Funkcialaj Ekvacioj, **37** (1994) 59–64

Painlevé-Umemura Extensions

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1. Introduction

Let K be a differential field of characteristic 0 and U be a universal extension of K. In what follows, unless otherwise stated, differential field extensions of K concerned will be assumed to be finitely generated, hence have U as a universal extension. For a differential field extension L of K by C_L we denote the field of constants of L.

Let R be a differential field extension of K. We say that R depends rationally on arbitrary constants over K if there exists a differential field extension E of K such that E and R are free over K and the equality $ER = EC_{ER}$ holds.

Recent discussions on Painlevé's first transcendent motivate the following (cf. [9], [10], [11]).

Definition. A differential field extension R of K is called a *Painlevé-Umemura* (PU-) extension of K if there exists a finite chain of differential field extensions of K: $K = R_0 \subset R_1 \subset \cdots \subset R_m$ such that $R_m = R$ and R_{i+1}/R_i depends rationally on arbitrary constants for each i. For instance, liouville extensions, or more widely, H-extensions defined in [4] are PU-extensions.

The aim of this note is to prove the following.

Theorem. *Suppose that a differential field extension R of K is contained in a certain PU-extension of K. Then R is itself a PU-extension of K.*

2. Invariant subfields

Let F be a field of characteristic 0 and t be an indeterminate over F. By $F[[t]]$ we denote the ring of formal power series in t with coefficients from F. Every ring homomorphism σ of F to $F[[t]]$ is always assumed to assign the unity to the unity. Let Σ be a set of such isomorphisms. For a subfield E of F with $\Sigma E \subset E[[t]]$, by E^{Σ} we denote the set of all e in E that are left invariant under every isomorphism in Σ. This set is a subfield of E. (cf. Kolchin [2, pp. 86–88].)

Lemma 1. *Let E be a subfield of F with $\Sigma E \subset E[[t]]$. Then F^{Σ} and E are linearly disjoint over E^{Σ}.*

Proof. Suppose a finite number of elements f_i of F^{Σ} to be linearly dependent over E. There exists a nontrivial linear combination of the f_i over E which vanishes. Among such linear combinations pick up the shortest one, say $\sum e_i f_i = 0$, where the e_i are nonzero elements of E. We may assume $e_i = 1$ for some i. Applying $\sigma \in \Sigma$, through subtraction, $\sum (e_i - \sigma e_i) f_i = 0$. By our assumption on the sum, the coefficients of the linear combination must be all zero. It thus follows every e_i lies in E^{Σ}. This is our assertion.

Lemma 2. *Let E be a subfield of F. Suppose $F = EF^{\Sigma}$. Then any intermediate field H between E and F satisfying $\Sigma H \subset H[[t]]$ has the property $H = EH^{\Sigma}$.*

Proof. We know that F^{Σ} and H are linearly disjoint over H^{Σ}. Hence EF^{Σ} and H are also linearly disjoint over EH^{Σ} since $H^{\Sigma} \subset EH^{\Sigma} \subset H$. Let h be in H. If $h \notin EH^{\Sigma}$, then $h \notin EF^{\Sigma}$. This contradicts the assumption $F = EF^{\Sigma}$. Thus $H \subset EH^{\Sigma}$, therefore $H = EH^{\Sigma}$.

Lemma 3. *Let I be a subfield of F and E, J be intermediate fields between I and F. Suppose that E and J are linearly disjoint over I, $\Sigma E \subset E[[t]]$ and $E^{\Sigma} = I$. Then every isomorphism in Σ can be extended to an isomorphism of JE to $JE[[t]]$ which is trivial on J and in this sense $(JE)^{\Sigma} = J$ holds.*

Proof. Clearly $J \subset (JE)^{\Sigma}$. To show the converse relation let x be an element of $(JE)^{\Sigma}$. It has the expression

$$x = \frac{\sum a_i x_i}{\sum b_j y_j},$$

where the a_i and the b_j are in E, the x_i and the y_j are in J, and $\sum b_j y_j \neq 0$. We may assume without losing generality that the x_i are linearly independent over I therefore J and $b_j = y_j = 1$ for some j. Among such expressions pick one such that the occurrence of nonzero b_j is the minimum in number. Applying $\sigma \in \Sigma$, we have

$$x \sum (a_i - \sigma a_i) x_i - \sum (b_j - \sigma b_j) y_j = 0 .$$

The minimality implies $b_j = \sigma b_j$ and then $a_i = \sigma a_i$ by the linear independence of the x_i. This completes the proof.

The results apply to the proof of the following known fact: Let L be a differential field extension of K and M be a differential field extension of L. Assume $M \subset LC_U$. Then $M = LC_M$. In fact let $\sigma = \exp tD$, where D

denotes the differentiation of M, and $\Sigma = \{\sigma\}$. By Lemma 2 $(LC_U)^{\Sigma} = C_U$ and Lemma 1 implies our assertion.

3. Rational dependence

Let K be a differential field of characteristic 0 and R be a differential field extension of K. Denote by $\Pi_K(R)$ the set of all intermediate differential fields between K and R that depend rationally on arbitrary constants over K. According to the last sentence in the previous section, the element P of this set is characterized by the property that there exists a differential field extension E of K such that P and E are free over K and the compositum EP is generated by constants over E. We shall here prove that this set has the maximum element in the inclusion order and discuss its fundamental property. At the end of this section a proof of the Theorem will be given.

Proposition 1. *For $P \in \Pi_K(R)$ there is a differential field E of K such that R and E are free over K and $EP = EC_{EP}$.*

Proof. By definition there is a differential field extension F of K such that P and F are free over K and FP/F is generated by constants. According to the following lemma there is a differential field extension Q of P such that FR and Q are free and differentially isomorphic over P. As E we may take the image of F.

Lemma 4. *There is a differential field extension S of K such that R and S are free and differentially isomorphic over K.*

Proof. Let L denote the algebraic closure of K in R. Then R/K is regular. The quotient field $\langle R \otimes_L R\rangle$ of the differential domain $R \otimes_L R$ is expressible as the compositum of two differential fields $R \otimes_L 1$ and $1 \otimes_L R$. Identifying the former with R and denoting the latter by S, we have the desired conclusion.

Proposition 2. *The set $\Pi_K(R)$ has the maximum element.*

Proof. Let P and Q be in $\Pi_K(R)$. Then there is a differential field extension E of K such that R and E are free over K and EP/E is generated by constants. Since $Q \in \Pi_K(ER)$ also holds there exists a differential field extension F of K such that ER and F are free over K and FQ/F is generated by constants. Hence $EFPQ/EF$ is generated by constants, therefore $PQ \in \Pi_K(R)$.

In the sequel this maximum element of $\Pi_K(R)$ will be denoted $P_K(R)$.

Proposition 3. *Let E be a differential field extension of K for which R*

and E are free over K. Then $EP \in \Pi_E(ER)$ for any $P \in \Pi_K(R)$ and $R \cap Q \in \Pi_K(R)$ for any $Q \in \Pi_E(ER)$.

Proof. Assume $P \in \Pi_K(R)$. Since $P\Pi_K(ER)$, by Proposition 1, there exists a differential field extension F of K such that ER and F are free over K and FP/F is generated by constants. This shows that EFP/EF is generated by constants, therefore $EP \in \Pi_E(ER)$. Assume next $Q \in \Pi_E(ER)$ and let $P = Q \cap R$. Then we have a differential field F of E such that ER and F are free over E and FQ/F, hence FP/F, is generated by constants. This completes the proof.

Proposition 4. *Let $P \in \Pi_K(R)$ and σ be a differential isomorphism of R into U. Then $\sigma P \in \Pi_{\sigma K}(\sigma R)$.*

Proof. Take a differential field extension E of K such that R and E are free over K and EP/E is generated by constants. It is known that there exists an extension τ of σ which is a differential isomorphism of ER into U. Clearly $\tau R = \sigma R$ and τE are free over $\tau K = \sigma K$. Since EP/E is generated by constants it follows that $\tau(E)\sigma(P)/\tau E$ is generated by constants as well. This completes the proof.

Proposition 5. *Let E be a differential field extension of K which is free from R over K. Then R and $P_E(ER)$ are linearly disjoint over $P_K(R)$. In particular*

$$P_E(ER) = EP_K(R), \ P_K(R) = R \cap P_E(ER).$$

Proof. Now suppose that nonzero elements $(a_i)_{1 \le i \le m}$ of R are linearly dependent over $P_E(ER)$. Let L denote the algebraic closure of K in R and Φ be the set of all differential field extension of L from which R is free over L. For each $A \in \Phi$ by $n(A)$ we denote the dimension of the $P_A(AR)$-vector space generated by the a_i. There is an $F \in \Phi$ with $n = n(F)$ being the minimum. We may suppose that the elements $(a_i)_{1 \le i \le n}$ constitute the base for the vector space generated by the a_i. Any other element $a = a_i$ has the expression

$$a = \sum_{i=1}^{n} x_i a_i, \ x_i \in P_F(FR).$$

Let M denote the algebraic closure of L in F. The quotient field $\langle F \otimes_M F \otimes_M MR \rangle$ is considered as a differential subfield of U, and written $F_1 F_2 R$, where F_1 stands for $F \otimes_M 1$ and F_2 stands for $1 \otimes_M F$. The mappings $\sigma_j \colon F \to F_j$ defined by

$$\sigma_1(x) = x \otimes 1, \ \sigma_2(x) = 1 \otimes x$$

are differential isomorphisms over M. These are also regarded as differential isomorphisms of FR into F_1F_2R over MR. Applying σ_j to the above equality, through subtracting, we have

$$\sum_{i=1}^{n} (\sigma_1 x_i - \sigma_2 x_i)a_i = 0 .$$

Since the elements x_i are all contained in $P_F(FR)$, by Proposition 4, it follows

$$\sigma_j x_i \in P_{F_j}(F_j R) ,$$

whence

$$\sigma_1 x_i - \sigma_2 x_i \in P_{F_1F_2}(F_1F_2R) .$$

By our assumption we see the $(a_i)_{1 \le i \le n}$ are linearly independent over $P_{F_1F_2}(F_1F_2R)$. This implies $\sigma_1 x_i = \sigma_2 x_i$, whence $x_i \in MR$, and $\in P_M(MR)$. (See Sublemma (25.5) in [10].) We must show $x_i \in P_K(R)$. To this end let N be a Galois extension of L including M. Since N and R are linearly disjoint over L, every automorphism σ of N/L is regarded as a differential automorphism of NR/R. By Proposition 4, we see $\sigma x_i \in P_N(NR)$. From the equality

$$\sum_{i=1}^{n} (x_i - \sigma x_i)a_i = 0$$

we find that $x_i = \sigma x_i$ for each σ, hence $x_i \in R$, using Lemma 3. This completes the proof.

Proposition 6. *Let S be a differential field extension of R. Set $R_0 = S_0 = K$ and for each $i \ge 1$*

$$R_i = P_{R_{i-1}}(R),\ S_i = P_{S_{i-1}}(S) .$$

Then for any i, R and S_i are linearly disjoint over R_i.

Proof. The proof proceeds by induction on i. In case $i = 0$, it is trivial. Let us first show the assertion in case $i = 1$. Let M denote the algebraic closure of K in S and take E to be a differential field extension of M such that S and E are linearly disjoint over M and $ES_1 = EC_{ES}$. Then $ER_1 = EC_{ER}$. Since ER and C_{ES} are linearly disjoint over C_{ER}, it follows that ER and $ES_1 = ER_1C_{ES}$ are linearly disjoint over ER_1. By the way, MR and E are linearly disjoint over M, whence MR and ES_1 are linearly disjoint over MR_1. We also know that R and MR_1 are linearly disjoint over R_1, since R_1 includes the algebraic closure of K in R. By this we conclude that R and ES_1 are linearly disjoint over R_1. Now assume i is positive and the assertion in case $i-1$ holds validly. Since R and S_{i-1} are linearly disjoint over R_{i-1}, by Proposition 5, R and $P_{S_{i-1}}(S_{i-1}R)$ are linearly disjoint over $R_i = P_{R_{i-1}}(R)$. By the fact obtained just above, noting $S_{i-1} \subset S_{i-1}R \subset S$, we have $S_{i-1}R$ and S_i are linearly disjoint over $P_{S_{i-1}}(S_{i-1}R)$. These verify the assertion in case i.

Proof of Theorem. By hypothesis there exists a PU-extension S of K containing R. As above we can construct the differential subfields of R and S:

$$R_i = P_{R_{i-1}}(R),\ S_i = P_{S_{i-1}}(S)\ (i \geq 1);\ R_0 = S_0 = K\,.$$

According to the above proposition $R_i = R \cap S_i$. Since $S = S_m$ for some integer m it follows that $R_m = R \cap S$, which shows that R is a PU-extension of K.

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(Ricevita la 11-an de oktobro, 1991)