sqrt{1+|\nabla u|^{2p}}}\nabla u \nabla v\,dx \\
&\quad + \int_{\Omega^-} \frac{|\nabla u|^{2p-2}}{\sqrt{1+|\nabla u|^{2p}}}\nabla u \nabla v\,dx \\
&= \int_{\Omega^-} \frac{|\nabla v|^{2p}}{\sqrt{1+|\nabla v|^{2p}}}dx = \int_\Omega \frac{|\nabla v|^{2p}}{\sqrt{1+|\nabla v|^{2p}}}dx\,,
\end{aligned}$$

where $\Omega^- = \{x \in \Omega \mid u(x) < 0\}$. Hence $\nabla v = 0$ almost everywhere in Ω. This implies that $u \geqq 0$ almost everywhere in Ω.

Theorem 2.2. *Let the assumptions* (A1) $\sim$ (A4) *be satisfied. If $1 \leqq q < p$ in* (A3), *then there exists a nonnegative, nonzero global minimizer u_λ of I_λ, i.e. a weak solution of* (1.1) (1.2) (1.3), *for any $\lambda > 0$.*

Further let us assume the following:

(A5) *There exists a constant* s *with* $1 < s$ *which satisfies the inequality*

$$f(x, \xi) \geqq d_4 \xi^{s-1} - d_5 \qquad \text{on } \Omega \times [0, \infty) \tag{2.9}$$

with some constants d_4, $d_5 > 0$.

Then this minimizer u_λ *satisfies*

$$\|\nabla u_\lambda\|_{L^p(\Omega)} \to \infty \qquad \text{as } \lambda \to \infty .$$

Proof. Noting the inequality (2.6), we see that I_λ is lower bounded in $W_0^{1,p}(\Omega)$ and $I_\lambda[u] \to \infty$ as $\|\nabla u\|_{L^p(\Omega)} \to \infty$. Taking notice of these facts, we easily see that there exists a global minimizer. Since $I^\lambda \equiv \min_{u \in W_0^{1,p}(\Omega)} I_\lambda[u] < 0$, u_λ is nonzero. The nonnegativity of u_λ follows from Lemma 2.1. Consider the infimum of $\|\nabla u\|_{L^p(\Omega)}$ under the constraint $\|u\|_{L^s(\Omega)} = 1$. Since $1 \leqq s < np/(n-p)$, it is well-known that there exists a function $\psi(x)$ in $W_0^{1,p}(\Omega)$ which attains the infimum under this constraint. Further by the maximum principle for the p-Laplace operator, $\psi(x)$ is positive in Ω. For the proof, see Guedda-Veron [9]. By the definition of ψ, the equality

$$\left(\int_\Omega |\nabla\psi|^p dx\right)^{1/p} = C_{p,s}\left(\int_\Omega |\psi|^s dx\right)^{1/s} \tag{2.10}$$

holds, where $C_{p,s}$ is the Poincare-Sobolev constant. For any given $\rho > 0$, put $v = \rho\psi/\|\nabla\psi\|_{L^p(\Omega)}$. Then, from (2.10) we have $\|\nabla v\|_{L^p(\Omega)} = \rho$ and $\|v\|_{L^s(\Omega)} = \rho/C_{p,s}$. Noting $v \geqq 0$ and using the assumption (A5) and Hölder's inequality, we have

$$\begin{aligned}
\inf_{\|\nabla u\|_{L^p(\Omega)} = \rho} I_\lambda[u] &\leqq I_\lambda[v] \\
&\leqq \int_\Omega |\nabla v|^p dx - c_4\lambda \int_\Omega |v|^s dx + c_5\lambda \int_\Omega |v| dx \\
&\leqq \int_\Omega |\nabla v|^p dx - c_4\lambda \int_\Omega |v|^s dx + c_5\lambda\left(\int_\Omega |v|^s dx\right)^{1/s} \\
&\leqq \rho^p - c_6\lambda\rho^s + c_7\lambda\rho
\end{aligned}$$

with some positive constant c_4, c_5, c_6 and c_7. Hence

$$I^\lambda \leqq \inf_{\rho > 0} (\rho^p - c_6\lambda\rho^s + c_7\lambda\rho) . \tag{2.11}$$

Putting

$$\rho = \left(\frac{c_6 s}{p}\lambda\right)^{1/(p-s)}$$

into (2.11), we have

$$I^{\lambda} \leqq -\gamma\lambda^{p/(p-s)} + c_8\lambda^{(1/(p-s))+1}, \tag{2.12}$$

where

$$\gamma = \left(\frac{c_6 s}{p}\right)^{p/(p-s)}\left(\frac{p}{s} - 1\right) > 0$$

and c_8 is some positive constant. On the other hand, from (2.6) the inequality

$$I_{\lambda}[u] \geqq \sqrt{1 + \rho^{2p}} - 1 - c_2\lambda(\rho + \rho^q) \tag{2.13}$$

holds on $\|\nabla u\|_{L^p(\Omega)} = \rho$. It is easy to check that the right hand in (2.13) is monotone decreasing on the interval $(0, (c_2\lambda/p)^{1/(p-1)})$ for each fixed $\lambda > p/c_2$. Hence, if a constant α is taken as $0 < \alpha < 1/(p-1)$, then the right hand in (2.13) is monotone decreasing on $(0, \lambda^{\alpha})$ for large λ. Thus, for each large λ, the inequality

$$I_{\lambda}[u] \geqq \sqrt{1 + \lambda^{2\alpha p}} - 1 - c_2\lambda(\lambda^{\alpha} + \lambda^{\alpha q}) \tag{2.14}$$

holds on the set $B(\lambda^{\alpha}) = \{u \in W_0^{1,p}(\Omega) \mid \|\nabla u\|_{L^p(\Omega)} \leqq \lambda^{\alpha}\}$. Moreover let us take $\alpha > 0$ so small that $1 + q\alpha < p/(p-s)$. Then there exists a $\lambda_0 > 0$ and the inequality

$$\sqrt{1 + \lambda^{2\alpha p}} - 1 - c_2\lambda(\lambda^{\alpha} + \lambda^{\alpha q}) > -\gamma\lambda^{p/(p-s)} + c_8\lambda^{(1/(p-s))+1} \tag{2.15}$$

holds for $\lambda \geqq \lambda_0$. Therefore, from (2.12) (2.14) (2.15), if $\lambda \geqq \lambda_0$, then $I^{\lambda} < I_{\lambda}[u]$ on the set $B(\lambda^{\alpha}) = \{u \in W_0^{1,p}(\Omega) \mid \|\nabla u\|_{L^p(\Omega)} \leqq \lambda^{\alpha}\}$. This shows that $u_{\lambda} \notin B(\lambda^{\alpha})$, i.e. $\|\nabla u_{\lambda}\|_{L^p(\Omega)} > \lambda^{\alpha}$. Hence we have

$$\|\nabla u_{\lambda}\|_{L^p(\Omega)} > \lambda^{\alpha} \to \infty \qquad \text{as } \lambda \to \infty .$$

3. The mountain pass lemma without P.-S. condition and the monotone operator method

In order to find out the second critical point of I_{λ}, we introduce a further assumption on $f(x, \xi)$.

(A6) There exist constants γ greather than p and $\xi_1 > 0$ such that the inequality

$$f(x, \xi)\xi \geqq \gamma \int_0^{\xi} f(x, \eta)d\eta \tag{3.1}$$

holds on $\Omega \times [\xi_1, \infty)$.

Then we have

Theorem 3.1. *Let the assumptions* (A1) (A2) (A3) (A6) *be satisfied. Further, if* $p < q < np/(n-p)$ *in* (A3), *then there exists a positive constant* λ_* *such that, for any* $0 < \lambda < \lambda_*$, *there exists a non-negative, nonzero critical point* v_λ *of* I_λ, *i.e. a weak solution of* (1.1) (1.2) (1.3). *Further, this critical point* v_λ *satisfies*

$$\|\nabla v_\lambda\|_{L^p(\Omega)} \to \infty \qquad as\ \lambda \to 0\,.$$

Remark 3.1. Integrating the inequality (3.1) shows that there exist constants d_6, $d_7 > 0$ such that

$$f(x, \xi) \geqq d_6 \xi^{\gamma-1} - d_7 \tag{3.2}$$

holds on $\Omega \times (0, \infty)$.

Remark 3.2. Let the constant q in (A.3) be greater than p and $f(x, \xi)$ be continuously differentiable in ξ for each x. Now consider the following assumption:

(A7) There exists a constant d_8 with $d_8 > d_1(p-1)$ such that the inequality

$$\frac{\partial f}{\partial \xi}(x, \xi) \geqq d_8 \xi^{q-2}$$

holds on $\Omega \times [\xi_2, \infty)$ for some constant $\xi_2 > 0$. Here d_1 is the constant stated in (A3).

Then the assumptions (A3) and (A7) imply (A6).

In fact, from (A3), for any $\delta > 0$ there exists $\xi_3 > 0$ such that

$$f(x, \xi) \leqq (d_1 + \delta)\xi^{q-1} \qquad \text{for } (x, \xi) \in \Omega \times [\xi_3, \infty)\,.$$

Since, from (A7)

$$\frac{\partial f}{\partial \xi}(x, \xi)\xi \geqq d_8 \xi^{q-1}$$

holds on $\Omega \times [\xi_2, \infty)$, we have

$$\frac{\partial f}{\partial \xi}(x, \xi)\xi \geqq d_8 \xi^{q-1} \geqq \frac{d_8}{d_1 + \delta} f(x, \xi)$$

for $x \in \Omega$ and $\xi \geqq \xi_4 \equiv \max(\xi_2, \xi_3)$. Hence the inequality

$$\frac{\partial f}{\partial \xi}(x, \xi)\xi + f(x, \xi) \geqq \left(1 + \frac{d_8}{d_1 + \delta}\right) f(x, \xi) \tag{3.3}$$

holds for $(x, \xi) \in \Omega \times [\xi_4, \infty)$. Now let us take $\varepsilon > 0$ sufficiently small and $\xi_4 > 0$ so large that $d_8/(d_1 + \delta) - \varepsilon > p - 1$ holds. Putting

$$\gamma = 1 + \frac{d_8}{d_1 + \delta} - \varepsilon > p$$

and integrating (3.3) over (ξ_4, ξ), we have

$$f(x, \xi)\xi - f(x, \xi_4)\xi_4 \geqq (\gamma + \varepsilon)\int_{\xi_4}^{\xi} f(x, \eta)d\eta$$

for $\xi \geqq \xi_4$. Noting the inequality in (A7), we easily see that there exists a constant $\xi_5 > 0$ such that

$$\gamma \int_0^{\xi_4} f(x, \eta)d\eta \leqq \varepsilon \int_{\xi_4}^{\xi} f(x, \eta)d\eta$$

holds for $(x, \xi) \in \Omega \times [\xi_5, \infty)$. Hence we have

$$f(x, \xi)\xi \geqq (\gamma + \varepsilon)\int_{\xi_4}^{\xi} f(x, \eta)d\eta \geqq \gamma \int_0^{\xi} f(x, \eta)d\eta$$

for $(x, \xi) \in \Omega \times [\xi_5, \infty)$.

Thus, in this case the whole assumptions in Theorem 3.1 are satisfied under (A1), (A2), (A3) and (A7).

In order to prove Theorem 3.1 we first recall the Ambrosetti-Rabinowitz mountain pass lemma without Palais-Smale condition.

Lemma 3.1 [1]. *Let I be a C^1-function on a Banach space E. Suppose there exist a neighborhood U of 0 in E and a constant α which satisfy the following:*

i) *$I[u] \geqq \alpha$ on the boundary of U,*

ii) *$I[0] < \alpha$,*

iii) *there exists a $v \notin U$ satisfying $I[v] < \alpha$.*

Then, for the constant

$$\mu \equiv \inf_{\gamma \in \Gamma} \max_{w \in \gamma} I[w] \qquad (\geqq \alpha)$$

where Γ denotes the class of paths joining 0 to v, there exists a sequence $\{u_j\}$ in E such that $I[u_j] \to \mu$ and $I'[u_j] \to 0$ in E^*.

Now we check the assumptions in Lemma 3.1 for the functional I_λ defined on $W_0^{1,p}(\Omega)$. Note that we assume $\operatorname{meas}(\Omega) = 1$. Recall that the functional I_λ satisfies the inequality

$$(2.6) \qquad I_\lambda[u] \geqq \sqrt{1 + \rho^{2p}} - 1 - c_2\lambda(\rho + \rho^q)$$

on $\|\nabla u\|_{L^p(\Omega)} = \rho$. Noting that

$$\frac{\sqrt{1+\rho^{2p}}-1}{c_2(\rho+\rho^q)} \to 0 \qquad \text{as } \rho \to 0 \text{ and } \infty$$

respectively, we put

$$\lambda_* \equiv \sup_{\rho>0} \frac{\sqrt{1+\rho^{2p}}-1}{c_2(\rho+\rho^q)} = \frac{\sqrt{1+\rho_0^{2p}}-1}{c_2(\rho_0+\rho_0^q)} > 0\,,$$

where $\rho_0 > 0$ is a constant at which the maximum λ_* is attained. Then, from (2.6) the inequality

$$I_\lambda[u] \geqq (\lambda_* - \lambda)c_2(\rho_0 + \rho_0^q) \tag{3.4}$$

holds on $\|\nabla u\|_{L^p(\Omega)} = \rho_0$. Putting

$$\alpha = (\lambda_* - \lambda)c_2(\rho_0 + \rho_0^q)\,, \tag{3.5}$$

we have $I_\lambda[u] \geqq \alpha > 0$ on $\|\nabla u\|_{L^p(\Omega)} = \rho_0$ for any $0 < \lambda < \lambda_*$. Clearly $I_\lambda[0] = 0$. Thus if $0 < \lambda < \lambda_*$, then the assumptions i) and ii) are satisfied by taking $U = \{u \in W_0^{1,p}(\Omega) \mid \|\nabla u\|_{L^p(\Omega)} < \rho_0\}$. Next let ϕ be a nonzero, nonnegative function in $C_0^\infty(\Omega)$. From the inequality (3.2), we have

$$F(x,\xi) \equiv \int_0^\xi f(x,\eta)d\eta \geqq \frac{d_6}{\gamma}\xi^\gamma - d_7\xi \qquad \text{for } \xi \geqq 0\,.$$

Hence

$$\begin{aligned} I_\lambda[r\phi] &= \int_\Omega (\sqrt{1+r^{2p}|\nabla\phi|^{2p}}-1)dx - \lambda\int_\Omega F(x,r\phi)dx \\ &\leqq r^p\left(\int_\Omega |\nabla\phi|^p dx\right) - \left(\frac{\lambda d_6}{\gamma}\int_\Omega \phi^\gamma\, dx\right)r^\gamma + \left(\lambda d_7\int_\Omega \phi\, dx\right)r \\ &\to -\infty \qquad \text{as } r \to \infty\,. \end{aligned}$$

Thus $I_\lambda[r\phi] < 0$ for large r. This implies the assumption (iii). Hence, if $0 < \lambda < \lambda_*$, applying Lemma 3.1 for the functional I_λ, we can find out a sequence $\{u_j\}$ in $W_0^{1,p}(\Omega)$ such that $I_\lambda[u_j] \to \mu$ and $I'_\lambda[u_j] \to 0$ in $W_0^{1,p}(\Omega)^*$, where μ is a constant stated in Lemma 3.1.

For this sequence $\{u_j\}$, we have the following.

Lemma 3.2. *Let the hypotheses in Theorem* 3.1 *be satisfied and* $\{u_j\}$ *be a sequence in* $W_0^{1,p}(\Omega)$ *obtained in Lemma* 3.1. *Namely it satisfies* $I_\lambda[u_j] \to \mu$ *and* $I'_\lambda[u_j] \to 0$ *in* $W_0^{1,p}(\Omega)^*$. *Then this sequence* $\{u_j\}$ *is bounded in* $W_0^{1,p}(\Omega)$.

Proof. By the assumptions, the equality

$$\int_\Omega \sqrt{1+|\nabla u_j|^{2p}}dx - \lambda\int_\Omega F(x,u_j)dx = \mu + 1 + o(1) \tag{3.6}$$

holds. Further, by putting $\zeta_j \equiv I'_\lambda[u_j]$, the equality

$$-p\,\mathrm{div}\left(\frac{|\nabla u_j|^{2p-2}}{\sqrt{1+|\nabla u_j|^{2p}}}\nabla u_j\right) - \lambda f(x, u_j) = \zeta_j \tag{3.7}$$

holds in the sense of $W_0^{1,p}(\Omega)^*$. Hence we have

$$p\int_\Omega \frac{|\nabla u_j|^{2p}}{\sqrt{1+|\nabla u_j|^{2p}}}dx - \lambda\int_\Omega f(x, u_j)u_j\,dx = \langle \zeta_j, u_j\rangle\,, \tag{3.8}$$

where $\langle\ ,\ \rangle$ denotes the duality between $W_0^{1,p}(\Omega)^*$ and $W_0^{1,p}(\Omega)$. Multiplying p by (3.6) and then subtracting (3.8), we obtain

$$\begin{aligned} &\lambda\int_\Omega \{f(x, u_j)u_j - pF(x, u_j)\}dx \\ &\leqq p\int_\Omega \frac{1}{\sqrt{1+|\nabla u_j|^{2p}}}dx + \lambda\int_\Omega \{f(x, u_j)u_j - pF(x, u_j)\}dx \\ &= p(\mu+1) - \langle \zeta_j, u_j\rangle + o(1)\,. \end{aligned} \tag{3.9}$$

On the other hand, from the assumption (A6) the inequality

$$\int_\Omega f(x, u)u\,dx \geqq \gamma\int_\Omega F(x, u)dx - d_9$$

holds for any $u \in W_0^{1,p}(\Omega)$ with some constant $d_9 > 0$. Hence we have

$$\int_\Omega \{f(x, u_j)u_j - pF(x, u_j)\}dx + d_9 \geqq \delta\int_\Omega F(x, u_j)dx\,, \tag{3.10}$$

where $\delta = \gamma - p > 0$ by the assumption. From (3.6), (3.9) and (3.10), we have

$$\begin{aligned} \int_\Omega |\nabla u_j|^p dx &\leqq \int_\Omega \sqrt{1+|\nabla u_j|^{2p}}\,dx \\ &= \lambda\int_\Omega F(x, u_j)dx + \mu + 1 + o(1) \\ &\leqq \frac{\lambda}{\delta}\int_\Omega \{f(x, u_j)u_j - pF(x, u_j)\}dx + \frac{\lambda}{\delta}d_9 + \mu + 1 + o(1) \\ &\leqq \frac{p}{\delta}(\mu+1) - \frac{1}{\delta}\langle \zeta_j, u_j\rangle + \frac{\lambda}{\delta}d_9 + \mu + 1 + o(1) \\ &\leqq \frac{p}{\delta}(\mu+1) + \frac{1}{\delta}\|\zeta_j\|_{W_0^{1,p}(\Omega)^*}\left(\int_\Omega |\nabla u_j|^p dx\right)^{1/p} \\ &\quad + \frac{\lambda}{\delta}d_9 + \mu + 1 + o(1)\,. \end{aligned}$$

Hence there exists a constant $C > 0$ such that

$$\int_\Omega |\nabla u_j|^p dx \leqq C$$

is valid.

From Lemma 3.2 there exist $v \in W_0^{1,p}(\Omega)$ and a subsequence of $\{u_j\}$, still denoted by $\{u_j\}$, such that

$$u_j \to v \quad \text{in } L^r(\Omega) \text{ for any } 1 \leqq r < np/(n-p)\,, \tag{3.11}$$

$$u_j \to v \quad \text{weakly in } W_0^{1,p}(\Omega)\,, \tag{3.12}$$

$$u_j \to v \quad \text{almost everywhere}\,. \tag{3.13}$$

Put

$$L[u] = \int_\Omega \sqrt{1 + |\nabla u|^{2p}}\, dx$$

and

$$A[u] = L'[u] = -p \operatorname{div}\left(\frac{|\nabla u|^{2p-2}}{\sqrt{1+|\nabla u|^{2p}}} \nabla u\right).$$

Noting that the operator L is convex, we see that the operator A is monotone and hemicontinuous. For the proof, see [11, Prop. 1.1 in Chap. 2]. From (3.11) and the assumption (A3), using Proposition B.1 in Rabinowitz [14], we see that

$$f(x, u_j) \to f(x, v) \qquad \text{in } L^{q/(q-1)}(\Omega)\,. \tag{3.14}$$

Since $W_0^{1,p}(\Omega)$ is imbedded continuously in $L^q(\Omega)$, (3.14) implies that

$$f(x, u_j) \to f(x, v) \qquad \text{in } W_0^{1,p}(\Omega)^*\,.$$

Further

$$\zeta_j \to 0 \qquad \text{in } W_0^{1,p}(\Omega)^*\,, \tag{3.15}$$

in the equation (3.7). Thus $A[u_j]$ converges to some χ in $W_0^{1,p}(\Omega)^*$. Letting $j \to \infty$ in (3.7), we have

$$\chi - \lambda f(x, v) = 0 \qquad \text{in } W_0^{1,p}(\Omega)^*\,. \tag{3.16}$$

Now making use of the monotonicity method of Minty and Browder (c.f. Lions [11]), we have

Lemma 3.3. *Let* χ *and* v *be the functions stated above. Then* $\chi = A[v]$ *holds.*

Proof. For any $w \in W_0^{1,p}(\Omega)$, put

$$X_j = \langle A[u_j] - A[w],\ u_j - w \rangle . \tag{3.17}$$

Since A is monotone, $X_j \geqq 0$. From (3.7), the equation

$$\langle A[u_j], u_j \rangle = \lambda \langle f(x, u_j), u_j \rangle + \langle \zeta_j, u_j \rangle \tag{3.18}$$

holds. Substituting (3.18) into (3.17), we have

$$X_j = \lambda \langle f(x, u_j), u_j \rangle + \langle \zeta_j, u_j \rangle - \langle A[u_j], w \rangle - \langle A[w], u_j - w \rangle . \tag{3.19}$$

Since $\{u_j\}$ is bounded in $W_0^{1,p}(\Omega)$ and $\zeta_j \to 0$ in $W_0^{1,p}(\Omega)^*$,

$$\lim_{j \to \infty} \langle \zeta_j, u_j \rangle = 0 .$$

Hence, taking notice of the equality (3.16), we have

$$\begin{aligned} \lim_{j \to \infty} X_j &= \lambda \langle f(x, v), v \rangle - \langle \chi, w \rangle - \langle A[w], v - w \rangle \\ &= \langle \chi - A[w], v - w \rangle . \end{aligned}$$

Therefore the inequality

$$\langle \chi - A[w], v - w \rangle \geqq 0 \tag{3.20}$$

is valid. Putting $w = v - tz$ in (3.20) for any $z \in W_0^{1,p}(\Omega)$, we have

$$t \langle \chi - A[v - tz], z \rangle \geqq 0 . \tag{3.21}$$

By dividing (3.21) by $t > 0$ and letting $t \to 0$, the inequality

$$\langle \chi - A[v], z \rangle \geqq 0$$

holds for any $z \in W_0^{1,p}(\Omega)$. Hence $\chi = A[v]$ is valid.

Proof of Theorem 3.1. We show that the limit function v given above, which denotes by v_λ, is the required solution stated in Theorem 3.1. From Lemma 3.3, it satisfies the equation

$$-p \operatorname{div}\left(\frac{|\nabla v|^{2p-2}}{\sqrt{1 + |\nabla v|^{2p}}} \nabla v \right) = \lambda f(x, v)$$

in the sense of $W_0^{1,p}(\Omega)^*$. The nonnegativity of v_λ is shown in Lemma 2.1. Hence we have only to show that this solution is nonzero and $\|\nabla v_\lambda\|_{L^p(\Omega)} \to \infty$ as $\lambda \to 0$. Since it is shown in the same way that the solution v_λ is nontrivial for any $0 < \lambda < \lambda_*$, we only show that $\|\nabla v_\lambda\|_{L^p(\Omega)} \to \infty$ as $\lambda \to 0$. Let $\{u_j\}$ be a sequence in Lemma 3.2 which converges to the solution v_λ. From (3.4) and (3.5),

$$
\begin{aligned}
(3.22)\qquad \alpha &= (\lambda_* - \lambda)c_2(\rho_0 + \rho_0^q) \\
&\leqq \mu = \lim_{j\to\infty} I_\lambda[u_j] \\
&= \lim_{j\to\infty} \left\{ \int_\Omega (\sqrt{1 + |\nabla u_j|^{2p}} - 1)dx - \lambda \int_\Omega F(x, u_j)dx \right\} \\
&\leqq \lim_{j\to\infty} \left\{ \int_\Omega \frac{|\nabla u_j|^{2p}}{\sqrt{1 + |\nabla u_j|^{2p}}}dx - \lambda \int_\Omega F(x, u_j)dx \right\} .
\end{aligned}
$$

Further, from (3.8), (3.11), (3.12) and (3.15), we have

$$
\begin{aligned}
(3.23)\qquad p \int_\Omega \frac{|\nabla u_j|^{2p}}{\sqrt{1 + |\nabla u_j|^{2p}}}dx &= \lambda \int_\Omega f(x, u_j)u_j \, dx + \langle \zeta_j, u_j \rangle \\
&\to \lambda \int_\Omega f(x, v_\lambda)v_\lambda \, dx
\end{aligned}
$$

as $j \to \infty$. Substituting (3.23) into (3.22) and noting

$$
\int_\Omega F(x, u)dx \geqq 0 \qquad \text{for any } u \in W_0^{1,p}(\Omega) ,
$$

we see

$$
\begin{aligned}
(3.24)\qquad \alpha &\leqq \lambda \left\{ \frac{1}{p} \int_\Omega f(x, v_\lambda)v_\lambda \, dx - \lim_{j\to\infty} \int_\Omega F(x, u_j)dx \right\} \\
&\leqq \frac{\lambda}{p} \int_\Omega f(x, v_\lambda)v_\lambda \, dx .
\end{aligned}
$$

From the assumption (A3), the inequality

$$
(3.25)\qquad \int_\Omega f(x, v_\lambda)v_\lambda \, dx \leqq (d_1 + d_2) \int_\Omega |v_\lambda|^q dx + d_2
$$

holds. Hence, from (3.24) and (3.25),

$$
\alpha \quad (= (\lambda_* - \lambda)c_2(\rho_0 + \rho_0^q)) \leqq \frac{\lambda}{p} \left\{ (d_1 + d_2) \int_\Omega |v_\lambda|^q dx + d_2 \right\}
$$

is valid. Therefore the inequality

$$
(3.26)\qquad \int_\Omega |v_\lambda|^q dx \geqq \frac{p(\lambda_* - \lambda)c_2(\rho_0 + \rho_0^q)}{\lambda(d_1 + d_2)} - \frac{d_2}{d_1 + d_2}
$$

holds. Noting that λ_* and ρ_0 are positive constants not depending on λ, we see that the right hand of (3.26) tends to ∞ as λ goes to 0. Hence we have

$\|v_\lambda\|_{L^q(\Omega)} \to \infty$ as $\lambda \to 0$. Using Sobolev's inequality, we complete the proof of Theorem 3.1.

In contradistinction to Theorem 2.2, we have

Theorem 3.2. *Let the assumptions* (A1), (A2), (A6) *be satisfied. If* (A3) *holds for some* $q > 2p$ *and* $d_2 = 0$, *then there exists a nonzero and nonnegative critical point* v_λ *for any* $\lambda > 0$. *Moreover, if the inequalities* (3.1) *and* (3.2) *with* $\gamma = q$, $d_7 = 0$ *hold on* $\Omega \times (0, \infty)$ *besides these conditions, then* v_λ *satisfies*

$$\|\nabla v_\lambda\|_{L^p(\Omega)} \to 0 \qquad \text{as } \lambda \to \infty .$$

Proof. Noting that the inequality

$$I_\lambda[u] \geqq \sqrt{1 + \rho^{2p}} - 1 - c_2 \lambda \rho^q$$

holds on $\|\nabla u\|_{L^p(\Omega)} = \rho$ instead of (2.6) and that

$$\frac{\sqrt{1 + \rho^{2p}} - 1}{c_2 \rho^q} \to \infty \qquad \text{as } \rho \to 0 ,$$

we easily obtain a critical point of I_λ for any $\lambda > 0$ in the same way as the proof of Theorem 3.1. Now we show that $\|\nabla v_\lambda\|_{L^p(\Omega)} \to 0$ as $\lambda \to \infty$ under the given assumptions. Let ϕ be the function stated in checking the assumption (iii) in Lemma 3.1 for I_λ. Namely, $\phi \in C_0^\infty(\Omega)$, $\phi \geqq 0$ and $\phi \not\equiv 0$. Further let us take $R > 0$ so large that $I_\lambda[R\phi] < 0$. Put

$$\Gamma \equiv \{\gamma \in C([0, 1]; W_0^{1,p}(\Omega)) | \gamma(0) = 0, \ \gamma(1) = R\phi\} .$$

For each $\lambda > 0$, μ_λ denotes the μ stated in Lemma 3.1 for the functional I_λ. Then.

$$\mu_\lambda \equiv \inf_{\gamma \in \Gamma} \max_{w \in \gamma} I_\lambda[w] \leqq \max_{0 \leqq r \leqq R} I_\lambda[r\phi] . \tag{3.27}$$

From (A3), (3.2),

$$\begin{aligned} I_\lambda[r\phi] &= \int_\Omega (\sqrt{1 + r^{2p}|\nabla\phi|^{2p}} - 1)dx - \lambda \int_\Omega F(x, r\phi)dx \\ &\leqq \left(\int_\Omega |\nabla\phi|^p dx\right) r^p - \frac{d_6}{q}\lambda \left(\int_\Omega \phi^q \, dx\right) r^q \\ &\equiv Ar^p - B\lambda r^q , \end{aligned} \tag{3.28}$$

where

$$A = \int_\Omega |\nabla\phi|^p dx , \qquad B = \frac{d_6}{q} \int_\Omega \phi^q \, dx .$$

The right hand of (3.28) attains the maximum

$$\left(1-\frac{p}{q}\right)A\left(\frac{pA}{qB}\right)^{p/(q-p)}\lambda^{-p/(q-p)}$$

at

$$r_0=\left(\frac{pA}{qB}\right)^{1/(q-p)}\lambda^{-1/(q-p)}.$$

Hence, from (3.27) and (3.28)

$$\mu_\lambda \leqq \left(1-\frac{p}{q}\right)A\left(\frac{pA}{qB}\right)^{p/(q-p)}\lambda^{-p/(q-p)} \tag{3.29}$$

holds. Thus

$$\mu_\lambda \to 0 \qquad \text{as } \lambda \to \infty\,.$$

Let $\{u_j\}$ be the sequence stated in Lemma 3.2 which satisfies (3.11)~(3.13). Namely,

$$I_\lambda[u_j] \to \mu_\lambda\,,$$

$$u_j \to v_\lambda \qquad \text{weakly in } W_0^{1,p}(\Omega) \text{ and strongly in } L^q(\Omega)\,.$$

Since I_λ is weakly lower semicontinuous in $W_0^{1,p}(\Omega)$, the inequality

$$\mu_\lambda \geqq I_\lambda[v_\lambda] \equiv \int_\Omega (\sqrt{1+|\nabla v_\lambda|^{2p}}-1)dx - \lambda\int_\Omega F(x, v_\lambda)dx \tag{3.30}$$

holds. From the assumption that

$$f(x,\xi)\xi \geqq \gamma\int_0^\xi f(x,\eta)d\eta$$

holds on $\Omega \times (0, \infty)$, we have

$$\int_\Omega F(x, v_\lambda)dx \leqq \frac{1}{\gamma}\int_\Omega f(x, v_\lambda)v_\lambda\, dx\,. \tag{3.31}$$

Further, since v_λ satisfies the equality

$$-p\operatorname{div}\left(\frac{|\nabla v_\lambda|^{2p-2}}{\sqrt{1+|\nabla v_\lambda|^{2p}}}\nabla v_\lambda\right) = \lambda f(x, v_\lambda)\,,$$

$$p\int_\Omega \frac{|\nabla v_\lambda|^{2p}}{\sqrt{1+|\nabla v_\lambda|^{2p}}}dx = \lambda\int_\Omega f(x, v_\lambda)v_\lambda\, dx \tag{3.32}$$

holds. From (3.30), (3.31) and (3.32),

$$\begin{aligned}\mu_\lambda &\geqq \int_\Omega (\sqrt{1+|\nabla v_\lambda|^{2p}}-1)dx - \frac{\lambda}{\gamma}\int_\Omega f(x, v_\lambda)v_\lambda\, dx \\ &= \int_\Omega (\sqrt{1+|\nabla v_\lambda|^{2p}}-1)dx - \frac{p}{\gamma}\int_\Omega \frac{|\nabla v_\lambda|^{2p}}{\sqrt{1+|\nabla v_\lambda|^{2p}}}dx \\ &= \int_\Omega \frac{1+\alpha|\nabla v_\lambda|^{2p}}{\sqrt{1+|\nabla v_\lambda|^{2p}}}dx - 1\,,\end{aligned} \tag{3.33}$$

where $\alpha = 1 - p/q$. Recall that we assume $\mathrm{meas}(\Omega) = 1$. Note $1/2 < \alpha < 1$, since $q > 2p$. Since the function $(1 + \alpha X^2)/\sqrt{1 + X^2}$ is convex when $1/2 < \alpha < 1$, from Jensen's inequality we have

$$\int_\Omega \frac{1+\alpha|\nabla v_\lambda|^{2p}}{\sqrt{1+|\nabla v_\lambda|^{2p}}}dx \geqq \frac{1+\alpha\left(\int_\Omega |\nabla v_\lambda|^p dx\right)^2}{\sqrt{1+\left(\int_\Omega |\nabla v_\lambda|^p dx\right)^2}}\,. \tag{3.34}$$

Putting

$$\rho_\lambda = \int_\Omega |\nabla v_\lambda|^p dx\,,$$

we have

$$\frac{1+\alpha\rho_\lambda^2}{\sqrt{1+\rho_\lambda^2}} - 1 \leqq \mu_\lambda \tag{3.35}$$

from (3.33) and (3.34). Since $1/2 < \alpha < 1$, the left hand of (3.35) is positive for $\rho_\lambda \neq 0$. Hence, letting $\lambda \to \infty$, we have $\rho_\lambda \to 0$. This completes the proof of Theorem 3.2.

4. A remark on the case $f(x, u) = qu^{q-1}$, $1 < q < np/(n-p)$

In this section we give a slight remark for the case $f(x, u) = qu^{q-1}$, $1 < q < np/(n-p)$. Since we consider only nonnegative solutions, the values of $f(x, u)$ for $u < 0$ are irrelevant and we may put $f(x, u) = q|u|^{q-2}u$ for $u \leqq 0$. As is noted in the introduction, the functional I_λ is approximated by

$$J_\lambda^1[u] = \int_\Omega |\nabla v|^p dx - \lambda\int_\Omega |u|^q dx \tag{4.1}$$

and

$$J_\lambda^2[u] = \frac{1}{2}\int_\Omega |\nabla u|^{2p} dx - \lambda\int_\Omega |u|^q dx\,, \tag{4.2}$$

when $|\nabla u|$ is large and small respectively. By finding out a minimizer w of $\|\nabla u\|_{L^p(\Omega)}$ on the sphere $\|u\|_{L^q(\Omega)} = 1$, we easily obtain a solution of

$$\begin{aligned} &-p \operatorname{div}(|\nabla w|^{p-2}\nabla w) = \kappa q w^{q-1} \quad \text{in } \Omega \\ &w = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.3}$$

where κ is a Lagrange multiplier. After stretching the Lagrange multiplier, i.e. by putting $w_\lambda = (\lambda/\kappa)^{1/(p-q)}w$, we obtain a solution of (4.3) in which κ is replaced with λ. This shows that there exists a solution w_λ of (4.3) with $\kappa = \lambda$ for any $\lambda > 0$, which tends to 0 (resp. to ∞) as $\lambda \to 0$ and to ∞ (resp. to 0) as $\lambda \to \infty$ when $q < p$ (resp. $p < q$). As for the functional (4.2), the consequences in which p is replaced with $2p$ hold justly. Roughly speaking, if (4.1) and (4.2) approximate I_λ properly, solutions u_λ of (1.1) (1.2) (1.3) ought to depend on λ like as solutions for (4.1) and (4.2) do when $|\nabla u_\lambda|$ is large and small in some sense respectively. The corollary in the following suggests the fact stated above implicitly.

Corollary 4.1. *Let* $f(x, u) = qu^{q-1}$, $1 < q < np/(n-p)$, *in the equation* (1.1). *Then the following holds.*

i) *If* $1 < q < p$, *then there exists a nonzero solution* u_λ *of* (1.1) (1.2) (1.3) *for any* $\lambda > 0$ *which satisfies*

$$\|\nabla u_\lambda\|_{L^p(\Omega)} \to 0 \qquad \text{as } \lambda \to 0$$

and

$$\|\nabla u_\lambda\|_{L^p(\Omega)} \to \infty \qquad \text{as } \lambda \to \infty .$$

ii) *If* $p < q < 2p$, *then there exists a constant* $\lambda_* > 0$ *and there exists at least two nonzero solutions* u_λ, v_λ *of* (1.1) (1.2) (1.3) *for any* $0 < \lambda < \lambda_*$ *which satisfy*

$$\|\nabla u_\lambda\|_{L^p(\Omega)} \to 0 \qquad \text{and} \qquad \|\nabla v_\lambda\|_{L^p(\Omega)} \to \infty$$

as $\lambda \to 0$.

iii) *If* $q > 2p$, *then there exists a nonzero solution* v_λ *of* (1.1) (1.2) (1.3) *for any* $\lambda > 0$ *which satisfies*

$$\|\nabla v_\lambda\|_{L^p(\Omega)} \to \infty \qquad \text{as } \lambda \to 0$$

and

$$\|\nabla v_\lambda\|_{L^p(\Omega)} \to 0 \qquad \text{as } \lambda \to \infty .$$

It is easy to see that the function $f(x, \xi) = q\xi^{q-1}$ satisfies the corresponding assumptions in Theorem 2.1, 2.2, 3.1 or 3.2 in each case. Hence the result follows simply.

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