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Global Existence and the Life Span of Solutions of Semilinear Wave Equations with Data of Non Compact Support in Three Space Dimensions

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1. Introduction

We consider the Cauchy problem for the semilinear wave equation

$$(1.1)\qquad \begin{cases} u_{tt} - \Delta u = |u|^p\,, & (x, t) \in \boldsymbol{R}^n \times (0, \infty)\,, \\ u(x, 0) = f(x),\ u_t(x, 0) = g(x)\,, & x \in \boldsymbol{R}^n \end{cases}$$

for small Cauchy data where $p > 1$ and $n = 2, 3$. The following remarkable results are classical now.

John [5] has shown that in three space dimensions global C^2-solutions exist if $p > 1 + \sqrt{2}$ and the small data with compact support are sufficiently smooth, and also that global solutions do not exist if $1 < p < 1 + \sqrt{2}$ for the data satisfying $f = 0$, $g \geq 0$ (see also [6]).

Glassey [3, 4] has proved that (1.1) has global C^2-solutions if $p > \frac{3 + \sqrt{17}}{2}$, provided the smooth data of compact support are small, and moreover that if $1 < p < \frac{3 + \sqrt{17}}{2}$, the solution blows up in finite time, provided the data of compact support satisfy $\int g(x)dx > 0$.

For the critical value, $p = 1 + \sqrt{2}$ in three space dimensions and $p = \frac{3 + \sqrt{17}}{2}$ in two space dimensions, Glassey [3], Schaeffer [8] have shown that the solutions blow up in finite time.

We note that in these results, the data are compactly supported. Then, the following natural question arises. Do global solutions exist for the data which are not of compact support if $p > 1 + \sqrt{2}$ in three space dimensions and $p > \frac{3 + \sqrt{17}}{2}$ in two space dimensions?

Asakura [2] has shown that in three space dimensions if $p > 1 + \sqrt{2}$, global solutions exist for small data $f(x) \in C^3(\boldsymbol{R}^3)$, $g(x) \in C^2(\boldsymbol{R}^3)$ satisfying

(1.2) $$D_x^{\alpha} f(x),\ D_x^{\beta} g(x) = O(|x|^{-1-k}) \text{ as } |x| \to \infty,\ |\alpha| \le 3,\ |\beta| \le 2,$$

provided $k > \dfrac{2}{p-1}$, and also that if the data satisfy

(1.3) $$f(x) = 0,\ g(x) \ge \frac{\varepsilon}{(1+|x|)^{1+k}},\ \varepsilon > 0$$

and $0 < k < \dfrac{2}{p-1}$, then the solution blows up in finite time even with $p > 1 + \sqrt{2}$.

In this paper, we shall prove that we can extend the global existence result above to include the borderline values of k, say $k = \dfrac{2}{p-1}$ in three space dimensions. We shall also show the existence of local solutions for the data satisfying (1.2) with $0 < k < \dfrac{2}{p-1}$, $p > 1 + \sqrt{2}$ and the lower bounds for the life span of the solution.

Independently, Kubota [7], [9] have recently shown in the different ways that in two space dimensions global solutions exist for the data satisfying (1.2) with $k > \dfrac{2}{p-1}$ and $p > \dfrac{3+\sqrt{17}}{2}$. Kubota [7] has also shown the global existence result for the case $k = \dfrac{2}{p-1}$ and $k \ne \dfrac{1}{2}$. Agemi and Takamura [1], [9] have shown also independently using the different methods that the solution blows up in finite time if the data satisfy (1.3), $0 < k < \dfrac{2}{p-1}$ and $p > \dfrac{3+\sqrt{17}}{2}$. We shall treat the case $k = \dfrac{2}{p-1}$ in two space dimensions separately in a future paper.

2. Main results

We study the Cauchy problem for the semilinear wave equation of the form

(2.1) $$\begin{cases} u_{tt} - \Delta u = F(u), & (x, t) \in \boldsymbol{R}^3 \times (0, \infty), \\ u(x, 0) = f(x),\ u_t(x, 0) = g(x), & x \in \boldsymbol{R}^3. \end{cases}$$

The hypotheses are the following:

(H1) $F(u) \in C^2(\boldsymbol{R})$ and there exist $p > 1 + \sqrt{2}$, $A > 0$ such that

$$|F^{(j)}(u)| \leq \left(\frac{d}{dz}\right)^j (Az^p)\Big|_{z=|u|} \qquad \text{for } |u| \leq 1,\ j = 0,\ 1,\ 2\,,$$

$$|F''(u) - F''(v)| \leq \begin{cases} Ap(p-1)|u-v|^{p-2}\,, & \text{if } 1+\sqrt{2} < p < 3\,, \\ Ap(p-1)|\phi|^{p-3}|u-v|\,, & \text{if } p \geq 3 \end{cases}$$

for $|u|,\ |v| \leq 1,\ \phi = \max\{|u|, |v|\}$.

(H2) $f(x) \in C^3(\boldsymbol{R}^3)$, $g(x) \in C^2(\boldsymbol{R}^3)$ satisfy

$$\sum_{|\alpha| \leq 3} |D_x^\alpha f(x)| + \sum_{|\beta| \leq 2} |D_x^\beta g(x)| \leq \frac{G}{(1+|x|)^{1+k}}$$

with $k > 0$, where G is a small parameter.

The following theorem improves the global existence result in [2].

Theorem 1. *Consider the problem* (2.1). *Assume the hypotheses* (H1) *and* (H2). *If* $k \geq \dfrac{2}{p-1}$ *and* G *is sufficiently small, depending on* A, p *and* k, *then there exists a unique global* C^2*-solution of* (2.1).

Remark 1. The global existence result for the case $k > \dfrac{2}{p-1}$ has been proved in [2]. Thus, in this paper we shall prove the theorem above for the case $k = \dfrac{2}{p-1}$.

We define the life span $T^* = T^*(G)$ as the supremum of all s such that a C^2-solution of (2.1) exists for all $x \in \boldsymbol{R}^3$ and $0 \leq t < s$. Our another aim of this paper is to prove the following.

Theorem 2. *Assume* (H1) *and* (H2). *If* $0 < k < \dfrac{2}{p-1}$, *then there exists a unique local* C^2*-solution of* (2.1). *Moreover there exist constants* G_0, C_0, *sufficiently small depending only on* A, p *and* k *such that for all* $0 < G \leq G_0$, *we have*

$$T^*(G) \geq C_0 G^{(p-1)/(k(p-1)-2)}\,. \tag{2.2}$$

Remark 2. If we set $F(u) = A|u|^p$, $f(x) = 0$, $g(x) \geq \dfrac{G}{(1+|x|)^{1+k}}$ with $0 < k < \dfrac{2}{p-1}$ and $p > 1 + \sqrt{2}$ instead of (H1) and (H2), then from the proof of the blow-up theorem in [2] we can derive

$$T^*(G) \leq C_0' G^{(p-1)/(k(p-1)-2)}\,,$$

where C_0' is a constant depending only on A, p and k.

Therefore we see that for $F(u) = A|u|^p$, $f(x) = 0$, $g(x) = \dfrac{G}{(1 + |x|)^{1+k}}$ with $0 < k < \dfrac{2}{p-1}$ and $p > 1 + \sqrt{2}$, $T^*(G)$ satisfies

$$C_0 G^{(p-1)/(k(p-1)-2)} \le T^*(G) \le C_0' G^{(p-1)/(k(p-1)-2)}$$

with certain constants C_0, C_0'.

In Section 3, we study the decay estimates of the solution for the homogeneous wave equation with the initial data satisfying (H2). The estimates have been derived in [2]. However, especially we need to improve the estimate for the case $k = 1$ in order to prove Theorem 1, 2. In Section 4, we shall prove the basic estimate for the existence proof following Asakura [2]. The main new tool is the use of Proposition 4.3. Finally, Theorem 1, 2 will be proved in Section 5, by using the basic estimate and the iteration as in [2], John [5].

We denote a constant in the estimate by C, which will change from step to step.

3. Decay estimate

We study the decay estimate of the solution u for the homogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = 0\,, & (x, t) \in \mathbf{R}^3 \times (0, \infty)\,, \\ u(x, 0) = f(x),\ u_t(x, 0) = g(x)\,, & x \in \mathbf{R}^3\,, \end{cases} \tag{3.1}$$

under the hypothesis (H2).

Lemma 3.1. *Let $f(x)$, $g(x)$ satisfy* (H2). *Then the solution u of* (3.1) *satisfies*

$$\sum_{|\alpha| \le 2} |D_x^\alpha u(x, t)| \le \begin{cases} \dfrac{C_k G}{(1 + t + r)(1 + |t - r|)^{k-1}}\,, & (k > 1)\,, \\ \dfrac{C_k G}{1 + t + r}\left(1 + \ln \dfrac{1 + t + r}{1 + |t - r|}\right), & (k = 1)\,, \\ \dfrac{C_k G}{(1 + t + r)^k}\,, & (0 < k < 1)\,, \end{cases} \tag{3.2}$$

where $r = |x|$, and a constant C_k depends only on k.

Remark 3.2. The estimates above for $k > 1$ and $0 < k < 1$ have been proved in [2]. Moreover, it has been shown by [2] that for the case $k = 1$,

$$(3.3)\qquad |D_x^\alpha u(x, t)| \le \frac{C_k G \ln(2 + t + r)}{1 + t + r}\,.$$

However, it seems impossible to prove Theorem 1, 2 by using (3.3) when $k = 1$. The estimate (3.2) for $k = 1$ is perfectly suitable for the proof of Theorem 1, 2.

Proof. It suffices to prove the estimate for $k = 1$ by Remark 3.2.

The solution of (3.1) is given by

$$(3.4)\qquad u(x, t) = \frac{t}{4\pi}\int_{|\omega|=1} g(x + t\omega)d\omega + \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{|\omega|=1} f(x + t\omega)d\omega\right),$$

where $d\omega$ denotes the surface measure on the unit sphere in $\boldsymbol{R}^3$.

Especially, if the data are spherically symmetric, i.e., $f(x) = f(r)$, $g(x) = g(r)$, $r = |x|$, then the solution of (3.1) is also spherically symmetric and can be written explicitly in the form

$$(3.5)\qquad u(r, t) = \frac{1}{2r}\{(r + t)f(r + t) + (r - t)f(|r - t|)\} + \frac{1}{2r}\int_{|r-t|}^{r+t} \rho g(\rho)d\rho\,.$$

Now, we shall prove the estimate. Differentiating (3.4) with respect to x, we have for $|\alpha| \le 2$

$$D_x^\alpha u(x, t) = \frac{t}{4\pi}\int_{|\omega|=1} (D_x^\alpha g(x + t\omega) + (\omega\cdot\nabla)(D_x^\alpha f(x + t\omega)))d\omega$$
$$+ \frac{1}{4\pi}\int_{|\omega|=1} D_x^\alpha f(x + t\omega)d\omega\,.$$

By (H2), we see that

$$(3.6)\qquad \sum_{|\alpha|\le 2} |D_x^\alpha u(x, t)| \le \frac{t + 1}{4\pi}\int_{|\omega|=1} \frac{G}{(1 + |x + t\omega|)^2}d\omega\,.$$

For $0 \le t \le \frac{1}{2}$, we have

$$(3.7)\qquad \sum_{|\alpha|\le 2} |D_x^\alpha u(x, t)| \le \frac{3G}{8\pi}\left(\frac{1 + t}{3} + |x|\right)^{-2}\int_{|\omega|=1} d\omega$$
$$\le \frac{27G}{2(1 + t + r)^2}\,,$$

since $1 - t \ge \frac{1 + t}{3}$ for $0 \le t \le \frac{1}{2}$.

For $t \geq \frac{1}{2}$, we proceed as follows.

From (3.6),

$$\sum_{|\alpha|\leq 2} |D_x^\alpha u(x, t)| \leq \frac{3t}{4\pi} \int_{|\omega|=1} \frac{G}{(1 + |x + t\omega|)^2} d\omega .$$

Then, since we can regard the right-hand side as the solution of (3.1) with $f = 0$, $g = G(1 + r)^{-2}$, it follows from (3.5) that

$$\begin{aligned} (3.8) \qquad \sum_{|\alpha|\leq 2} |D_x^\alpha u(x, t)| &\leq \frac{3G}{2r} \int_{|r-t|}^{r+t} \frac{\rho}{(1+\rho)^2} d\rho \\ &\leq \frac{3G}{2r} \int_{|r-t|}^{r+t} \frac{1}{1+\rho} d\rho \\ &= \frac{3G}{2r} \ln \frac{1 + r + t}{1 + |r - t|} . \end{aligned}$$

We shall estimate the last term on the right for $t \geq 0$ to avoid repeating the same argument in the next section (see Lemma 4.2). We distinguish two cases, following the way in [2].

Case 1. $t \geq 2r$.

If $t \geq 2r$, then $t - r \geq \frac{t}{2} \geq \frac{t+r}{3}$. Since $\frac{1+t+r}{1+t-r} = 1 + \frac{2r}{1+t-r}$, we have

$$\begin{aligned} (3.9) \qquad \sum_{|\alpha|\leq 2} |D_x^\alpha u(x, t)| &\leq \frac{3G}{1 + t - r} \\ &\leq \frac{9G}{1 + t + r} . \end{aligned}$$

Case 2. $2r \geq t$.

We see that $r \geq \frac{t+r}{3}$. Thus,

$$(3.10) \qquad \sum_{|\alpha|\leq 2} |D_x^\alpha u(x, t)| \leq \frac{9G}{2(t + r)} \ln \frac{1 + r + t}{1 + |r - t|} .$$

If $t + r \geq 1$, we have

$$(3.11) \qquad \sum_{|\alpha|\leq 2} |D_x^\alpha u(x, t)| \leq \frac{9G}{1 + t + r} \ln \frac{1 + r + t}{1 + |r - t|} .$$

Since the right-hand side of (3.10) is uniformly bounded, also for $0 \leq t + r \leq 1$, (3.11) holds.

Thus combining (3.9) and (3.11), we obtain

$$\sum_{|\alpha|\le 2} |D_x^\alpha u(x,t)| \le \frac{9G}{1+t+r}\left(1+\ln\frac{1+r+t}{1+|r-t|}\right), \tag{3.12}$$

which concludes the proof of Lemma 3.1. ■

4. The basic estimate

We study the inhomogeneous Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = w(x,t), & (x,t) \in \boldsymbol{R}^3 \times (0,\infty), \\ u(x,0) = u_t(x,0) = 0, & x \in \boldsymbol{R}^3. \end{cases} \tag{4.1}$$

Then, the solution of (4.1) is given by

$$u(x,t) = \frac{1}{4\pi}\int_0^t (t-\tau)\int_{|\omega|=1} w(x+(t-\tau)\omega,\tau)\,d\omega\,d\tau. \tag{4.2}$$

We denote (4.2) by $u = Lw$. Note that $w \ge 0$ implies $Lw \ge 0$. If w is spherically symmetric, i.e., $w(x,t) = w(r,t)$, then we can write the solution of (4.1) in the form

$$u(r,t) = \frac{1}{2r}\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \rho w(\rho,\tau)\,d\rho\,d\tau. \tag{4.3}$$

We denote (4.3) by $u = Pw$.

Following John [5], we define the function

$$\tilde{u}(r,t) = \sup_{|x|=r} |u(x,t)|.$$

Then, we have

$$|(Lw)(x,t)| \le (P\tilde{w})(r,t) \tag{4.4}$$

(see [2], p. 1470 and also [5], Lemma II).

The following lemma is the basic estimate for the existence proof.

Lemma 4.1. *Assume that* $u(x,t) \in C(\boldsymbol{R}^3 \times [0,\infty))$ *satisfies*

$$|u(x,t)| \le \begin{cases} \dfrac{M}{(1+t+r)(1+|t-r|)^{k-1}}, & (k>1), \\[2ex] \dfrac{M}{1+t+r}\left(1+\ln\dfrac{1+t+r}{1+|t-r|}\right), & (k=1), \\[2ex] \dfrac{M}{(1+t+r)^k}, & (0<k<1), \end{cases} \tag{4.5}$$

where $r = |x|$, $M > 0$. *Let* $p > 1+\sqrt{2}$.

(i) *If* $k = \dfrac{2}{p-1}$, *then there exists a constant* C'_k *depending only on* k, *not on* M *such that*

$$(4.6)\qquad |L|u|^p(x, t)| \le \begin{cases} \dfrac{C'_k M^p}{(1+t+r)(1+|t-r|)^{k-1}}, & (k > 1), \\ \dfrac{C'_k M^3}{1+t+r}\left(1 + \ln \dfrac{1+t+r}{1+|t-r|}\right), & (k = 1), \\ \dfrac{C'_k M^p}{(1+t+r)^k}, & (0 < k < 1). \end{cases}$$

(ii) *If* $0 < k < \dfrac{2}{p-1}$, *then there exists a constant* $C'_{p,k}$ *depending only on* p *and* k, *not on* M *such that*

$$(4.7)\qquad |L|u|^p(x, t)| \le \begin{cases} \dfrac{C'_{p,k} M^p (1+t)^{2-k(p-1)}}{(1+t+r)(1+|t-r|)^{k-1}}, & (k > 1), \\ \dfrac{C'_{p,k} M^p (1+t)^{3-p}}{1+t+r}, & (k = 1), \\ \dfrac{C'_{p,k} M^p (1+t)^{2-k(p-1)}}{(1+t+r)^k}, & (0 < k < 1). \end{cases}$$

Proof. We shall prove (4.6) and (4.7) following [2].

Since $u = \tilde{u}$ satisfies (4.5), by (4.4), it is enough to estimate

$$(4.8)\qquad (P\tilde{u}^p)(r, t) = \frac{1}{2r}\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \rho \tilde{u}(\rho, \tau)^p d\rho d\tau .$$

The following lemma will be used repeatedly in this section.

Lemma 4.2.

$$(4.9)\qquad \frac{1}{2r}\int_{|r-t|}^{r+t} \frac{1}{(1+\rho)^k} d\rho \le \begin{cases} \dfrac{\max(1, k-1)\min(r, t)}{(k-1)r(1+t+r)(1+|t-r|)^{k-1}}, & (k > 1), \\ \dfrac{3}{1+t+r}\left(1 + \ln \dfrac{1+t+r}{1+|t-r|}\right), & (k = 1), \\ \dfrac{(1-k)\min(r, t)}{r(1+t+r)^k}, & (0 < k < 1) \end{cases}$$

for all t, $r \ge 0$.

Proof. We have proved (4.9) for $k = 1$ in Section 1. The estimates for $k > 1$ and $0 < k < 1$ are proved in [2]. ■

We distinguish three cases.

Case 1. $k > 1$.

We have now to consider two cases, $r \geq t$ and $t \geq r$.

(i) $r \geq t$.

By (4.5), we get

$$(P\tilde{u}^p)(r, t) \leq \frac{1}{2r} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \frac{\rho M^p}{(1 + \tau + \rho)^p (1 + \rho - \tau)^{p(k-1)}} d\rho d\tau .$$

Changing variables by

$$\alpha = \rho + \tau, \ \beta = \rho - \tau , \tag{4.10}$$

we have

$$(P\tilde{u}^p)(r, t) \leq \frac{M^p}{4r} \int_{r-t}^{r+t} \frac{1}{(1 + \alpha)^{p-1}} \int_{r-t}^{\alpha} \frac{1}{(1 + \beta)^{pk-p}} d\beta d\alpha .$$

Note that $pk - p < 1$, since $p > 1 + \sqrt{2}$ and $k \leq \dfrac{2}{p-1}$. Then

$$(P\tilde{u}^p)(r, t) \leq \frac{CM^p}{r} \int_{r-t}^{r+t} \frac{1}{(1 + \alpha)^{pk-2}} d\alpha . \tag{4.11}$$

If $k = \dfrac{2}{p-1}$, then $pk - 2 = k > 1$. Hence using Lemma 4.2, we obtain

$$(P\tilde{u}^p)(r, t) \leq \frac{CM^p}{(1 + t + r)(1 + r - t)^{k-1}} . \tag{4.12}$$

If $k < \dfrac{2}{p-1}$, then from (4.11)

$$\begin{aligned} (P\tilde{u}^p)(r, t) &\leq \frac{CM^p(1 + t + r)^{2-k(p-1)}}{r} \int_{r-t}^{r+t} \frac{1}{(1 + \alpha)^k} d\alpha \\ &\leq \frac{CM^p t}{r(1 + t + r)^{k(p-1)-1}(1 + r - t)^{k-1}} . \end{aligned} \tag{4.13}$$

We have used here Lemma 4.2. Note that $k(p-1) - 1 > 0$ since $p > 1 + \sqrt{2}$ and $k > 1$. We shall distinguish again two cases.

If $r \geq 1$, from (4.13)

$$(4.14) \qquad (P\tilde{u}^p)(r,t) \le \frac{CM^p(1+t)^{2-k(p-1)}}{(1+t+r)(1+r-t)^{k-1}},$$

since $3r \ge 1 + t + r$.

The right-hand side of (4.13) is bounded, since $r \ge t$. Hence, also for $0 \le r \le 1$, (4.14) holds. Therefore, we have

$$(P\tilde{u}^p)(r,t) \le \frac{CM^p(1+t)^{2-k(p-1)}}{(1+t+r)(1+r-t)^{k-1}}.$$

(ii) $t \ge r$.

We see that

$$(P\tilde{u}^p)(r,t) \le \frac{1}{2r}\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \frac{\rho M^p}{(1+\tau+\rho)^p(1+|\tau-\rho|)^{p(k-1)}}\, d\rho d\tau .$$

Then, by (4.10) we get

$$\begin{aligned}
(4.15) \qquad (P\tilde{u}^p)(r,t) &\le \frac{M^p}{4r}\int_{t-r}^{t+r}\int_{r-t}^{\alpha} \frac{1}{(1+\alpha)^{p-1}(1+|\beta|)^{pk-p}}\, d\beta d\alpha \\
&= \frac{M^p}{4r}\int_{t-r}^{t+r} \frac{1}{(1+\alpha)^{p-1}} \int_{r-t}^{t-r} \frac{1}{(1+|\beta|)^{pk-p}}\, d\beta d\alpha \\
&\quad + \frac{M^p}{4r}\int_{t-r}^{t+r} \frac{1}{(1+\alpha)^{p-1}} \int_{t-r}^{\alpha} \frac{1}{(1+\beta)^{pk-p}}\, d\beta d\alpha \\
&= \frac{M^p}{2r}\int_{t-r}^{t+r} \frac{1}{(1+\alpha)^{p-1}} \int_{0}^{t-r} \frac{1}{(1+\beta)^{pk-p}}\, d\beta d\alpha \\
&\quad + \frac{M^p}{4r}\int_{t-r}^{t+r} \frac{1}{(1+\alpha)^{p-1}} \int_{t-r}^{\alpha} \frac{1}{(1+\beta)^{pk-p}}\, d\beta d\alpha \\
&\le \frac{CM^p}{r}\int_{t-r}^{t+r} \frac{1}{(1+\alpha)^{pk-2}}\, d\beta d\alpha .
\end{aligned}$$

If $k = \dfrac{2}{p-1}$, using Lemma 4.2 we obtain

$$(4.16) \qquad (P\tilde{u}^p)(r,t) \le \frac{CM^p}{(1+t+r)(1+t-r)^{k-1}}.$$

If $k < \dfrac{2}{p-1}$, it follows from Lemma 4.2 and (4.15) that

(4.17)
$$(P\tilde{u}^p)(r,t) \le \frac{CM^p(1+t+r)^{2-k(p-1)}}{r}\int_{t-r}^{t+r}\frac{1}{(1+\alpha)^k}d\alpha$$
$$\le \frac{CM^p(1+t)^{2-k(p-1)}}{(1+t+r)(1+t-r)^{k-1}}\,.$$

Case 2. $k=1$.
(i) $r \ge t$.
We proceed as before.

(4.18)
$$(P\tilde{u}^p)(r,t) \le \frac{1}{2r}\int_0^t\int_{r-t+\tau}^{r+t-\tau}\frac{\rho M^p}{(1+\tau+\rho)^p}\left(1+\ln\frac{1+\rho+\tau}{1+\rho-\tau}\right)^p d\rho d\tau$$
$$\le \frac{M^p}{4r}\int_{r-t}^{r+t}\frac{1}{(1+\alpha)^{p-1}}\int_{r-t}^{\alpha}\left(1+\ln\frac{1+\alpha}{1+\beta}\right)^p d\beta d\alpha\,.$$

Now, we make use of the following proposition which is important in order to prove Lemma 4.1 for the case $k=1$.

Proposition 4.3. *Let n be a positive integer. Then,*

(4.19)
$$\int_0^{\alpha}\left(1+\ln\frac{1+\alpha}{1+\beta}\right)^n d\beta$$
$$=\alpha+\sum_{m=1}^{n}\binom{n}{m}\left\{m!\alpha-\sum_{l=0}^{m-1}\frac{m!}{(m-l)!}(\ln(1+\alpha))^{m-l}\right\}.$$

We shall prove the proposition later on in this section.

Since $k \le \dfrac{2}{p-1}$ and $k=1$, $p \le 3$. By Proposition 4.3,

(4.20)
$$\int_0^{\alpha}\left(1+\ln\frac{1+\alpha}{1+\beta}\right)^3 d\beta = 16\alpha-15\ln(1+\alpha)-6(\ln(1+\alpha))^2$$
$$-(\ln(1+\alpha))^3\,.$$

Combining this with (4.18), we have

(4.21)
$$(P\tilde{u}^p)(r,t) \le \frac{4M^p}{r}\int_{r-t}^{r+t}\frac{1}{(1+\alpha)^{p-2}}d\alpha\,.$$

If $k=\dfrac{2}{p-1}$, $p=3$. Thus, using Lemma 4.2, we get

(4.22)
$$(P\tilde{u}^p)(r,t) \le \frac{24M^3}{1+t+r}\left(1+\ln\frac{1+r+t}{1+r-t}\right).$$

If $k < \frac{2}{p-1}$, $p < 3$. From (4.21), using Lemma 4.2 we have

$$(P\tilde{u}^p)(r, t) \leq \frac{CM^p t}{r(1 + r + t)^{p-2}} .$$

Then, proceeding as the case 1. $k > 1$, (i) $r \geq t$, $0 < k < \frac{2}{p-1}$, we obtain

$$(P\tilde{u}^p)(r, t) \leq \frac{CM^p(1 + t)^{3-p}}{1 + t + r} . \tag{4.23}$$

(ii) $t \geq r$.

As before

$$\begin{aligned} (P\tilde{u}^p)(r, t) &\leq \frac{M^p}{4r} \int_{t-r}^{t+r} \int_{r-t}^{\alpha} \frac{1}{(1 + \alpha)^{p-1}} \left(1 + \ln \frac{1 + \alpha}{1 + |\beta|}\right)^p d\beta d\alpha \\ &= \frac{M^p}{2r} \int_{t-r}^{t+r} \frac{1}{(1 + \alpha)^{p-1}} \int_0^{t-r} \left(1 + \ln \frac{1 + \alpha}{1 + \beta}\right)^p d\beta d\alpha \\ &\quad + \frac{M^p}{4r} \int_{t-r}^{t+r} \frac{1}{(1 + \alpha)^{p-1}} \int_{t-r}^{\alpha} \left(1 + \ln \frac{1 + \alpha}{1 + \beta}\right)^p d\beta d\alpha \\ &\leq \frac{12M^p}{r} \int_{t-r}^{t+r} \frac{1}{(1 + \alpha)^{p-2}} d\alpha . \end{aligned} \tag{4.24}$$

We have used (4.20) in the last estimate.

If $k = \frac{2}{p-1}$ $(p = 3)$, using Lemma 4.2 we obtain

$$(P\tilde{u}^p)(r, t) \leq \frac{72M^3}{1 + t + r}\left(1 + \ln \frac{1 + t + r}{1 + t - r}\right) . \tag{4.25}$$

If $k < \frac{2}{p-1}$ $(p < 3)$, it follows from Lemma 4.2 and (4.24) that

$$\begin{aligned} (P\tilde{u}^p)(r, t) &\leq \frac{CM^p}{(1 + t + r)^{p-2}} \\ &\leq \frac{CM^p(1 + t)^{3-p}}{1 + t + r} , \end{aligned} \tag{4.26}$$

since $p > 1 + \sqrt{2}$.

Case 3. $0 < k < 1$.

(i) $r \geq t$.

As before

$$(4.27)\qquad (P\tilde{u}^p)(r, t) \le \frac{1}{2r}\int_0^t\int_{r-t+\tau}^{r+t-\tau}\frac{\rho M^p}{(1+\tau+\rho)^{pk}}d\rho d\tau$$
$$\le \frac{M^p}{4r}\int_{r-t}^{r+t}\int_{r-t}^{\alpha}\frac{1}{(1+\alpha)^{pk-1}}d\beta d\alpha\,,$$

or

$$(4.28)\qquad (P\tilde{u}^p)(r, t) \le \frac{M^p}{4r}\int_{r-t}^{r+t}\frac{1}{(1+\alpha)^{pk-2}}d\alpha\,.$$

If $k = \frac{2}{p-1}$, using Lemma 4.2 we have

$$(4.29)\qquad (P\tilde{u}^p)(r, t) \le \frac{CM^p}{(1+t+r)^k}\,.$$

If $k < \frac{2}{p-1}$, we distinguish three cases.

(a) $r \ge 1$ and $2t \ge r \ge t$.

Note that $pk - 2 < k < 1$. Then, it follows from Lemma 4.2 and (4.28) that

$$(4.30)\qquad (P\tilde{u}^p)(r, t) \le \frac{CM^p t}{r(1+r+t)^{pk-2}}$$
$$\le \frac{CM^p(1+t)^{2-k(p-1)}}{(1+t+r)^k}\,.$$

(b) $r \ge 1$ and $r \ge 2t$.

The β-integral in (4.27) is dominated by $\alpha - (r-t) \le 2t$. Then, using Lemma 4.2 we have

$$(P\tilde{u}^p)(r, t) \le \frac{M^p t}{2r}\int_{r-t}^{r+t}\frac{1}{(1+\alpha)^{pk-1}}d\alpha$$
$$\le \frac{M^p t(1+t+r)^2}{2r}\int_{r-t}^{r+t}\frac{1}{(1+\alpha)^{pk+1}}d\alpha$$
$$\le \frac{CM^p t^2(1+t+r)}{r(1+r-t)^{pk}}\,.$$

If $r \ge 1$ and $r \ge 2t$, then $r - t \ge \frac{r}{2} \ge \frac{r+t}{3}$ and $\frac{5r}{2} \ge 1 + t + r$. Note that $pk - k > 0$. Thus, (4.31) $$(P\tilde{u}^p)(r, t) \leq \frac{CM^p(1+t)^{2-k(p-1)}}{(1+t+r)^k}.$$

(c) $0 \leq r \leq 1$.

As the case (a) $r \geq 1$ and $2t \geq r \geq t$,

$$(P\tilde{u}^p)(r, t) \leq \frac{CM^p t}{r(1+r+t)^{pk-2}}.$$

Since $r \geq t$, the right-hand side is bounded. Hence, also for $0 \leq r \leq 1$, (4.31) holds.

Therefore, we obtain

$$(P\tilde{u}^p)(r, t) \leq \frac{CM^p(1+t)^{2-k(p-1)}}{(1+t+r)^k}.$$

(ii) $t \geq r$.

$$(P\tilde{u}^p)(r, t) \leq \frac{M^p}{4r}\int_{t-r}^{t+r}\int_{r-t}^{t-r}\frac{1}{(1+\alpha)^{pk-1}}d\beta d\alpha + \frac{M^p}{4r}\int_{t-r}^{t+r}\int_{t-r}^{\alpha}\frac{1}{(1+\alpha)^{pk-1}}d\beta d\alpha$$
$$\leq \frac{3M^p}{4r}\int_{t-r}^{t+r}\frac{1}{(1+\alpha)^{pk-2}}d\alpha.$$

If $k = \frac{2}{p-1}$, using Lemma 4.2 we obtain

(4.32) $$(P\tilde{u}^p)(r, t) \leq \frac{CM^p}{(1+t+r)^k}.$$

If $k < \frac{2}{p-1}$, $pk - 2 < k < 1$. Using Lemma 4.2 again, we obtain

(4.33) $$(P\tilde{u}^p)(r, t) \leq \frac{CM^p}{(1+t+r)^{pk-2}} \leq \frac{CM^p(1+t)^{2-k(p-1)}}{(1+t+r)^k}.$$

This completes the proof of Lemma 4.1. ■

In [2], all the β-integral except for the case $k > 1 + \frac{1}{p}$ in the proof of Lemma 2.3 of [2] is dominated by a constant. For this reason, Lemma 2.3 in [2] holds only for $k > \frac{2}{p-1}$.

Proof of Proposition 4.3. Using the binominal theorem, we have

$$\int_0^\alpha \left(1 + \ln \frac{1+\alpha}{1+\beta}\right)^n d\beta = \sum_{m=0}^n \binom{n}{m} \int_0^\alpha \left(\ln \frac{1+\alpha}{1+\beta}\right)^m d\beta \tag{4.34}$$

$$= \alpha + \sum_{m=1}^n \binom{n}{m} \int_0^\alpha \left(\ln \frac{1+\alpha}{1+\beta}\right)^m d\beta .$$

Moreover for the integral in the last term

$$\int_0^\alpha (\ln(1+\alpha) - \ln(1+\beta))^m d\beta \tag{4.35}$$

$$= \sum_{l=0}^m (-1)^l \binom{m}{l} (\ln(1+\alpha))^{m-l} \int_0^\alpha (\ln(1+\beta))^l d\beta .$$

By induction on l and integrating by parts, we easily derive that

$$\int (\ln x)^l dx = \sum_{i=0}^l (-1)^i \frac{l!}{(l-i)!} x (\ln x)^{l-i} .$$

Applying this to (4.35), we get

(4.36)

$$\int_0^\alpha (\ln(1+\alpha) - \ln(1+\beta))^m d\beta$$

$$= \sum_{l=0}^m (-1)^l \binom{m}{l} (\ln(1+\alpha))^{m-l}$$

$$\times \left\{ \sum_{i=0}^l (-1)^i \frac{l!}{(l-i)!} (1+\alpha)(\ln(1+\alpha))^{l-i} - (-1)^l l! \right\}$$

$$= \alpha(\ln(1+\alpha))^m + \sum_{l=1}^m (-1)^l \binom{m}{l} (\ln(1+\alpha))^{m-l}$$

$$\times \left\{ (1+\alpha)(\ln(1+\alpha))^l + \sum_{i=1}^l (-1)^i \frac{l!}{(l-i)!} (1+\alpha)(\ln(1+\alpha))^{l-i} - (-1)^l l! \right\}$$

$$= \alpha(\ln(1+\alpha))^m + (1+\alpha)(\ln(1+\alpha))^m \sum_{l=1}^m (-1)^l \binom{m}{l}$$

$$+ \sum_{i=1}^m (1+\alpha)(\ln(1+\alpha))^{m-i} \sum_{l=i}^m (-1)^{l+i}$$

$$\times \frac{m!}{(m-l)!(l-i)!} - \sum_{l=1}^m \frac{m!}{(m-l)!} (\ln(1+\alpha))^{m-l}$$

$$= m!(1+\alpha) - (\ln(1+\alpha))^m - \sum_{l=1}^m \frac{m!}{(m-l)!} (\ln(1+\alpha))^{m-l}$$

in the last equality we have used that

$$\sum_{l=0}^{m} (-1)^l \binom{m}{l} = 0$$

and

$$\sum_{l=i}^{m} (-1)^{l+i} \frac{m!}{(m-l)!(l-i)!} = 0 \qquad \text{for } m \geq i+1 .$$

Thus, inserting (4.36) into (4.34), we obtain (4.19). ∎

5. Proof of Theorem 1, 2

To prove Theorem 1, 2 we have to make some preparations.

A solution $u \in C^2(\boldsymbol{R}^3 \times I_{p,k})$ of (2.1) satisfies the integral equation

$$u = u^0 + LF(u) , \tag{5.1}$$

where

$$I_{p,k} = \begin{cases} [0, \infty) , & \text{if } k = \dfrac{2}{p-1} , \\[2ex] [0, T) , & \text{if } 0 < k < \dfrac{2}{p-1} , \end{cases}$$

and u^0 is the solution of (3.1) and L is given by (4.2).

Conversely the following proposition holds. In order to find a solution of (2.1), we need the proposition (see [2], [5]).

Proposition 5.1. *If u satisfies* (5.1) *and $D_x^\alpha u$ is continuous on $\boldsymbol{R}^3 \times I_{p,k}$ for $|\alpha| \leq 2$, then u is the solution of* (2.1).

Proof of Theorem 1, 2. We define the norm for functions $u(x, t)$ which are continuous on $\boldsymbol{R}^3 \times I_{p,k}$

(5.2)

$$\|u\|_{p,k} = \begin{cases} \sup\limits_{\substack{x \in \boldsymbol{R}^3 \\ t \in I_{p,k}}} (1 + t + |x|)(1 + |t - |x||)^{k-1} |u(x, t)| , & (k > 1) , \\[3ex] \sup\limits_{\substack{x \in \boldsymbol{R}^3 \\ t \in I_{p,k}}} (1 + t + |x|)\left(1 + \ln \dfrac{1 + t + |x|}{1 + |t - |x||}\right)^{-1} |u(x, t)| , & (k = 1) , \\[3ex] \sup\limits_{\substack{x \in \boldsymbol{R}^3 \\ t \in I_{p,k}}} (1 + t + |x|)^k |u(x, t)| , & (0 < k < 1) . \end{cases}$$

Then, we have

$$|u(x, t)| \le \|u\|_{p,k}\,, \tag{5.3}$$

$$\|\,|u|^{\theta}|v|^{1-\theta}\|_{p,k} \le \|u\|_{p,k}^{\theta}\|v\|_{p,k}^{1-\theta}\,, \qquad \text{for } 0 \le \theta \le 1\,, \tag{5.4}$$

and by Lemma 4.1,

$$\|L|u|^{p}\|_{p,k} \le B_{p,k}\|u\|_{p,k}^{p}\,, \tag{5.5}$$

$$\|L|u|^{\theta p}|v|^{(1-\theta)p}\|_{p,k} \le B_{p,k}\|u\|_{p,k}^{\theta p}\|v\|_{p,k}^{(1-\theta)p}\,, \tag{5.6}$$

where

$$B_{p,k} = \begin{cases} C'_k\,, & \left(k = \dfrac{2}{p-1}\right), \\ C'_{p,k}(1+T)^{2-k(p-1)}\,, & \left(0 < k < \dfrac{2}{p-1}\right), \end{cases} \tag{5.7}$$

and C'_k, $C'_{p,k}$ are constants given in Lemma 4.1.

Let $X_{p,k}$ be the linear space defined by

$$X_{p,k} = \{u(x, t) \colon D_x^{\alpha}u(x, t) \in C(\boldsymbol{R}^3 \times I_{p,k}),\ \|D_x^{\alpha}u\|_{p,k} < \infty \text{ for } |\alpha| \le 2\}\,. \tag{5.8}$$

We can verify easily that $X_{p,k}$ is complete with respect to the norm

$$\|u\|_{X_{p,k}} = \sum_{|\alpha| \le 2} \|D_x^{\alpha}u\|_{p,k}\,.$$

We define the sequence of functions $\{u_n\}$ by

$$u_0 = u^0,\ u_{n+1} = u^0 + LF(u_n)\,.$$

It follows from Lemma 3.1 that

$$\|u^0\|_{X_{p,k}} \le C_k G\,. \tag{5.9}$$

Hence, $u^0 \in X_{p,k}$.

Now, we assume that

$$2^p pAB_{p,k}(C_k G)^{p-1} \le 1 \qquad \text{and} \qquad C_k G \le \frac{1}{2}\,. \tag{5.10}$$

Then, we have

$$2^p pAB_{p,k}\|u^0\|_{p,k}^{p-1} \le 1 \qquad \text{and} \qquad \|u^0\|_{p,k} \le \frac{1}{2}\,. \tag{5.11}$$

As in [2], [5], we see that if u^0 satisfies (5.11), then $\{u_n\}$ is a Cauchy sequence in $X_{p,k}$. Since $X_{p,k}$ is complete, there exists $u \in X_{p,k}$ such that $D_x^{\alpha}u_n$ converges uniformly to $D_x^{\alpha}u$ for $|\alpha| \le 2$, $n \to \infty$. Clearly u satisfies (5.1).

Therefore if $k = \frac{2}{p-1}$, choosing G sufficiently small by (5.10), it follows from Proposition 5.1 that there exists a unique global solution of (2.1) since $B_{p,k}$ is a constant as given in (5.7).

On the other hand, if $0 < k < \frac{2}{p-1}$, there exists a unique local solution of (2.1). Combining (5.10) with (5.7) we obtain (2.2). This completes the proof of Theorem 1, 2. ■

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