

## A Comparison Result for Hamilton-Jacobi Equations and Applications to some Differential Games Lacking Controllability\*

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### 0. Introduction

In this paper we consider viscosity solutions to Hamilton-Jacobi equations

$$(0.1) \quad H(x, u, Du) = 0 \quad \text{in } \mathcal{O},$$

where  $\mathcal{O}$  is an unbounded open set in  $\mathbf{R}^N$  and the Hamiltonian  $H$  satisfies structural conditions which include the cases of the Bellman equation in optimal control theory and the Isaacs equation for the value function of differential games. We study comparison and uniqueness questions for boundary value problems where the boundary condition is the classical Dirichlet condition on some part of  $\partial\mathcal{O}$  (even singular, i.e.  $+\infty$ ) and the weaker “Dirichlet type” condition

$$(0.2) \quad u = g \quad \text{or} \quad H(x, u, Du) = 0 \quad \text{on } \Gamma \subseteq \partial\mathcal{O},$$

which has to be interpreted in the viscosity sense. Condition (0.2) was first studied by Ishii [I4] and Barles & Perthame [BP1, BP2], who showed that it is the natural condition for the limit of the vanishing viscosity approximation to the classical Dirichlet boundary value problem, as well as for the value functions of deterministic control problems whose payoff involves the exit time from  $\mathcal{O}$ .

We first prove a comparison theorem extending a result of Ishii [I4] to unbounded domains with minimal regularity assumptions and to Hamiltonians verifying structural conditions inspired by an older work of Ishii [I2]. Both these extensions are strongly motivated by optimal control theory, as we show on several examples. Then we derive from it a uniqueness theorem for bounded viscosity solutions, continuous at the boundary  $\partial\mathcal{O}$ , of (0.1), coupled

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with mixed boundary condition that can be Dirichlet, or Dirichlet-type (0.2), or “constrained”, that is

$$(0.3) \quad H(x, u, Du) \geq 0 \quad \text{on } A \subseteq \partial\mathcal{O}.$$

The boundary condition (0.3) is the correct one for value functions of control problems with the “state constraint” that the system cannot exit  $\bar{\mathcal{O}}$  through  $A$ , see Soner [Sn] and Capuzzo Dolcetta & P. L. Lions [CDL]. For the portion of boundary where either (0.2) or (0.3) is verified we need some regularity of the boundary itself, and we assume a condition introduced by Soner [Sn] which we show to be equivalent to the uniform Lipschitz continuity of  $\partial\mathcal{O}$ . We also show on counterexamples that comparison principle and uniqueness of the solution fail if such a condition is violated even at a single point.

Next we apply the comparison theorem to study the free boundary problem

$$(0.4) \quad \begin{cases} H(x, DU) = 0 & \text{in } \Omega \setminus \mathcal{T}, \\ U \geq 0 & \text{on } \partial\mathcal{T}, \\ H(x, DU) \leq 0 & \text{if } x \in \partial\mathcal{T} \text{ and } U(x) > 0, \\ U(x) \rightarrow +\infty & \text{as } x \rightarrow x_0 \in \partial\Omega. \end{cases}$$

where  $\mathcal{T}$  is a given closed set,  $\Omega$  is an unknown open set containing  $\mathcal{T}$ , and the Hamiltonian is of the form

$$(0.5) \quad H(x, p) := \min_{b \in B} \max_{a \in A} \{-f(x, a, b) \cdot p - h(x, a, b)\}.$$

We prove that if  $(\Omega, U)$  is a solution pair of (0.4) with  $U$  continuous and bounded below, then  $U$  is the value function of a suitable differential game and  $\Omega$  is the set where such function is finite.

This completes the main result of our previous paper [BS2] where we considered a similar free boundary problem with the stronger classical Dirichlet condition  $U = 0$  on  $\partial\mathcal{T}$ . This condition is always satisfied by the value function  $V$  of generalized pursuit-evasion games, but it is not very useful if  $V$  is discontinuous at points of  $\partial\mathcal{T}$ . The continuity of  $V$  at such points holds if the system is better locally controllable by the pursuer than by the evader in a neighbourhood of  $\mathcal{T}$ , a property that is often not verified in practice, see e.g. the books of Isaacs [Is] and Hajek [H]. Under a regularity assumption on  $\partial\mathcal{T}$ , the result of this paper allows  $V$  to be discontinuous on  $\partial\mathcal{T}$  provided it can be extended continuously to it, which implies that  $V$  is continuous off  $\mathcal{T}$ . (In particular we cannot treat games where “barriers” occur, where the term “barrier” is in the sense of the theory of differential games, that is,

roughly speaking, a surface of discontinuity of  $V$  off  $\mathcal{T}$ ). This regularity of  $V$  may hold even when the system has very bad controllability properties near  $\mathcal{T}$ , but we show that a sufficient condition for it is, roughly speaking, the controllability of the system near some suitable subset of  $\mathcal{T}$ . To our knowledge this result is new also in the case of time-optimal control by a single player.

We have already used the comparison theorem of next section for various other applications. In [BS3] it plays a crucial role in proving the convergence of discrete approximations to the capture time in pursuit-evasion games, a question left open since the classical paper of Fleming [F1]. In [S1] it is employed to prove the equivalence of three seemingly different concepts of value in survival games, namely Fleming's [F1], Friedman's [F], and Varaya-Roxin-Elliott-Kalton's [EK]. In [BS4] it implies the coincidence of value functions with different sets of admissible controls. In [S2] it is used to prove super and sub-optimality principles for survival games. In the very recent work of one of the authors and Staicu [BS $\sigma$ ] it is exploited to prove, via an approximation argument, the uniqueness of discontinuous solutions (also off  $\mathcal{T}$ ) for a boundary value problem associated to the minimum time function (where in particular the Hamiltonian (0.5) is convex in  $p$ , i.e.  $B$  is a singleton).

Finally we recall that the fundamental results on the theory of viscosity solutions of Hamilton-Jacobi equations can be found in Crandall & Lions [CL] and Crandall, Evans & Lions [CEL] and that the application of this theory to control problems was first made by Lions [LL], see also [B] and [BF1] for time-optimal control. For the connection to differential games we refer to Barron, Evans & Jensen [BEJ], Evans & Souganidis [ES], Souganidis [So], our papers [BS1, BS2], and the references therein. For a general introduction to differential games we refer to the books of Isaacs [Is], Friedman [F] and Hajek [H], and to Fleming [F1] and Elliott & Kalton [EK].

The paper is organized as follows: in *Section 1* we state and prove the comparison and uniqueness results for boundary value problems with given boundary, in *Section 2* we study the free boundary problem (0.4), in *Section 3* we prove the sufficient condition for the continuity off  $\mathcal{T}$  of the value function  $V$ . In the *Appendix* we show that Soner's regularity condition of  $\partial\mathcal{O}$  is equivalent to the Lipschitz continuity of  $\partial\mathcal{O}$ .

## 1. Comparison and uniqueness theorems

We begin by recalling the definition of viscosity solution for a fully non-linear first order partial differential equation

$$(1.0) \quad H(x, u, Du) = 0 \quad \text{in } \mathcal{O},$$

where  $H: \mathcal{O} \times \mathbf{R} \times \mathbf{R}^N \rightarrow \mathbf{R}$  is locally bounded and  $\mathcal{O}$  is open. First of all for a function  $G: \mathcal{Q} \rightarrow \mathbf{R}$ ,  $\mathcal{Q} \subset \mathbf{R}^k$  the upper and lower semicontinuous envelopes are defined by

$$G^*(x) := \limsup_{r \downarrow 0} \{G(y): y \in \mathcal{Q}, |y - x| \leq r\},$$

$$G_*(x) := \liminf_{r \downarrow 0} \{G(y): y \in \mathcal{Q}, |y - x| \leq r\}, \quad x \in \bar{\mathcal{Q}}.$$

Let  $u_1, u_2: \mathcal{Q} \rightarrow \mathbf{R}$  be respectively an upper and a lower semicontinuous function. We say that  $u_1, u_2$  are respectively a viscosity subsolution and supersolution of (1.0) if for all  $\varphi \in C^1(\mathcal{O})$  such that  $u_1 - \varphi$  has a local maximum at  $y$ , respectively  $u_2 - \varphi$  has a local minimum at  $y$ , we have

$$H_*(y, u_1(y), D\varphi(y)) \leq 0,$$

and respectively

$$H^*(y, u_2(y), D\varphi(y)) \geq 0.$$

We say that  $u$  is a viscosity solution of (1.0) if  $u^*$  is a subsolution and  $u_*$  is a supersolution.

In the following we will also deal with the so called ‘‘Dirichlet type boundary conditions’’ and say that an upper semicontinuous function  $u_1$  satisfies

$$u_1 \leq g \quad \text{or} \quad H(x, u_1, Du_1) \leq 0 \quad \text{on} \quad \partial\mathcal{O}$$

in the viscosity sense if for all  $\varphi \in C^1(\mathcal{O})$  such that  $u_1 - \varphi$  has a local maximum at  $y$ , we have

$$u_1(y) \leq g(y) \quad \text{or} \quad H(y, u_1(y), D\varphi(y)) \leq 0 \quad \text{on} \quad \partial\mathcal{O}.$$

The results of this Section extend in some way several comparison and uniqueness theorems for viscosity solutions of Hamilton-Jacobi equations, e.g. Theorem 4.1 in [I1], Theorem 2 in [I2], Theorem 2.2 in [Sn], Corollary 2.1 in [I4], Theorem IX.1 in [CDL], Theorem 2.1 in [BS2], Theorem 1 in [CIL], and its proof uses several arguments of the proofs of those results, in particular of [I4]. We consider an open set  $\mathcal{O} \subseteq \mathbf{R}^N$  and a Hamiltonian  $H$  satisfying the conditions

$$(1.1a) \quad \begin{cases} H: \bar{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^N \rightarrow \mathbf{R} \text{ is continuous,} \\ r \rightarrow H(x, r, p) \text{ is nondecreasing for all } x \in \bar{\mathcal{O}}, p \in \mathbf{R}^N, \end{cases}$$

$$(1.1b) \quad \begin{cases} H(y, r, p - \tau y) - H(x, r, p + \lambda x) \leq |p|L|x - y| + \rho(|x - y|, R) \\ \quad + \lambda K(1 + |x|^2) + \tau K(1 + |y|^2), \\ \text{for all } x, y \in \bar{\mathcal{O}}, r \in \mathbf{R}, |x|, |y|, |r| \leq R, p = \mu(x - y), \mu, \lambda, \tau, R \geq 0, \\ \text{and for all } p \in \mathbf{R}^N \text{ if } x, y \in B(\Gamma, c'), \end{cases}$$

$$(1.1c) \quad \begin{cases} H(x, r, p) - H(x, r, p + x - w) \leq \sigma(|x - w|), \\ \text{for all } p \in \mathbf{R}^N, x \in B(\Gamma, c'), w \in \Gamma, \end{cases}$$

for some  $c' > 0$ ,  $\rho(\cdot, R)$  and  $\sigma(\cdot)$  continuous, increasing and null in 0 for all  $R \geq 0$ , and where  $\Gamma \subset \partial\mathcal{O}$ ; and

$$(1.2) \quad \begin{cases} \text{there is a constant } c > 0 \text{ and a bounded, uniformly} \\ \text{continuous function } \eta: \bar{\mathcal{O}} \cap B(\partial\mathcal{O}, c) \rightarrow \mathbf{R}^N \\ \text{such that } B(x + t\eta(x), ct) \subseteq \mathcal{O} \text{ for all } x \in \bar{\mathcal{O}} \cap B(\partial\mathcal{O}, c) \\ \text{and } 0 < t \leq c; \end{cases}$$

(here and throughout the paper  $B(y, r)$  denotes the open ball of radius  $r$  centered at  $y$ ,  $B(X, r) := \{x: \text{dist}(x, X) < r\}$  and  $B(\emptyset, R) = \emptyset$ ). These assumptions look rather technical and deserve some comments. The one sided estimates of the continuity of  $H$  in (1.1b–c) are designed to be as weak as possible on the Hamiltonians arising in the theory of optimal control and differential games described in Example 1.6, see also Example 1.7 for a remark about their optimality. Condition (1.2) was introduced by Soner [Sn] in the study of constrained viscosity solutions and then used by all authors who studied weak boundary conditions in the viscosity sense. It says essentially that the boundary of  $\mathcal{O}$  is a Lipschitz surface: this is made precise in the Appendix. This condition is relaxed in Theorem 1.2. Its optimality is discussed at the end of the present Section: see Examples 1.11 and 1.12. We also give some examples of classical problems in control theory whose Dynamic Programming equation has to be solved in a set with unbounded boundary and discuss (1.2) in these cases, see Examples 1.8, 1.9, 1.10.

**Theorem 1.1.** *Let  $H$  and  $\mathcal{O}$  satisfy (1.1) with  $\Gamma = \partial\mathcal{O}$  and (1.2),  $v_i: \bar{\mathcal{O}} \rightarrow \mathbf{R}$ ,  $i = 1, 2$ ,  $v_1$  upper semicontinuous and bounded from above,  $v_2$  lower semicontinuous, bounded from below and continuous at each point of  $\partial\mathcal{O}$ . Assume  $v_1$  and  $v_2$  are respectively viscosity subsolution and supersolution of*

$$v + H(x, v, Dv) = 0 \quad \text{in } \mathcal{O},$$

and

$$v_1 \leq v_2 \quad \text{or} \quad v_1 + H(x, v_1, Dv_1) \leq 0 \quad \text{on } \partial\mathcal{O}.$$

Then  $v_1 \leq v_2$  in  $\bar{\mathcal{O}}$ . The same conclusion holds if  $v_1$  instead of  $v_2$  is assumed to be continuous at each point of  $\partial\mathcal{O}$ , and  $v_1 \leq v_2$  or  $v_2 + H(x, v_2, Dv_2) \geq 0$  on  $\partial\mathcal{O}$ .

*Proof.* We only prove the first statement, the second being similar. We proceed by contradiction and assume that  $\sup_{\bar{\mathcal{O}}} (v_1 - v_2) > 0$ . Let  $z(x, y) := v_1(x) - v_2(y)$ .

First case:  $A := \sup_{\bar{c}} (v_1 - v_2)^+ < \sup_{\bar{c}} (v_1 - v_2)$ . We define

$$\tilde{H}(x, y, r, p, q) := H(x, r + v_2(y), p) - H(y, v_1(x) - r, -q),$$

and observe that  $\tilde{H}$  is lower semicontinuous and  $z$  is a viscosity subsolution of

$$z + \tilde{H}(x, y, z, D_x z, D_y z) = 0 \quad \text{in } \mathcal{O} \times \mathcal{O}.$$

Let  $\delta > 0$  be such that  $\sup_{\bar{c}} (v_1 - v_2) > A + 2\delta$ . We also consider the following function

$$\phi(x, y) := A + \delta + \varepsilon^{-1}(\varepsilon^{4L/\varepsilon} + |x - y|^{2\varepsilon/2L}) + \beta(\langle x \rangle^m + \langle y \rangle^m)$$

where  $\varepsilon, \beta, m \in (0, 1)$  and  $\langle x \rangle = (1 + |x|^2)^{1/2}$ . By the definition of  $\phi$  we observe the following:

$$(1.3) \quad \phi(x, y) \geq A + \delta;$$

$$(1.4) \quad \phi(x, x) = A + \delta + \varepsilon + 2\beta\langle x \rangle^m;$$

$$(1.5) \quad \phi(x, y) \geq \varepsilon^{-1+\varepsilon/L} \quad \text{if } |x - y| = \varepsilon.$$

Moreover if we set  $R(\beta) = \left\{ \left( \sup_{\bar{c} \times \bar{c}} z + 1 \right) / \beta \right\}^{1/m}$ , then we get

$$(1.6) \quad z(x, y) - \phi(x, y) \leq -1 \quad \text{if } |x| \text{ or } |y| = R(\beta).$$

Let  $\tilde{x} \in \mathcal{O}$  be such that  $v_1(\tilde{x}) - v_2(\tilde{x}) > A + 2\delta$ , then we fix first  $\beta$  and then  $\varepsilon$  small enough such that

$$(1.7) \quad \varepsilon + 2\beta\langle \tilde{x} \rangle^m \leq \delta/2;$$

$$(1.8) \quad \varepsilon^{-1+\varepsilon/L} \geq \sup_{\bar{c} \times \bar{c}} z;$$

$$(1.9) \quad |\tilde{x}| < R(\beta);$$

$$(1.10) \quad z(x, y) \leq A + \delta \quad \text{if } x \text{ or } y \in \partial\mathcal{O}, |x - y| < \varepsilon \text{ and } |x|, |y| < R(\beta);$$

$$(1.11) \quad \rho(\varepsilon, R(\beta)) \leq \delta/2.$$

where (1.10) holds for any fixed  $\beta$  because the upper semicontinuity of  $z$  implies

$$\limsup_{\varepsilon \rightarrow 0} \sup \{z(x, y) - A\} : (x, y) \in \partial(\mathcal{O} \times \mathcal{O}), |x - y| < \varepsilon \text{ and } |x|, |y| < R(\beta) \} \leq 0.$$

Now if we consider the open set  $\Omega(\varepsilon, \beta) := \{(x, y) \in \mathcal{O} \times \mathcal{O}; |x - y| < \varepsilon \text{ and } |x|, |y| < R(\beta)\}$ , then

$$\max_{\partial\Omega(\varepsilon, \beta)} (z - \phi) \leq 0,$$

by (1.3), (1.10) if either  $x$  or  $y$  is on  $\partial\mathcal{O}$ , by (1.5), (1.8) if  $|x - y| = \varepsilon$  and by (1.6) if  $|x|$  or  $|y| = R(\beta)$ . Moreover (1.9) implies  $(\tilde{x}, \tilde{x}) \in \Omega(\varepsilon, \beta)$  and then by (1.4), (1.7) and the choice of  $\tilde{x}$  we obtain  $\max_{\overline{\Omega(\varepsilon, \beta)}} (z - \phi) \geq \delta/2$ . Therefore let  $(\bar{x}, \bar{y}) \in \mathcal{O} \times \mathcal{O}$  be a point where the last maximum is attained. By the definition of viscosity subsolution, setting

$$p := (e^{4L/\varepsilon} + |\bar{x} - \bar{y}|^2)^{\varepsilon/2L-1}(\bar{x} - \bar{y})/L,$$

we get

$$\begin{aligned} 0 &\geq z(\bar{x}, \bar{y}) + \tilde{H}(\bar{x}, \bar{y}, z(\bar{x}, \bar{y}), D_x\phi(\bar{x}, \bar{y}), D_y\phi(\bar{x}, \bar{y})) \\ &\geq \delta/2 + \phi(\bar{x}, \bar{y}) + H(\bar{x}, v_2(\bar{y}), D_x\phi(\bar{x}, \bar{y})) - H(\bar{y}, v_2(\bar{y}), -D_y\phi(\bar{x}, \bar{y})) \\ &\geq \delta/2 + \phi(\bar{x}, \bar{y}) + H(\bar{x}, v_2(\bar{y}), p + m\beta\langle\bar{x}\rangle^{m-2}(\bar{x})) \\ &\quad - H(\bar{y}, v_2(\bar{y}), p - m\beta\langle\bar{y}\rangle^{m-2}(\bar{y})) \\ &\geq 3\delta/2 + \varepsilon^{-1}(e^{4L/\varepsilon} + |\bar{x} - \bar{y}|^2)^{\varepsilon/2L} + \beta(\langle\bar{x}\rangle^m + \langle\bar{y}\rangle^m) - |p|L|\bar{x} - \bar{y}| \\ &\quad - \rho(|\bar{x} - \bar{y}|, R(\beta)) - K(1 + |\bar{x}|^2)m\beta\langle\bar{x}\rangle^{m-2} - K(1 + |\bar{y}|^2)m\beta\langle\bar{y}\rangle^{m-2} \\ &\geq \delta + \beta(\langle\bar{x}\rangle^m + \langle\bar{y}\rangle^m)(1 - mK) \\ &> 0 \end{aligned}$$

for  $m \leq K^{-1}$ , where the second inequality follows from  $v_1(\bar{x}) - v_2(\bar{y}) = z(\bar{x}, \bar{y}) \geq \delta/2 + \phi(\bar{y}, \bar{y}) \geq 0$  and the monotonicity of  $H$  in the second variable, the fourth inequality follows from the definition of  $\phi$  and (1.1), and the fifth from an easy computation and (1.11). This gives the required contradiction.

*Second case:*  $\sup_{\partial\mathcal{C}} (v_1 - v_2) = \sup_{\bar{\mathcal{C}}} (v_1 - v_2) (> 0)$ . Then

$$\max_{\bar{\mathcal{C}}} (v_1(x) - v_2(x) - 2\beta\langle x \rangle^m) = \max_{\partial\mathcal{C}} (v_1(x) - v_2(x) - 2\beta\langle x \rangle^m) + \omega(\beta)$$

where  $\omega(\beta) \geq 0$  and  $\omega(\beta) \rightarrow 0$  as  $\beta \rightarrow 0^+$  (such a condition will hold for each function  $\omega_i$  which appears in the following).

Let  $w$  be a point, depending on  $\beta$ , at which  $\max_{\partial\mathcal{C}} (v_1(x) - v_2(x) - 2\beta\langle x \rangle^m)$  is attained and choose  $\beta < \bar{\beta}$  so that such a max is positive. We consider the function

$$\phi(x, y) := v_1(x) - v_2(y) - |(y - x)/\varepsilon - \eta(w)|^2 - \frac{1}{2}|x - w|^2 - \beta(\langle x \rangle^m + \langle y \rangle^m)$$

where  $\eta$  is the function that appears in (1.2), and, for  $\beta$  fixed, we take a

compact set  $B = \overline{B(0, R(\beta))} \cap \bar{\mathcal{O}}$ , such that  $w \in B$  and  $\phi(x, y) \leq 0$  for all  $(x, y) \in (\mathcal{O} \times \mathcal{O}) \setminus (B \times B)$ . By (1.2)  $w + \varepsilon\eta(w) \in \mathcal{O}$  if  $\varepsilon \leq c$ . Then if  $\phi(x, y)$  attains its maximum value in  $\bar{\mathcal{O}} \times \bar{\mathcal{O}}$  at  $(\bar{x}, \bar{y})$  we obtain, for  $\beta \leq \bar{\beta}$

$$(1.12) \quad \phi(\bar{x}, \bar{y}) \geq \phi(w, w + \varepsilon\eta(w)) \geq z(w, w) - \omega_1(E\varepsilon) - 2\beta\langle w \rangle^m - E\varepsilon \geq \delta > 0$$

where  $E = \sup \eta$ ,  $\omega_1$  denotes the modulus of continuity of  $v_2$  on  $\partial\mathcal{O} \cap B$  and the last inequality holds for  $\varepsilon \leq \varepsilon(\beta)$  small enough. Note that this also implies  $(\bar{x}, \bar{y}) \in B \times B$ . Observe also that

$$(1.13) \quad v_1(\bar{x}) - v_2(\bar{y}) \geq \delta + \beta(\langle \bar{x} \rangle^m + \langle \bar{y} \rangle^m) > 0.$$

Moreover by the first two inequalities in (1.12)

$$(1.14) \quad |(\bar{y} - \bar{x})/\varepsilon - \eta(w)|^2 + \frac{1}{2}|\bar{x} - w|^2 \leq z(\bar{x}, \bar{y}) - \beta(\langle \bar{x} \rangle^m + \langle \bar{y} \rangle^m) - z(w, w) \\ + 2\beta\langle w \rangle^m + \omega_2(\varepsilon).$$

It follows from (1.14) that  $|\bar{x} - \bar{y}| \leq C\varepsilon$ ,  $C$  independent of  $\varepsilon$ . Thus again by (1.14) and since  $z$  is u.s.c.

$$|(\bar{y} - \bar{x})/\varepsilon - \eta(w)|^2 + \frac{1}{2}|\bar{x} - w|^2 \leq \sup_{\substack{|x-y| \leq C\varepsilon \\ (x,y) \in \bar{B} \times \bar{B}}} \{z(x, y) - \beta(\langle x \rangle^m + \langle y \rangle^m)\} \\ - \max_{\bar{\mathcal{O}}} (v_1(x) - v_2(x) - 2\beta\langle x \rangle^m) \\ + \omega(\beta) + \omega_2(\varepsilon) \\ \leq \omega_3(\varepsilon) + \omega(\beta).$$

Therefore we get

$$(1.15) \quad |(\bar{y} - \bar{x})/\varepsilon - \eta(w)| \leq \omega_4(\varepsilon) + \omega(\beta)^{1/2};$$

$$(1.16) \quad |\bar{x} - w| \leq \omega_4(\varepsilon) + \omega(\beta)^{1/2}.$$

From (1.15) we deduce

$$(1.17) \quad |\bar{y} - \bar{x}| \leq C_1\varepsilon$$

where  $C_1$  is independent of  $\varepsilon$  and  $\beta$  for all  $\beta$  small enough and  $\varepsilon \leq \varepsilon(\beta)$ . Moreover

$$(1.18) \quad |\bar{y} - (\bar{x} + \varepsilon\eta(\bar{x}))| \leq \varepsilon(|(\bar{y} - \bar{x})/\varepsilon - \eta(w)| + |\eta(w) - \eta(\bar{x})|)$$

which implies by (1.2) that  $\bar{y} \in \mathcal{O}$  if  $\varepsilon \leq c$  and  $\omega_4(\varepsilon) + \omega(\beta)^{1/2} + |\eta(w) - \eta(\bar{x})| \leq c$  (by (1.15)). The last inequality holds if we choose first  $\beta$

and then  $\varepsilon$  small by the uniform continuity of  $\eta$  and (1.16). By the definition of  $w$  and (1.12) we also have that, if  $\bar{x} \in \partial\mathcal{O}$ , then

$$\begin{aligned}
 (1.19) \quad 0 &< \max_{\partial\mathcal{O}} (v_1(x) - v_2(x) - 2\beta\langle x \rangle^m) \\
 &= z(w, w) - 2\beta\langle w \rangle^m \\
 &\leq z(\bar{x}, \bar{y}) + \omega_5(\varepsilon) \\
 &\leq z(\bar{x}, \bar{x}) + \omega_1(C_1\varepsilon) + \omega_5(\varepsilon)
 \end{aligned}$$

where we used (1.17) in the last inequality. Taking  $\varepsilon$  small enough this implies  $v_1(\bar{x}) > v_2(\bar{x})$ . Now we can use the PDE for  $v_2$  and the boundary condition if  $\bar{x} \in \partial\mathcal{O}$ , or the PDE if  $\bar{x} \in \mathcal{O}$ , for  $v_1$ , and get

$$v_2(\bar{y}) + H(\bar{y}, v_2(\bar{y}), -2((\bar{y} - \bar{x})/\varepsilon - \eta(w))/\varepsilon - \beta m \langle \bar{y} \rangle^{m-2} \bar{y}) \geq 0;$$

$$v_1(\bar{x}) + H(\bar{x}, v_1(\bar{x}), -2((\bar{y} - \bar{x})/\varepsilon - \eta(w))/\varepsilon + \beta m \langle \bar{x} \rangle^{m-2} \bar{x} + \bar{x} - w) \leq 0.$$

Therefore by (1.1), setting  $p = -2((\bar{y} - \bar{x})/\varepsilon - \eta(w))/\varepsilon$ ,  $r = v_2(\bar{y})$ , and using (1.13)

$$\begin{aligned}
 0 &\geq v_1(\bar{x}) - v_2(\bar{y}) + H(\bar{x}, r, p + \beta m \langle \bar{x} \rangle^{m-2} \bar{x} + \bar{x} - w) \pm H(\bar{x}, r, p + \beta m \langle \bar{x} \rangle^{m-2} \bar{x}) \\
 &\quad - H(\bar{y}, r, p - \beta m \langle \bar{y} \rangle^{m-2} \bar{y}) \\
 &\geq \delta + \beta(\langle \bar{x} \rangle^m + \langle \bar{y} \rangle^m) - \sigma(|\bar{x} - w|) - |p|L|\bar{x} - \bar{y}| - \rho(|\bar{x} - \bar{y}|, R(\beta)) \\
 &\quad - K(1 + |\bar{x}|^2)m\beta\langle \bar{x} \rangle^{m-2} - K(1 + |\bar{y}|^2)m\beta\langle \bar{y} \rangle^{m-2} \\
 &\geq \delta + \beta(1 - mK)(\langle \bar{x} \rangle^m + \langle \bar{y} \rangle^m) - \sigma(|\bar{x} - w|) \\
 &\quad - 2L\varepsilon^{-1}|\bar{x} - \bar{y}|(|\bar{y} - \bar{x})/\varepsilon - \eta(w)| - \rho(|\bar{x} - \bar{y}|, R(\beta)) \\
 &\geq \delta + \beta(1 - mK)(\langle \bar{x} \rangle^m + \langle \bar{y} \rangle^m) - \sigma(\omega_4(\varepsilon) + 2\omega(\beta)^{1/2}) \\
 &\quad - 2LC_1(\omega_4(\varepsilon) + \omega(\beta)^{1/2}) - \rho(C_1\varepsilon, R(\beta))
 \end{aligned}$$

where in the last inequality we have used (1.15), (1.16) and (1.17). Now we let first  $\varepsilon$  go to 0 and observe that for  $m \leq K^{-1}$  we get

$$0 \geq \delta - \sigma(\omega(\beta)^{1/2}) - 2LC_1\omega(\beta)^{1/2}$$

which gives the required contradiction for  $\beta$  small enough and concludes the proof. ///

Theorem 1.1 covers the problems with mixed boundary conditions studied by Capuzzo Dolcetta & Lions [CDL], where one knows a priori that the classical Dirichlet condition  $v_1 \leq v_2$  is attained on a given part of the boundary  $\Gamma_2$ , and the ‘‘constrained’’ condition  $v_2 + H(x, v_2, Dv_2) \geq 0$  is attained on the remaining part of the boundary  $\Gamma_1$ . However simple modifications of the

previous proof allow to show that for these problems no regularity of  $\Gamma_2$  is needed. This is made precise by next result where we assume that  $\Gamma_1$  and  $\Gamma_2$  “do not touch at  $\infty$ ”, namely

$$(1.20) \quad \text{for some } R > 0 \text{ dist}(\Gamma_1 \setminus B(0, R), \Gamma_2 \setminus B(0, R)) > 0,$$

and that  $\Gamma_1$  “is Lipschitz” (satisfies (1.2)) in the following way:

$$(1.21) \quad \left\{ \begin{array}{l} \text{there is a constant } c > 0, \text{ an open set } \Omega \supset \Gamma_1 \cup B(\Gamma_1 \setminus B(0, R), c), \\ \text{and a bounded, uniformly continuous function } \eta: \bar{\mathcal{O}} \cap \Omega \rightarrow \mathbf{R}^N \\ \text{such that } B(x + t\eta(x), ct) \subseteq \mathcal{O} \text{ for all } x \in \bar{\mathcal{O}} \cap \Omega \text{ and } 0 < t \leq c. \end{array} \right.$$

Note that (1.21) allows  $\Gamma_1$  and  $\Gamma_2$  to touch in  $B(0, R)$  without any regularity request on  $\Gamma_2$ , while choosing  $\Omega = B(\Gamma_1, c)$  (which looks more similar to (1.2)) would force some regularity of  $\Gamma_2$  near  $\Gamma_1$ , see also Remark 1.3 below.

**Theorem 1.2.** *Assume that  $\partial\mathcal{O} = \Gamma_1 \cup \Gamma_2$ , (1.20) and (1.21) hold,  $\Gamma_1$  is relatively open in  $\partial\mathcal{O}$  (and possibly empty), and  $H$  satisfies (1.1) with  $\Gamma = \Gamma_1$ . Let  $v_i: \bar{\mathcal{O}} \rightarrow \mathbf{R}$ ,  $i = 1, 2$ , be bounded functions such that  $v_1$  is upper semicontinuous,  $v_2$  is lower semicontinuous and continuous at each point of  $\Gamma_1$ . Assume moreover that  $v_1$  and  $v_2$  are respectively viscosity subsolution and supersolution of*

$$v + H(x, v, Dv) = 0 \quad \text{in } \mathcal{O},$$

and satisfy the following boundary condition

$$\begin{array}{ll} v_1 \leq v_2 & \text{on } \Gamma_2 \\ v_1 \leq v_2 \quad \text{or} \quad v_1 + H(x, v_1, Dv_1) \leq 0 & \text{on } \Gamma_1. \end{array}$$

Then  $v_1 \leq v_2$  in  $\bar{\mathcal{O}}$ . The same conclusion holds if we assume  $v_1$  continuous at each point of  $\Gamma_1$  and  $v_2 \geq v_1$  or  $v_2 + H(x, v_2, Dv_2) \geq 0$  on  $\Gamma_1$ .

*Proof.* The proof is obtained by a refinement to that of Theorem 1.1. We first remark that the regularity of the boundary is used only in the “Second case”. We first choose  $\gamma > 0$  such that  $\sup_{\partial\mathcal{O}} (v_1 - v_2) = 3\gamma$  and observe that

$$(1.22) \quad \max_{\partial\mathcal{O}} (v_1(x) - v_2(x) - 2\beta\langle x \rangle^m) \geq 2\gamma \text{ for all } \beta \leq \bar{\beta}.$$

Since  $v_1 - v_2 \leq 0$  on  $\Gamma_2$ , a point  $w$  where the max in (1.22) is attained is in  $\Gamma_1$ . Moreover (1.19) implies that if  $\bar{x} \in \partial\mathcal{O}$  then  $\bar{x} \in \Gamma_1$ , by taking  $\varepsilon$  small enough. We claim that  $\text{dist}(w, \Gamma_2) \geq \delta > 0$ , at least along a sequence  $\beta_n \rightarrow 0^+$ . Then, by choosing  $\beta$  and  $\varepsilon$  small enough, we can make  $\bar{x}$  arbitrarily close to  $w$  by (1.16), so that  $\bar{x}$  is in the domain of  $\eta$  by (1.21), we can write (1.18) and get  $\bar{y} \in \mathcal{O}$ . The rest of the proof is now the same as in Theorem 1.1.

To prove the claim we first observe that, by the choice of  $w$  and (1.22), we have  $v_1(w) - v_2(w) \geq 2\gamma > 0$  if  $\beta$  is small enough. Next we note that if  $|w(\beta)| > R$  as  $\beta \rightarrow 0^+$ , then  $\text{dist}(w, \Gamma_2) \geq \delta' > 0$  for  $\beta$  small enough by assumption (1.20). If instead we can find a subsequence  $\beta_n \rightarrow 0^+$  such that  $w(\beta_n) \rightarrow \bar{w} \in \bar{\Gamma}_1$ , then by the semicontinuity of  $v_1$  and  $v_2$  we have that

$$v_1(\bar{w}) - v_2(\bar{w}) \geq 2\gamma > 0.$$

Since  $v_1 \leq v_2$  on  $\Gamma_2 = \partial\mathcal{O} \setminus \Gamma_1$ ,  $\bar{w} \in \Gamma_1$ , and since  $\Gamma_1$  is relatively open we can assume again that  $\text{dist}(w, \Gamma_2) \geq \delta > 0$ , at least along a subsequence  $\beta_n \rightarrow 0^+$ .  
///

*Remark 1.3.* The assumption (1.20) can be avoided if we assume instead either some regularity of  $v_1 - v_2$  near  $\Gamma_2$ , or Lipschitz regularity of  $\Gamma_2$  near  $\Gamma_1$ . More precisely we have the following variants of Theorem 1.2.

(i) The conclusions of Theorem 1.2 hold if (1.20) is replaced by

(1.23) for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\text{dist}(x, \Gamma_2) < \delta$  implies

$$v_1(x) - v_2(x) < \varepsilon.$$

In fact, by (1.22), if we take  $\varepsilon = 2\gamma$  we have that  $\text{dist}(x, \Gamma_2) \geq \delta$  (where  $w$  is any point where the maximum in (1.22) is attained) and then the proof of Theorem 1.2 works without assuming (1.20).

(ii) The conclusions of Theorem 1.2 hold if (1.20) is dropped and (1.21) holds with  $\Omega = B(\Gamma_1, c)$ .

In fact in the proof above  $\bar{x}$  is in the domain of  $\eta$  for  $\beta$  and  $\varepsilon$  small enough and we do not have to prove that  $\text{dist}(w, \Gamma_2) \geq \delta > 0$  independently of  $\beta$ .  
///

Next results apply the previous arguments to the uniqueness and continuity everywhere of bounded viscosity solutions continuous at the boundary of two mixed boundary value problems.

**Theorem 1.4.** *Under the assumptions of Theorem 1.2, for any bounded and continuous  $g: \partial\mathcal{O} \rightarrow \mathbf{R}$  uniformly continuous on  $\Gamma_1$ , there is at most one bounded viscosity solution, uniformly continuous on  $\Gamma_1$  and continuous at each point of  $\partial\mathcal{O}$ , of*

$$(1.24) \quad \begin{cases} v + H(x, v, Dv) = 0 & \text{in } \mathcal{O}, \\ v = g & \text{on } \Gamma_2, \\ v = g \text{ or } v + H(x, v, Dv) = 0 & \text{on } \Gamma_1, \end{cases}$$

and it must be continuous in  $\bar{\mathcal{O}}$ . The same conclusions hold if either (1.20) is replaced by

(1.25) for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\text{dist}(x, \Gamma_2) < \delta$  implies  
 $|v(x) - g(x)| < \varepsilon$

(“ $v$  attains the boundary data uniformly”), or (1.20) is dropped and (1.21) holds with  $\Omega = B(\Gamma_1, c)$ .

**Corollary 1.5.** *Under the assumptions of Theorem 1.2, for any bounded and continuous  $g: \Gamma_2 \rightarrow \mathbf{R}$ , there is at most one bounded viscosity solution, continuous at each point of  $\partial\mathcal{O}$ , of*

$$(1.26) \quad \begin{cases} v + H(x, v, Dv) = 0 & \text{in } \mathcal{O}, \\ v = g & \text{on } \Gamma_2, \\ v + H(x, v, Dv) \geq 0 & \text{on } \Gamma_1, \end{cases}$$

and it must be continuous on  $\bar{\mathcal{O}}$ . The same conclusions hold if either (1.20) is replaced by (1.25), or (1.20) is dropped and (1.21) holds with  $\Omega = B(\Gamma_1, c)$ .

*Proofs.* Let  $u$  and  $v$  be two such solutions of (1.24). It's enough to prove that  $u^* \leq v_*$ , because the other inequality is obtained by exchanging the role of  $u$  and  $v$ , and the continuity by taking  $u = v$ .

We set  $v_1 = u^*$  and  $v_2 = v_*$  and observe that they are both continuous at points of  $\Gamma_1$  because  $u$  and  $v$  are, and they coincide with  $g$  at points of  $\Gamma_2$ . We have only to make some variants to the proofs of Theorems 1.1 and 1.2. In the “Second case” by (1.22) we can choose the point  $w \in \Gamma_1$  so that one of the two following possibilities occurs

- (a)  $v_1(w) > g(w) + \gamma$ ,
- (b)  $v_2(w) < g(w) - \gamma$ .

In case (a) we keep proceeding as in the proof of Theorem 1.1 until formula (1.19). As in the proof of Theorem 1.2, if  $\bar{x} \in \partial\mathcal{O}$  then  $\bar{x} \in \Gamma_1$ , and the uniform continuity of  $g$  and  $v_1$  on  $\Gamma_1$  gives

$$g(\bar{x}) - \omega_7(\beta, \varepsilon) + \gamma \leq g(w) + \gamma < v_1(w) \leq v_1(\bar{x}) + \omega_8(\beta, \varepsilon),$$

where  $\omega_i$ ,  $i = 7, 8$ , can be made as small as we please by choosing first  $\beta$  and then  $\varepsilon$ , both small. Therefore for a suitable choice of these parameters we have  $g(\bar{x}) < v_1(\bar{x})$ , so that we can use the boundary condition for  $v_1$  and conclude as in the proof of Theorem 1.1.

To treat case (b) we modify the auxiliary function  $\phi$  by replacing the term  $|x - w|^2$  with  $|y - w|^2$  and repeat the proof of case (a) with the roles of the variables  $x$  and  $y$  interchanged, which is possible because  $v_2$  is also continuous at the boundary and uniformly continuous on  $\Gamma_1$ .

The last statements follow from Remark 1.3 because (1.25) implies (1.23), and this completes the proof of Theorem 1.4.

The proof of Corollary 1.5 follows directly from Theorem 1.2 and Remark 1.3. ///

Next we describe the class of Hamiltonians we are interested in for the applications and we see when they satisfy condition (1.1).

*Example 1.6.* Let

$$(1.27) \quad H(x, r, p) = \min_{b \in B} \max_{a \in A} \{-f(x, a, b) \cdot p - h(x, a, b) + m(x, a, b)r\},$$

where  $A$  and  $B$  are compact sets. It is easy to show by a standard argument (see e.g. Lemma 1.2 in [BS2]) that (1.1) is satisfied if  $h$ ,  $m$  and  $f$  are continuous in  $\mathcal{O} \times A \times B$ ,  $m \geq 0$ ,  $f$  is bounded on  $\Gamma \times A \times B$ , and

$$(1.28) \quad (f(x, a, b) - f(y, a, b)) \cdot (x - y) \leq L|x - y|^2 \quad \text{for all } x, y, a, b,$$

(1.29)

$$|f(x, a, b) - f(y, a, b)| \leq L|x - y| \quad \text{for all } x, y \in B(\Gamma, c'), a \in A, b \in B.$$

These Hamiltonians arise in the theory of differential games and in the classical optimal control theory if  $B$  is a singleton. The typical assumption that the controlled vector fields  $f$  are uniformly Lipschitz, i.e. (1.29) for all  $x, y$  (see e.g. [BS2], [CDL], [ES], [LL], [Sn]), is too restrictive for certain applications where one is interested in polynomial fields. Condition (1.28), that the fields  $f(\cdot, a, b) - LI$  ( $I = \text{identity}$ ) be monotone "decreasing" for some constant  $L$  independent of  $a$  and  $b$ , is weaker, and allows  $f$  to grow faster than linearly as  $x$  goes to infinity. ///

If the vector fields  $f$  in the Hamiltonian (1.27) grow at infinity faster than linearly and they are not nondecreasing, then uniqueness is immediately lost even for classical solutions and linear Hamiltonians, as next example (which was pointed out to us by P. Souganidis) shows.

*Example 1.7.* The constant 0 is solution of

$$\begin{cases} u - x^\alpha u_x = 0 & \text{in } ]0, +\infty[ , \\ u(0) = 0. \end{cases}$$

If  $\alpha \leq 1$  this solution is unique in the class of Theorem 1.4, because (1.28) holds (note that for  $\alpha = 1$   $u(x) = x$  is also a solution, but it is unbounded). If  $\alpha > 1$  then

$$u(x) = \exp\left(\frac{x^{1-\alpha}}{1-\alpha}\right) \quad \text{for } x \neq 0, u(0) = 0,$$

is another bounded classical solution.

Note that if we change sign to  $x^\alpha u_x$ , then (1.28), (1.1) and the uniqueness Theorem 1.4 hold for all  $\alpha \geq 0$ . ///

Now we give some examples from control theory to motivate the study of Hamilton-Jacobi equations in sets with unbounded boundary.

*Example 1.8.* (The moon landing problem) In the classical moon landing problem, see e.g. [FR],  $N = 3$ ,  $\mathcal{O} = \{(h, m, v): h > 0, m > 0\}$  where  $h$  is the height from the ground,  $m$  the mass and  $v$  the velocity of the aircraft, and the target is  $\mathcal{T} := \{h = 0, v = 0\}$ . If the system is “locally controllable” the minimum time function  $T(h, m, v)$ , i.e. the infimum of the times necessary to reach the target starting at  $(h, m, v)$  and staying within  $\bar{\mathcal{O}}$ , is expected to satisfy the Dirichlet condition  $T = 0$  on  $\mathcal{T} = \Gamma_2$  and the boundary condition associated to the state constraint on  $\Gamma_1 = \{h = 0 \text{ or } m = 0\} \setminus \mathcal{T}$ . Note that  $\mathcal{O}$  satisfies (1.21) with  $\Omega = \mathbf{R}^3$ , while (1.20) does not hold. By Remark 1.3 the Kruzkov transform of  $T$ ,  $v := \psi(T)$ , is the unique solution of (1.26) (with a suitable Hamiltonian of the form (1.27) and  $g = 0$ ), where

$$(1.30) \quad \psi(r) := \begin{cases} 1 - e^{-r} & \text{if } r < +\infty; \\ 1 & \text{if } r = +\infty. \end{cases} ///$$

*Example 1.9.* (Time-optimal control of nonautonomous systems) Consider a nonautonomous system in  $\mathbf{R}^N$

$$y'(t) = f(y(t), t, a(t))$$

where  $a(\cdot)$  is the control, and the minimum time  $T(x, s)$  necessary to drive the system to a given closed target  $\mathcal{T}$  starting at the initial time  $s$  from the initial position  $x$ . By adding the equation  $t' = 1$  to the system we get an autonomous system in  $\mathbf{R}^{N+1}$ , and the target now is  $\mathcal{T} \times \mathbf{R}$  which is unbounded even if  $\mathcal{T}$  is bounded. Then  $v = \psi(T)$  (where  $\psi$  is given by (1.30)) satisfies (1.24) with  $\mathcal{O} = \mathbf{R}^{N+1} \setminus (\mathcal{T} \times \mathbf{R})$ . If the regularity condition (1.2) holds for  $\mathbf{R}^N \setminus \mathcal{T}$  then it holds for  $\partial\mathcal{O}$  as well, and we can apply Theorem 1.6 with  $\Gamma_2 = \emptyset$ . In the important special case  $\mathcal{T} = \{x_0\}$  the comparison principle for the weak “Dirichlet type” boundary condition does not hold: see Example 1.11 below. In this case we can apply Theorem 1.6 with  $\Gamma_1 = \emptyset$ ,  $\Gamma_2 = \partial\mathcal{O}$ , and  $v$  is the unique solution of (1.24) if it satisfies (1.25). ///

*Example 1.10.* (Pursuit-evasion games) In this case  $N = 2k$ , there are two players whose states  $y_1, y_2$  are in  $\mathbf{R}^k$ , and the game stops when either the coordinates of the players coincide, or they are very close by, i.e.  $|y_1 - y_2| \leq \varepsilon$  for some given  $\varepsilon$ . Then the Kruzkov transform  $\psi(T)$  of the “capture time”  $T(x_1, x_2)$  ( $x_1, x_2$  are the initial positions of the players) satisfies (1.24) with  $\mathcal{O} = \mathbf{R}^N \setminus \mathcal{T}$ , where  $\mathcal{T}$  is the diagonal  $\{(y, y): y \in \mathbf{R}^k\}$  in the former case, and

a neighbourhood of it,  $\{(y_1, y_2): |y_1 - y_2| \leq \varepsilon\}$ , in the latter case. Note that in the former case  $\mathcal{O}$  does not satisfy (1.2) and we have to take  $\Gamma_1 = \emptyset$  in (1.24), while in the latter case it does and we can take  $\Gamma_1 = \partial\mathcal{O}$ . ///

We end this Section by discussing the optimality of the Lipschitz regularity assumption (1.2). Usually the comparison theorems for the Dirichlet problem for Hamilton-Jacobi equations hold for general open sets. However for the boundary conditions considered here, some regularity condition on  $\mathcal{O}$  must be assumed. For open sets of the form  $\mathcal{O} = \mathbf{R}^N \setminus \mathcal{F}$  it is crucial at least that  $\mathcal{F}$  have non empty interior, as the following example shows.

*Example 1.11.* Take  $\mathcal{F}$  closed with empty interior,  $\Omega$  open,  $\Omega \supset \mathcal{F}$ ,  $d(x) := \text{dist}(x, \mathcal{F})$  and assume  $\inf \{d(x): x \in \partial\Omega\} > 0$ . Consider the problem

$$(1.31) \quad \begin{cases} |DT| = 1 & \text{in } \mathcal{O} = \Omega \setminus \mathcal{F}, \\ T = 0 \text{ or } |DT| = 1 & \text{on } \partial\mathcal{F}, \\ T = d & \text{on } \partial\Omega. \end{cases}$$

Clearly the distance function  $d$  from  $\mathcal{F}$  is a Lipschitz continuous viscosity solution and, by e.g. the results of [BS2], it is larger than or equal to any continuous subsolution bounded below of the Dirichlet problem, that is a subsolution  $S$  of the PDE such that  $S \leq 0$  on  $\partial\mathcal{F}$  and  $S \leq d$  on  $\partial\Omega$ . This does not hold any more if we relax the classical Dirichlet condition to the ‘‘Dirichlet type’’ condition on  $\partial\mathcal{F}$  in (1.31) because any positive constant  $c$  such that  $c < \inf \{d(x): x \in \partial\Omega\}$  is a subsolution of this problem. In fact  $\bar{\mathcal{O}} = \bar{\Omega}$  implies that at any maximum point  $y \in \partial\mathcal{F}$  for  $c - \varphi(x)$  in  $\bar{\mathcal{O}}$ , we have  $D\varphi(y) = 0$ . By taking the Kruzkov transform defined by (1.30) of the solution and the subsolution we can get a counterexample to the conclusions of Theorem 1.1 and of Corollary 2.1 of Ishii [I1] when no hypotheses on  $\mathcal{O}$  are assumed.

To understand why the weak ‘‘Dirichlet type’’ boundary condition enforces the comparison principle in (1.31) when the boundary is more regular, note that when  $\text{int}(\mathcal{F}) \neq \emptyset$  there are many more test functions, and a positive constant does not satisfy the ‘‘Dirichlet type’’ condition on  $\partial\mathcal{F}$  even if it is a classical solution! This is easily seen if for instance  $\mathcal{F}$  is (locally) a half-space, say  $\{x: x \cdot n \leq 0\}$ , because  $\varphi(x) = ax \cdot n$  is such that  $c - \varphi(x)$  has a maximum in  $\bar{\mathcal{O}}$  at each point  $y$  of  $\partial\mathcal{O}$  if  $a > 0$ , but  $D\varphi(y) = an$  has modulus larger than 1 for suitable  $a$ . ///

The last example shows that the conclusion of Theorem 1.1 may be false even if the regularity condition (1.2) fails at only one point and  $\text{int}(\mathcal{F}) \neq \emptyset$ .

*Example 1.12.* In  $\mathbf{R}^2$  take  $\mathcal{F} = \{(x, y): |x| \leq |y|\}$ ,  $v(x, y) = \psi(|x| - |y|)$ ,

where  $\psi$  is given by (1.30),  $u(x, 0) = 1$  and  $u = v$  if  $y \neq 0$ . Then it is not hard to check that  $v$  is a bounded Lipschitz continuous solution of

$$(1.32) \quad v + |\partial v / \partial x| - 1 = 0 \quad \text{in } \mathbf{R}^2 \setminus \mathcal{T},$$

and it is null on  $\partial \mathcal{T}$ , and  $u$  is an upper semicontinuous solution of (1.32), it is null on  $\partial \mathcal{T} \setminus (0, 0)$ ,  $u_*(0, 0) = 0$ , and  $u^*$  is a supersolution of (1.32) at  $(0, 0)$ . Note that (1.2) fails only at  $(0, 0)$  and that in this case  $\mathcal{T} = \overline{\text{int } \mathcal{T}}$ . ///

## 2. Uniqueness for a free boundary problem

In the second part of the paper we will study an application of the Theorems 1.1 and 1.2 to a free boundary problem arising in differential games of generalized pursuit and evasion. Consider a dynamical system controlled by two players

$$(2.1) \quad \begin{cases} y' = f(y, a, b) \\ y(0) = x \in \mathbf{R}^N \end{cases}$$

where the vector field  $f: \mathbf{R}^N \times A \times B \rightarrow \mathbf{R}^N$  is continuous and satisfies at least (1.28). We will suppose that the control sets  $A, B$  are compact subsets of  $\mathbf{R}^M$  and that the control functions  $a(\cdot)$  and  $b(\cdot)$  belong respectively to the following sets of admissible controls:

$$\mathcal{A} := \{a: \mathbf{R}_+ \rightarrow A \text{ measurable}\}, \quad \mathcal{B} := \{b: \mathbf{R}_+ \rightarrow B \text{ measurable}\}.$$

We will denote by  $y_x(\cdot; a, b)$  or simply by  $y_x(\cdot)$ ,  $y(\cdot)$  a solution of the system (2.1) corresponding to a choice of the functions  $a$  and  $b$ . We are also given a closed ‘‘target’’ (or ‘‘terminal’’) set  $\mathcal{T} \subset \mathbf{R}^N$  and the two following functions: a running cost  $h: \mathbf{R}^N \times A \times B \rightarrow \mathbf{R}$  which we will suppose to be continuous, positive and bounded away from 0, i.e., without loss of generality

$$(2.2) \quad h(x, a, b) \geq 1 \quad \text{for all } x, a, b$$

and a final cost  $g: \mathcal{T} \rightarrow [G; +\infty[$  which will be bounded below ( $G > -\infty$ ) and continuous.

For each choice of  $a$  and  $b$  the game starts at the point  $x$  at time 0 and ends at the first time the trajectory hits the target, i.e. at the time

$$(2.3) \quad t_x = t_x(a, b) := \inf \{t: y(t) \in \mathcal{T}\} \leq +\infty,$$

where  $t_x = +\infty$  if  $y_x(\cdot)$  never reaches the target. The payoff of such a trajectory is defined by

$$(2.4) \quad P(x, a, b) := \int_0^{t_x} h(y_x(t), a(t), b(t)) dt + g(y_x(t_x)).$$

We introduce the set of admissible strategies for the first player in the sense of Varaya, Roxin, Elliott and Kalton (see [EK]),

$$\Delta := \{\alpha: \mathcal{B} \rightarrow \mathcal{A}: b(t) = b'(t) \text{ a.e. for } t \leq t' \text{ implies } \alpha[b](t) = \alpha[b'](t) \text{ a.e. for } t \leq t'\},$$

and we will indicate by  $\Gamma$  the corresponding set of strategies for the second player. The lower VREK value is defined by

$$(2.5) \quad V(x) := \inf_{\alpha \in \Delta} \sup_{b \in \mathcal{B}} P(x, \alpha[b], b)$$

and in the same way  $\tilde{V}(x) = \sup_{\beta \in \Gamma} \inf_{a \in \mathcal{A}} P(x, a, \beta[a])$  is the upper VREK value.

The following results in this section will deal with the lower value  $V$ , but everything can be translated in terms of  $\tilde{V}$ . We also introduce the (lower) capturability set, namely the set of starting points of the game such that the first player has a choice for the strategy which forces the system into the target in time less than some positive constant  $C$ , no matter which is the control selected by the second, i.e.

$$\mathcal{R} := \{x: V(x) < +\infty\}.$$

It is not difficult to prove, see Lemma 1.1 in [BS2], that  $\mathcal{R}$  is independent of the choice of  $h$  and  $g$ . As in [CL, B, BF1, BS1, BS2] we use Kruzkov's change of unknown variable given by (1.30) and consider the function defined by

$$v(x) := \psi(V(x)).$$

It is important to remark that  $v$  is bounded and it is itself the (lower) value function of a differential game. In fact, as it is easily seen

$$v(x) := \inf_{\alpha \in \Delta} \sup_{b \in \mathcal{B}} \psi(P(x, \alpha|b, b)), \quad \text{for all } x \in \mathbf{R}^N \setminus \mathcal{T},$$

where the payoff is

$$\begin{aligned} \psi(P(x, a, b)) &= \int_0^{t_x} h(y_x, a, b) \exp\left(-\int_0^t h(y_x, a, b) ds\right) dt \\ &\quad + \exp\left(-\int_0^{t_x} h(y_x, a, b) ds\right) \psi(g(y_x(t_x))). \end{aligned}$$

The relevant hamiltonians for our problem are:

$$(2.6) \quad H(x, p) := \min_{b \in \mathcal{B}} \max_{a \in \mathcal{A}} \{-f(x, a, b) \cdot p - h(x, a, b)\},$$

and

$$(2.7) \quad \mathcal{H}(x, r, p) := \min_{b \in B} \max_{a \in A} \{-f(x, a, b) \cdot p - h(x, a, b) + (h(x, a, b) - 1)r\}.$$

We also define  $\tilde{H}$  and  $\tilde{\mathcal{H}}$  interchanging min and max in (2.6) and (2.7).

Next result is based on the dynamic programming principle and relates the value functions  $V$ ,  $v$  and the Hamiltonians (2.6) and (2.7). Its proof is got by combining the arguments in [ES] and [I4].

**Proposition 2.1.** (i) *If  $\mathcal{R}$  is open and  $V$  is locally bounded, then  $V$  is a viscosity solution of*

$$(2.8) \quad \begin{cases} H(x, DV) = 0 & \text{in } \mathcal{R} \setminus \mathcal{T}, \\ V = g \text{ or } H(x, DV) = 0 & \text{on } \partial\mathcal{T}. \end{cases}$$

(ii)  *$v$  is a viscosity solution of*

$$(2.9) \quad v(x) + \mathcal{H}(x, v(x), Dv(x)) = 0 \quad \text{in } \mathbf{R}^N \setminus \mathcal{T}. \quad ///$$

*Remark 2.2.* We recall that the upper value functions satisfy the above equations with  $\tilde{H}$ ,  $\tilde{\mathcal{H}}$  in place of  $H$  and  $\mathcal{H}$  respectively. We also want to remark that, if the value function  $V$  is not continuous at the points of the boundary  $\partial\mathcal{T}$ , then the natural Dirichlet boundary condition  $V(x) = g(x)$  for all  $x \in \partial\mathcal{T}$  is not very meaningful nor useful, and the correct boundary condition is the one, due to Barles & Perthame [BP1, BP2] and Ishii [I4], which appears in (2.8). Of course this has to be interpreted in the viscosity sense.

We will consider the following assumptions:

$$(2.10) \quad g \equiv 0,$$

(2.11) the restriction of  $V$  to  $\mathcal{T}^c$  can be extended to a function  $\bar{V}$ , defined in  $\overline{\mathcal{T}^c}$ , which is continuous at the points of  $\partial\mathcal{T}$ .

Note that (2.10) implies  $V \geq 0$  and then  $\bar{V} = V^*$ . Next result completes the boundary value problem (2.8) by giving some informations on the unknown set  $\mathcal{R}$ , whose boundary  $\partial\mathcal{R}$  is the “free boundary”, and on the boundary condition verified by  $V$  on  $\partial\mathcal{R}$ .

**Proposition 2.3.** *Assume (1.28), (2.2) and (2.11). Then  $\mathcal{R} \supset \mathcal{T}$  is open,  $V$  is continuous in  $\mathcal{R} \setminus \mathcal{T}$  and satisfies the singular boundary condition*

$$V(x) \rightarrow +\infty \quad \text{as } x \rightarrow x_0 \in \partial\mathcal{R}.$$

*Proof.* Let  $\bar{v} := \psi(\bar{V})$ . Proposition 2.1 and Theorem 1.4 imply that  $\bar{v}$  is continuous in  $\overline{\mathbf{R}^N \setminus \mathcal{T}}$  and  $\mathcal{R} = \{x: \bar{v}(x) < 1\}$ . ///

Next result is essentially Theorem 3.1 of [BS2]. In that paper the vector

field  $f$  was supposed to satisfy a stronger assumption, namely a global Lipschitz condition uniformly in  $a, b$ . However condition (1.28) is enough to have the estimate

$$|y_x(t; a, b) - y_z(t; a, b)| \leq e^{Lt} |x - z| \text{ for all } t \geq 0, x, z \in \mathbf{R}^N \text{ and } a \in \mathcal{A}, b \in \mathcal{B},$$

and to have the comparison principle for classical Dirichlet boundary conditions, that is Theorem 1.4 with  $\Gamma_2 = \partial\mathcal{O}$ . Then it is easy to check that the results of [BS2] still hold under the present assumptions.

**Theorem 2.4.** *Assume (1.28), (2.2) and that there exists a pair  $(\Omega, U)$  such that  $\Omega \supset \mathcal{T}$  is open,  $U \in C(\Omega \setminus \text{int } \mathcal{T})$  is bounded below and satisfies*

$$\begin{cases} H(x, DU) = 0 & \text{in } \Omega \setminus \mathcal{T}, \\ U = g & \text{on } \partial\mathcal{T}, \\ U(x) \rightarrow +\infty & \text{as } x \rightarrow x_0 \in \partial\Omega. \end{cases}$$

*in the viscosity sense. Then  $\Omega = \mathcal{R}$  and  $U = V$  in  $\Omega \setminus \text{int } \mathcal{T}$ . ///*

Now we give the main result of this section, which is a uniqueness theorem analogous to Theorem 2.4 but holding also when the value function  $V$  has discontinuity points on  $\partial\mathcal{T}$ . It says that the pair  $(\mathcal{R}, V)$  is the unique solution of the free boundary value problem described by Propositions 2.1 and 2.3.

**Theorem 2.5.** *Assume (1.28), (1.29) with  $\Gamma = \partial\mathcal{T}$ , (2.2), (2.10), (2.11), that  $\mathcal{T}^c$  satisfy (1.2) and  $f$  be bounded on  $\partial\mathcal{T} \times A \times B$ . Let  $\Omega \supset \mathcal{T}$  be an open set and  $S \in C(\Omega \setminus \text{int } \mathcal{T})$  be bounded below and viscosity solution of*

$$\begin{cases} H(x, DS) = 0 & \text{in } \Omega \setminus \mathcal{T}, \\ S \geq 0 & \text{on } \partial\mathcal{T}, \\ H(x, DS) \leq 0 & \text{if } x \in \partial\mathcal{T} \text{ and } S(x) > 0, \\ S(x) \rightarrow +\infty & \text{as } x \rightarrow x_0 \in \partial\Omega. \end{cases}$$

*Then  $\Omega = \mathcal{R}$  and  $S = \bar{V}$  in  $\Omega \setminus \text{int } \mathcal{T}$ .*

*Proof.* Let  $g_1$  and  $g_2$  be the restriction to  $\partial\mathcal{T}$  of  $\bar{V}$  and  $S$  respectively. Observe that by Proposition 2.3 the two functions  $\bar{V}$  and  $S$  satisfy, in the viscosity sense,

$$\begin{cases} H(x, DW) = 0 & \text{in } \mathcal{O} \setminus \mathcal{T}, \\ W = \theta & \text{on } \partial\mathcal{T}, \\ W(x) \rightarrow +\infty & \text{as } x \rightarrow x_0 \in \partial\mathcal{O}, \end{cases}$$

where, respectively,  $(\mathcal{O}, \theta) = (\Omega, g_2)$  if we consider  $S$  and  $(\mathcal{R}, g_1)$  if we consider  $\bar{V}$ . By Theorem 2.4 and since the set  $\mathcal{R}$  does not depend on the final cost, we deduce that  $\Omega = \mathcal{R}$ , and that  $\bar{V}$  and  $S$  are the lower value of a differential game with terminal cost  $g_1$  and  $g_2$  respectively, that is, they are given by (2.5) (2.4) with  $g$  replaced by  $g_1$  and  $g_2$  respectively. We now consider the change of variables (1.30) and introduce the two functions  $\bar{v} = \psi(\bar{V})$  and  $u = \psi(S)$  both extended by continuity to the constant value 1 off  $\partial\mathcal{R}$ . By Proposition 2.1, they are both viscosity solutions of (2.9) and, since  $g_1 \geq 0$ ,  $g_2 \geq 0$ , they satisfy the following boundary conditions on  $\partial\mathcal{T}$ , in the viscosity sense:

$$(2.12) \quad \bar{v} \leq \psi(g_2) = u \quad \text{or} \quad \bar{v} + \mathcal{H}(x, \bar{v}, D\bar{v}) \leq 0,$$

$$(2.13) \quad u \leq \psi(g_1) = \bar{v} \quad \text{or} \quad u + \mathcal{H}(x, u, Du) \leq 0.$$

Now, as observed in Example 1.6, our assumptions allow us to apply twice Theorem 1.1 and obtain, using (2.9) and (2.11) that  $u \leq \bar{v}$ , and by means of (2.9) and (2.12) that  $\bar{v} \leq u$ . Then the conclusion is reached. ///

The last result of this section says that the continuous extension  $\bar{V}$  of  $V$  is itself the value function of a slightly different differential game. We denote by  $\hat{t}_x$  the first entry time in the interior of  $\mathcal{T}$ , that is,

$$\hat{t}_x(a, b) := \inf \{t: y(t) \in \text{int } \mathcal{T}\} (\leq +\infty).$$

**Corollary 2.6.** *Under the assumption of Theorem 2.5, let  $\hat{V}$  be given by (2.5) (2.4) where  $t_x$  is replaced by  $\hat{t}_x$ . Then  $\bar{V} = \hat{V}$ .*

*Proof.* By the arguments of [ES] and [I4],  $\hat{v} := \psi(\hat{V})$  is a solution of (2.9) with the boundary condition

$$\hat{v} = 0 \quad \text{or} \quad \hat{v}(x) + \mathcal{H}(x, \hat{v}(x), D\hat{v}(x)) = 0 \quad \text{on } \partial\mathcal{T}.$$

Since  $g_1 \geq 0$ ,  $\hat{v}$  satisfies also

$$\hat{v}(x) + \mathcal{H}(x, \hat{v}(x), D\hat{v}(x)) \leq 0 \quad \text{if} \quad \hat{v}^* > \bar{v} = \psi(g_1) \quad \text{on } \partial\mathcal{T}.$$

Then Theorem 1.1 implies  $\hat{v}^* \leq \bar{v}$ . On the other hand  $v \leq \hat{v}$  everywhere by definition, because  $t_x \leq \hat{t}_x$ ,  $g = 0$  and  $h \geq 0$ , and then  $\bar{v} = v^* \leq \hat{v}^*$ . ///

### 3. Pursuit-evasion games with value functions discontinuous only on the target

If condition (2.11) is verified, we are considering a differential game of pursuit and evasion which has no “barriers”, that is its value has no surfaces of discontinuity off  $\mathcal{T}$ . A trivial example where this situation occurs is a constant  $f$  and  $\mathcal{T}$  the exterior of a convex set. The result of this Section gives a sufficient condition for (2.11).

First we need some more notations and follow the book of Hajek [H] for the definitions. Let  $x \in \mathbf{R}^N$ , we say that a strategy  $\alpha$  is winning at the position  $x$  if  $t_x(\alpha[b], b) < +\infty$  for all  $b \in \mathcal{B}$ . If, in particular,  $x \in \mathcal{R}$ , then there exist winning strategies at  $x$ . We also introduce the set of “usable” points of the target by means of the position:

$$\mathcal{U} := \bigcup_{\substack{x \in \mathbf{R}^N \setminus \mathcal{T} \\ \alpha \text{ winning at } x}} \overline{\{y_x(t_x(\alpha[b], b)): b \in \mathcal{B}\}}.$$

In particular, if we are considering a control problem, i.e. if  $B$  is a singleton, then  $\mathcal{U}$  coincides with the set of ending points of trajectories hitting the target.

**Theorem 3.1.** *Assume (1.28), (2.2) and  $g \equiv 0$ . For a  $\mathcal{T}' \subset \mathcal{T}$  closed such that  $\mathcal{T}' \supset \mathcal{U}$ , consider the value function*

$$V'(x) := \inf_{\alpha \in \Delta} \sup_{b \in \mathcal{B}} \int_0^{t'_x} h(y_x(t), \alpha[b](t), b(t)) dt$$

where  $t'_x$  is the first time the trajectory hits the target  $\mathcal{T}'$ . If  $V'$  is continuous in  $\mathcal{R}' \setminus \text{int } \mathcal{T}'$  and  $\mathcal{R}' \supset \mathcal{T}$ , then  $\mathcal{R} = \mathcal{R}'$  and  $V = V'$  in  $\mathbf{R}^N \setminus \mathcal{T}$ . Thus the restriction of  $V$  to  $\mathcal{R} \setminus \mathcal{T}$  has a continuous extension in  $\mathcal{R} \setminus \text{int } \mathcal{T}$ .

*Proof.* Let  $g'$  be the restriction of  $V'$  to  $\partial \mathcal{T}$ , then  $V'$  satisfies in the viscosity sense the following Dirichlet problem:

$$\begin{cases} H(x, DV') = 0 & \text{in } \mathcal{R}' \setminus \mathcal{T}, \\ V' = g' & \text{on } \partial \mathcal{T}, \\ V(x) \rightarrow +\infty & \text{as } x \rightarrow x_0 \in \partial \mathcal{R}'. \end{cases}$$

Therefore, by Theorem 2.4 and the remark that  $\mathcal{R}$  does not depend on the final cost function, we get  $\mathcal{R}' = \mathcal{R}$  and the representation formula

$$V'(x) := \inf_{\alpha \in \Delta} \sup_{b \in \mathcal{B}} \left\{ \int_0^{t'_x} h(y_x(t), \alpha[b](t), b(t)) dt + g'(y_x(t'_x)) \right\}.$$

We now observe that  $g' \geq 0$  implies  $V' \geq V$ . If we had  $V'(x) > V(x)$  at some point  $x \in \mathcal{R} \setminus \mathcal{T}$ , then, by the definition of  $V$ , we could find a winning strategy  $\alpha$  such that

$$\sup_{b \in \mathcal{B}} \int_0^{t_x} h(y_x, \alpha|b|, b) dt < \sup_{b \in \mathcal{B}} \int_0^{t_x} h(y_x, \alpha[b], b) dt + g'(y_x(t_x))$$

and then, for a suitable choice of  $b$ , we would obtain

$$\int_0^{t_x} h(y_x, \alpha|b|, b) dt < \int_0^{t_x} h(y_x, \alpha[b], b) dt + g'(y_x(t_x))$$

and then  $g'(y_x(t_x)) > 0$ , which implies  $y_x(t_x) \notin \mathcal{T}'$ . On the other hand, by the definition of  $\mathcal{U}$ , we also have, as it is easily checked,  $y_x(t_x) \in \mathcal{U}$ , and this contradicts the assumption  $\mathcal{T}' \supset \mathcal{U}$ .  $///$

*Remark 3.2.* It is not possible to prove the converse of the previous proposition, i.e. condition (2.11) does not imply, in general, the existence of a target  $\mathcal{T}' \subset \mathcal{T}$  such that  $\mathcal{T}' \supset \mathcal{U}$ , the corresponding value function  $V'$  is continuous and  $V = V'$  in  $\mathbf{R}^N \setminus \mathcal{T}$ . In fact if we introduce the set of essential points of  $\mathcal{T}$  as

$$\text{ess } \mathcal{T} := \bigcap_{t>0} \overline{\mathcal{R}(t) \setminus \mathcal{T}},$$

where  $\mathcal{R}(t) := \{x: T(x) < t\}$  and  $T$  is defined as in (2.5) with  $b = 1$ ,  $g \equiv 0$ . Then it is easy to check that (2.11) implies  $\text{ess } \mathcal{T} = \{x \in \mathcal{T}: V'(x) = 0\}$  and many examples are known in which  $\text{ess } \mathcal{T} \subsetneq \mathcal{U}$  (see Hajek [H]). The interest of Theorem 3.1 is in the fact that, to our knowledge, in the literature there are not explicit conditions on the vector fields implying (2.10), and our result links the proof of condition (2.11) with the well known problem of local controllability.

### Appendix: Domains with Lipschitz boundary and condition (1.2)

In this section we show that the technical condition (1.2) on the open set  $\Omega$  is essentially equivalent to having a Lipschitz boundary, in the sense that  $\partial\Omega$  is locally the graph of Lipschitz functions satisfying a uniform bound on their Lipschitz constants.

**Proposition A.1.** *Let  $\Omega \subset \mathbf{R}^N$  be an open set with the following properties: there exist  $x_i \in \partial\Omega$ ,  $i \in I$  (a set of indexes),  $\varepsilon, r > 0$ ,  $M \in \mathbf{N}$ ,  $\eta_i \in \mathbf{R}^N$ ,  $|\eta_i| = 1$ , and Lipschitz continuous functions  $f_i: \{v: v \cdot \eta_i = 0\} \rightarrow \mathbf{R}$  such that  $\text{Lip}(f_i) \leq L$  for all  $i \in I$  and*

$$\bigcup_{i \in I} B(x_i, r/2) \supset B(\partial\Omega, \varepsilon) \cap \Omega.$$

$$\Omega \cap B(x_i, 2r) = \{t\eta_i + v: t > f_i(v)\} \cap B(x_i, 2r) \quad \text{for all } i \in I,$$

(A.1) *for all  $x \in \Omega$  the set  $I(x)$  of indexes  $i \in I$  such that  $x \in B(x_i, r)$  has at most  $M$  elements.*

*Then  $\Omega$  satisfies the condition (1.2).*

*Proof.* It is easy to check that if  $c := \sin \arctan(1/L) \wedge (r/4) \wedge (\varepsilon/4)$ , then  $B(t\eta_i + x, ct) \subset \Omega$  for all  $0 < t \leq c$  and  $x \in B(x_i, 3r/2) \cap \overline{\Omega}$ ,  $i \in I$ . Now we consider equi uniformly continuous functions  $\theta_i: \mathbf{R}^N \rightarrow \mathbf{R}$ ,  $i \in I$  such that  $0 \leq \theta_i \leq 1$ ,  $\theta_i(x) = 1$  if  $x \in B(x_i, r/2)$ ,  $\text{supp } \theta_i \subset B(x_i, r)$  and let  $\overline{\theta}_i: \overline{B(\partial\Omega, \varepsilon)} \rightarrow \mathbf{R}$  be defined

by means of the position  $\bar{\theta}_i = \theta_i / \left( \sum_{i \in I} \theta_i \right)$ . Therefore if we define  $\eta(x) := \sum_{i \in I} \bar{\theta}_i(x) \eta_i$ , we obtain by (A.1) that the function  $\eta: \overline{B(\partial\Omega, \varepsilon)} \cap \partial\Omega \rightarrow \mathbf{R}^N$  is uniformly continuous and bounded.

Now let  $x \in \overline{B(\partial\Omega, \varepsilon)}$ , we want to prove that  $w := x + t(\eta(x) + cv) \in \Omega$  for all  $v \in B(0, 1)$  and  $0 < t \leq c$ . To this end we observe that

$$w = x + t(\bar{\theta}_{i_1}(x)\eta_{i_1} + \cdots + \bar{\theta}_{i_s}(x)\eta_{i_s} + cv) \quad \text{for } i_1, \dots, i_s \in I(x)$$

and that, by the choice of  $c$ ,  $w_1 := x + t\bar{\theta}_{i_1}(x)(\eta_{i_1} + cv) \in \Omega \cap B(x_i, r + (c + c^2)\bar{\theta}_{i_1}(x))$ , for  $i \in I(x)$ , which also implies  $w_1 \in B(x_i, 3r/2)$ . Moreover we have that  $w_2 := w_1 + t\bar{\theta}_{i_2}(x)(\eta_{i_2} + cv) \in \Omega \cap B(x_i, r + (c + c^2)[\bar{\theta}_{i_1}(x) + \bar{\theta}_{i_2}(x)]) \subset B(x_i, 3r/2)$ , for  $i \in I(x)$ . In general

$$\begin{aligned} w_k &:= w_{k-1} + t\bar{\theta}_{i_k}(x)(\eta_{i_k} + cv) \in \Omega \cap B(x_i, r + (c + c^2)[\bar{\theta}_{i_1}(x) + \cdots + \bar{\theta}_{i_k}(x)]) \\ &\subset B(x_i, 3r/2), \end{aligned}$$

for  $i \in I(x)$  and thus we conclude that  $w \in \Omega \cap B(x_i, r + (c + c^2))$ .  $///$

**Proposition A.2.** *Assume the open set  $\Omega$  satisfies (1.2). Then there exist  $x_i \in \partial\Omega$ ,  $i \in N$ ,  $\delta, r > 0$ ,  $\eta_i \in \mathbf{R}^N$ ,  $|\eta_i| = 1$ , and Lipschitz continuous functions  $f_i: \{v: |v| < \delta, v \cdot \eta_i = 0\} \rightarrow \mathbf{R}$ , such that  $\text{Lip}(f_i) \leq L$  for all  $i \in N$ , the sets  $B(x_i, r/2)$ ,  $i \in N$ , form a locally finite covering of  $\partial\Omega$ , and*

$$\Omega \cap B(x_i, 2r) = \{x_i + t\eta_i + v: t > f_i(v)\} \cap B(x_i, 2r) \quad \text{for all } i \in N.$$

*Proof (sketch).* Fix  $x_0 \in \partial\Omega$ , assume for simplicity that  $|\eta(x_0)| = 1$ , and set  $\eta_0 = \eta(x_0)$ . Observe that (1.2) implies that  $\Omega$  contains a truncated open cone of vertex  $x_0$ , axis  $\eta_0$ , height  $c$ , and opening depending only on  $c$ . Moreover  $x_0 \notin B(x + t\eta(x), ct)$ , that is,  $|x - (x_0 - t\eta(x))| > ct$ , for all  $x \in \Omega$  and  $t \leq c$ , and then the continuity of  $\eta$  implies  $B(x_0 - t\eta_0, ct/2) \cap B(x_0, \varepsilon) \subseteq \Omega^c := \mathbf{R}^N \setminus \Omega$  for all  $t \leq c$ , for a suitable  $\varepsilon$  depending only on the modulus of continuity of  $\eta$  and on  $c$ . Thus  $\Omega^c$  contains a closed truncated cone of vertex  $x_0$ , axis  $-\eta_0$ , and height and opening independent of  $x_0$ .

Next observe that the continuity of  $\eta$  implies also the existence, for each  $x \in \partial\Omega$  near  $x_0$ , of two truncated cones of vertex  $x$ , height and opening independent of  $x$  and  $x_0$ , one with axis  $\eta_0$  and contained in  $\Omega$ , the other with axis  $-\eta_0$  and contained in  $\Omega^c$ .

Now we define  $f_0: \{v: |v| < \delta, v \cdot \eta_0 = 0\} \rightarrow \mathbf{R}$  as follows:  $f_0(v) = s$  if  $x_0 + s\eta_0 + v \in \partial\Omega$ . The existence of the cones described above implies, for a suitably small  $\delta$ , that  $f_0$  is well defined, that  $x_0 + s\eta_0 + v \in \Omega$  (respectively  $\in \text{int}(\Omega^c)$ ) if and only if  $s > f_0(v)$  (resp.  $<$ ), and that  $\text{Lip}(f_0) \leq 2/c$ . One can also show that  $\delta$  can be chosen independent of  $x_0$ .

The conclusion is easily reached by a standard compactness argument on a suitable sequence of compact sets invading  $\partial\Omega$ . ///

*Remark A.3.* If  $\partial\Omega$  is bounded then the statement of Proposition (A.2) holds with the index  $i$  varying in a finite set. ///

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