

A Normalization of a Nonlinear n -System Not Satisfying the Poincaré Condition at an Irregular Singularity

By

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§1. Introduction

Let $(t, x) = (t, x_1, \dots, x_n)$ be the coordinates of $\mathbf{C} \times \mathbf{C}^n$, where \mathbf{C} is the complex plane and \mathbf{C}^n is the n -dimensional complex space with the norm

$$|x| = \max_{1 \leq i \leq n} \{|x_i|\}.$$

Consider a singular system of nonlinear differential equations

$$(1.1) \quad tdx/dt = g(t, x),$$

where $g(t, x)$ is an n -dimensional row vector function meromorphic on a polydisk U centered at $(t, x) = (0, 0)$ and holomorphic on $U \setminus \{t = 0\}$. The least nonnegative integer σ such that $t^\sigma g(t, x)$ is holomorphic on U is called the *Poincaré rank* of (1.1). We study the problem of reducing system (1.1) with $\sigma > 0$ and

$$t^\sigma g(t, x)|_{t=0, x=0} = 0$$

into a system solvable by quadrature by a suitable biholomorphic transformation.

Let $\lambda_1, \dots, \lambda_n$ be the characteristic values of the Jacobian matrix $(\partial G/\partial x)(0, 0)$, where $G = G(t, x) = t^\sigma g(t, x)$. We call these characteristic values the characteristic exponents of system (1, 1). The condition that all characteristic exponents of (1.1) are located on one side of a line in \mathbf{C} passing through the origin is called the *Poincaré condition*. The following theorem under the condition is well known.

Malmquist's theorem ([7], [8], [9]). *Suppose that the Poincaré condition is satisfied. Then there exists a transformation $x = \varphi(t, y)$ which takes system (1.1) into a linear system*

$$tdy/dt = y \operatorname{diag}(a_1(t), \dots, a_n(t)),$$

where $a_i(t)$, $1 \leq i \leq n$, is a polynomial in t^{-1} of degree σ :

$$a_i(t) = \lambda_i t^{-\sigma} + \cdots + \alpha_i,$$

and $\varphi(t, y)$ is holomorphic on $S \times V$, $S \subset \mathbf{C}$ being a sectorial domain $\{t \in \mathbf{C}; \theta_1 < \arg t < \theta_2, 0 < |t| < r\}$, $V \subset \mathbf{C}^n$ being a polydisk centered at $y = 0$.

We remark that, although the Poincaré condition is a generic one, there are many important systems which do not satisfy the condition such as the Painlevé equations, the restrictions of the Garnier system on lines, and the system for the spherical pendulum. As far as the author knows, the first who studied the case where the Poincaré condition is completely violated is M. Iwano ([2]). He considered a 2-system of Poincaré rank 1 of the form

$$(1.2) \quad tdx/dt = x \operatorname{diag}(\mu_1 t^{-1} + \alpha_1, -\mu_2 t^{-1} + \alpha_2) + f(t, x)$$

where μ_1 and μ_2 are coprime positive integers, α_1 and α_2 are nonnegative real numbers such that $\operatorname{Re}(\mu_1 \alpha_2 + \mu_2 \alpha_1) > 0$ and $f(t, x)$ is holomorphic at $(t, x) = (0, 0)$ with $f(t, x) = O(|x|^2)$, O being Landau's symbol; and he showed the existence of a transformation which changes system (1.2) into the system:

$$(1.3) \quad tdy/dt = y \operatorname{diag}(\mu_1 t^{-1} + \alpha_1, -\mu_2 t^{-1} + \alpha_2).$$

After the work of M. Iwano, his idea was made use of in the study of the Painlevé equations ([11], [12], [13], [15]), the assumption on α_1 and α_2 was improved ([3], [10]), and many studies on this subject appeared, for example, ([4], [5], [6], [12]). Among them, the author was most interested in the article of M. Iwano ([4]) which investigated a 4-system with the characteristic exponents $\sqrt{-1}, 1, 0, -1$ in connection with the study of the spherical pendulum. The purpose of this paper is to study an n -system of which some of the characteristic exponents are on a line in \mathbf{C} passing through the origin and the others are located on one side of the line.

We now explain our problem. Set, for preassigned positive constants $\underline{\theta}, \bar{\theta}, r$, and δ ,

$$S(\underline{\theta}, \bar{\theta}; r) = \{t \in \mathbf{C}; \underline{\theta} < \arg t < \bar{\theta}, |t| < r\},$$

$$D(\underline{\theta}, \bar{\theta}; r, \delta) = \{(t, x) \in \mathbf{C} \times \mathbf{C}^n; t \in S(\underline{\theta}, \bar{\theta}; r), |x| < \delta\},$$

and denote by $\mathcal{A}(\underline{\theta}, \bar{\theta}; r)$ the set of holomorphic functions from $S(\underline{\theta}, \bar{\theta}; r)$ into \mathbf{C}^n admitting asymptotic expansions in powers of t as $t \rightarrow 0$, $t \in S(\underline{\theta}, \bar{\theta}; r)$. We consider the system

$$(1.4) \quad tdx_i/dt = x_i a_i(t) + f_i(t, x_1, \dots, x_n), \quad 1 \leq i \leq n,$$

or in vector expression,

$$\Sigma: \quad \begin{aligned} tdx/dt &= xA(t) + f(t, x), \\ A(t) &= \text{diag}(a_1(t), \dots, a_n(t)), \quad f = (f_1, \dots, f_n), \end{aligned}$$

where

(i) $a_i(t) = \lambda_i t^{-1} + \alpha_i$ are linear functions in t^{-1} with coefficients in \mathbf{C} such that

$$(1.5) \quad \text{Im } \lambda_i > 0 \quad \text{if } 1 \leq i \leq p, \quad \text{Im } \lambda_{p+j} = 0 \quad \text{if } 1 \leq j \leq q,$$

($p + q = n$) and

$$\text{Im } \alpha_i = 0, \quad 1 \leq i \leq n,$$

and furthermore

$$(1.6) \quad (k, \lambda) - \lambda_i \equiv k_1 \lambda_1 + \dots + k_n \lambda_n - \lambda_i \neq 0, \quad 1 \leq i \leq n,$$

for all $k = (k_1, \dots, k_n) \in \mathbf{Z}_{\geq 0}^n$ with $|k| \geq 2$, where $\mathbf{Z}_{\geq 0}^n$ is the direct sum of n copies of the set of all nonnegative rational integers $\mathbf{Z}_{\geq 0}$ and $|k| = k_1 + \dots + k_n$;

(ii) There exists a positive constant κ such that

$$(1.7) \quad \lambda_{p+j} + \kappa \alpha_{p+j} > 0, \quad 1 \leq j \leq q;$$

(iii) f is holomorphic and bounded on $D(\theta, \bar{\theta}; r, \delta)$ with $f = O(|x|^2)$ and that the coefficients $f_k(t)$ in the power series expansion of $f(t, x)$ as $f(t, x) = \sum_{|k| \geq 2} f_k(t) x^k$ are in $\mathcal{A}(\theta, \bar{\theta}; r)$. Here, for $k = (k_1, \dots, k_n) \in \mathbf{Z}_{\geq 0}^n$, x^k stands for $x_1^{k_1} \dots x_n^{k_n}$.

Set

$$(1.8) \quad A_i(t) = \int_{\infty}^t \lambda_i t^{-2} dt = -\lambda_i t^{-1}, \quad 1 \leq i \leq n.$$

We say that θ is a singular direction of $A_i(t)$ if

$$(1.9) \quad \cos(\theta + \arg \lambda_i) = 0.$$

A half line passing through $t = 0$ with argument θ satisfying (1.9) is called a singular line of $A_i(t)$.

We suppose moreover the following assumptions:

(A₁) For each i with $1 \leq i \leq n$, $S(\theta, \bar{\theta}; r)$ contains one and only one singular line of $A_i(t)$ and neither θ nor $\bar{\theta}$ is a singular direction of $A_i(t)$.

(A₂) For each i with $1 \leq i \leq p$, the i -th entry of f is factored by x_i .

We state our main theorem.

Theorem. *Under the assumptions (A₁) and (A₂), there exists a unique vector function u holomorphic and bounded on $D(\theta, \bar{\theta}; r', \delta')$ with $u(t, y) = O(|y|^2)$ such that a change of coordinates $(t, x) \rightarrow (t, y)$ of the form*

$$T: \quad x = y + u(t, y)$$

reduces the system Σ to

$$\Sigma': \quad tdy/dt = yA(t),$$

provided that r' and $\delta' > 0$ are sufficiently small. Here the coefficients $u_k(t)$ in the power series expansion of $u(t, y)$ as $u(t, y) = \sum_{|k| \geq 2} u_k(t)y^k$ are in $\mathcal{A}(\underline{\theta}, \bar{\theta}; r')$.

By virtue of the theorem, we can construct an n -parameter family of solutions to the system Σ . In fact, since the general solution to the reduced system Σ' is

$$(1.10) \quad \begin{aligned} y(t; C) &= (y_1(t; C_1), \dots, y_n(t; C_n)), \\ y_i(t; C_i) &= C_i e^{-\lambda_i/t} t^{\alpha_i}, \quad 1 \leq i \leq n, \end{aligned}$$

an n -parameter family of solutions to the original system Σ is given by

$$x(t; C) = y(t; C) + u(t, y(t; C)).$$

Here $C = (C_1, \dots, C_n)$ is an arbitrary constant vector.

We use the following abbreviation:

$$S(r) = S(\underline{\theta}, \bar{\theta}; r), \quad D(r, \delta) = D(\underline{\theta}, \bar{\theta}; r, \delta),$$

since $\underline{\theta}$ and $\bar{\theta}$ are fixed hereafter.

§2. The construction of the formal transformation T

Let

$$\Sigma: \quad tdx/dt = xA(t) + f(t, x)$$

be changed to a system of the form

$$(2.1) \quad tdy/dt = yA(t) + g(t, y),$$

by a change of coordinates $(t, x) \rightarrow (t, y)$ of the form

$$T: \quad x = y + u(t, y).$$

We have, then,

$$(2.2) \quad g(t, y) = -(t\partial u/\partial t + L_A u - f(t, y + u))(\mathbf{1}_n + \partial u/\partial y)^{-1},$$

where $\mathbf{1}_n$ is an n -by- n unit matrix and L_A is an operator associated with A defined by

$$(2.3) \quad (L_A u)(t, y) = yA(t)\partial u(t, y)/\partial y - u(t, y)A(t).$$

Therefore, in order that (2.1) is linearized by T , it is necessary and sufficient that u satisfies the following sytem of partial differential equations:

$$(2.4) \quad t\partial u/\partial t + L_A u = f(t, y + u).$$

We want to obtain a formal solution to (2.4) of the form

$$(2.5) \quad u(t, y) = \sum_{|k| \geq 2} u_k(t) y^k.$$

Substituting (2.5) to (2.4) and equating the coefficients of like powers in y , we have

$$(2.6) \quad \begin{aligned} & tdu_{ik}/dt + u_{ik}[\{(k, \lambda) - \lambda_i\}t^{-1} - \{(k, \alpha) - \alpha_i\}] \\ & = \sum f_{il} \cdot u_{1\mu^1} \cdots u_{1\mu^1} \cdots u_{n\nu^1} \cdots u_{n\nu^1 n}, \end{aligned}$$

where the summation are taken with respect to $l = (l_1, \dots, l_n) \in \mathbf{Z}_{\geq 0}^n$ with $|l| \geq 2$ such that

$$l + \mu^1 + \cdots + \mu^{l^1} + \cdots + \nu^1 + \cdots + \nu^{l^n} = k, \quad |k| \geq 2,$$

and $u_k = (u_{1k}, \dots, u_{nk})$, $\mu^i, \dots, \nu^j \in \mathbf{Z}_{\geq 0}^n$, $1 \leq i \leq l_1, \dots, 1 \leq j \leq l_n$, $f_l = (f_{1l}, \dots, f_{nl})$, and further

$$\begin{aligned} u_{i(0, \dots, 0)} &\equiv 0, & 1 \leq i \leq n, \\ u_{i(0, \dots, 0, 1, 0, \dots, 0)} &\equiv \delta_{ij}, & 1 \leq i, j \leq n, \end{aligned}$$

by convention, δ_{ij} being Kronecker's delta.

Note that, by (A_2) , $f_{ik}(t)$ vanish identically for $k_i = 0$.

We can uniquely determine $u_{ik} \in \mathcal{A}(\theta, \bar{\theta}; r)$, $|x| \geq 2$ so that they satisfy (2.6) with $u_{ik} \equiv 0$ for $k_i = 0$, $1 \leq i \leq p$, by making use of the following lemma.

Lemma 2.1. (Hukuhara) *For preassigned constants θ_1, θ_2 and $r > 0$, we consider a differential equation of the form:*

$$tdu/dt = uc(t) + f(t),$$

where $c(t)$ is a linear function in t^{-1} and $f \in \mathcal{A}(\theta_1, \theta_2; r)$. Suppose that $S(\theta_1, \theta_2; r)$ contains one and only one singular line of $C(t) = \int_{\infty}^t c(t)t^{-1} dt$ and

neither θ_1 nor θ_2 is a singular direction of $C(t)$, then the equation has a unique solution $u \in \mathcal{A}(\theta_1, \theta_2; r)$.

Therefore we can construct a formal transformation T given by $x = y + \sum_{|k| \geq 2} u_k(t) y^k$.

§3. Sketch of the proof of Theorem

3.1 Truncated systems for the formal power series $\sum u_k(t)y^k$.
Recalling the assumption (A_2), we write Σ as

$$\begin{aligned}tdx^+ / dt &= x^+ (A^+(t) + F^+(t, x^+, x^*)), \\tdx^* / dt &= x^* A^*(t) + f^*(t, x^+, x^*),\end{aligned}$$

where

$$\begin{aligned}x^+ &= (x_1, \dots, x_p), \quad x^* = (x_{p+1}, \dots, x_n), \\A^+(t) &= \text{diag}(a_1(t), \dots, a_p(t)), \quad A^*(t) = \text{diag}(a_{p+1}(t), \dots, a_n(t)), \\f^+ &= (f_1, \dots, f_p), \quad f^* = (f_{p+1}, \dots, f_n), \\F^+ &= \text{diag}(x_1^{-1}f_1, \dots, x_p^{-1}f_p).\end{aligned}$$

Let $\sum u_k(t)y^k$ be the formal power series constructed in §2 and write the change of coordinates T as follows:

$$\begin{aligned}x^+ &= y^+ (\mathbf{1}_p + \text{diag}(u^+)), \quad u^+ = (y_1^{-1}u_1, \dots, y_p^{-1}u_p), \\x^* &= y^* + u^*, \quad u^* = (u_{p+1}, \dots, u_n),\end{aligned}$$

where $y^+ = (y_1, \dots, y_p)$, $y^* = (y_{p+1}, \dots, y_n)$.

Set, for each $N \geq 1$,

$$(3.1)^+ \quad U_{[N]}^+(t, y) = \text{diag}(u_{[N]}^+),$$

and set, for each $N \geq 2$,

$$(3.1)^* \quad u_{[N]}^*(t, y) = y^* + \sum_{2 \leq |k| \leq N} u_k^*(t)y^k,$$

where

$$\begin{aligned}u_{[N]}^* &= (1, \dots, 1) + \sum_{1 \leq |k| \leq N} u_k^+(t)y^k, \\u_k^+ &= (u_{1, k-(1, 0, \dots, 0)}, \dots, u_{p, k-(0, \dots, 0, 1, 0, \dots, 0)}), \\u_k^* &= (u_{p+1, k}, \dots, u_{n, k}).\end{aligned}$$

Then, in order that a change of coordinates $(t, x) \rightarrow (t, y)$ of the form

$$(3.2)^+ \quad x^+ = y^+ (U_{[N]}^+(t, y) + \Phi^+(t, y)),$$

$$(3.2)^* \quad x^* = u_{[N]}^*(t, y) + \varphi^*(t, y),$$

where $\Phi^+ = \text{diag}(\varphi_1^+, \dots, \varphi_p^+)$, transforms the system Σ into the system Σ' , it is necessary and sufficient that the vector function $\varphi = (\varphi^+, \varphi^*)$ satisfies the following system of partial differential equations:

$$(3.3)_N^+ \quad t\partial\varphi^+/\partial t + y^+ A^+(t)\partial\varphi^+/\partial y^+ = f_{[N]}^+(t, y, \varphi),$$

$$(3.4)_N^* \quad t\partial\varphi^*/\partial t + L_{A^*}\varphi_{[N]}^*(t, y, \varphi),$$

where $\varphi^+ = (\varphi_1^+, \dots, \varphi_p^+)$ and

$$(3.4)^+ \quad f_{[N]}^+(t, y, \varphi) = -t\partial u_{[N]}^+/\partial t - y^+ A^+(t)\partial u_{[N]}^+/\partial y^+ + (u_{[N]}^+ + \varphi^+)F^+(t, y^+(U_{[N]}^+ + \Phi^+), u_{[N]}^* + \varphi^*),$$

$$(3.4)^* \quad f_{[N]}^*(t, y, \varphi) = -t\partial u_{[N]}^*/\partial t - L_{A^*}u_{[N]}^* + f^*(t, y^+(U_{[N]}^+ + \Phi^+), u_{[N]}^* + \varphi^*).$$

Put $f_{[N]} = (f_{[N]}^+, f_{[N]}^*)$. Then we can verify

Lemma 3.1. *There exist positive constants C_N and M such that*

$$(3.5) \quad |f_{[N]}(t, y, 0)| \leq C_N|y|^{N+1};$$

$$(3.6) \quad |f_{[N]}(t, y, \varphi) - f_{[N]}(t, y, \psi)| \leq M(|y| + |\varphi| + |\psi|)|\varphi - \psi|,$$

hold for $(t, y) \in D(r_N, \delta_N)$ and $|\varphi|, |\psi| < \rho_N$, provided r_N, δ_N , and $\rho_N > 0$ are sufficiently small.

We note that the inequality

$$(3.7) \quad |f_{[N]}(t, y, \varphi)| \leq M(|y| + |\varphi|)|\varphi| + C_N|y|^{N+1}$$

for $(t, y) \in D(r_N, \delta_N)$ and $|\varphi| < \rho_N$, is derived from (3.5) and (3.6).

Define $e_i(t)$ and $e_j^*(t)$ by

$$e_i(t) = 1, \quad 1 \leq i \leq p,$$

$$e_j^*(t) = \exp \left[\int_{\infty}^t a_j(t)t^{-1} dt \right] = \exp [-\lambda_j t^{-1} + \alpha_j \log t], \quad 1 \leq j \leq q,$$

and set

$$e(t) = (e_1(t), \dots, e_p(t), e_1^*(t), \dots, e_q^*(t)),$$

$$E(t) = \text{diag}(e_1(t), \dots, e_p(t), e_1^*(t), \dots, e_q^*(t)).$$

Moreover define ψ^* by

$$\varphi^* = \psi^* \text{diag}(e_1^*(t), \dots, e_q^*(t)).$$

Then φ is the solution to (3.3)_N⁺ and (3.3)_N^{*} if and only if $\psi = \psi(t, y(t)) =$

$(\psi^+(t, y(t)), \psi^*(t, y(t)))$ satisfies the following system:

$$(3.8)_N \quad t d\psi/dt = f_{[N]}(t, y(t), \psi E(t))E(t)^{-1},$$

where $y(t)$ is the general solution to Σ' and

$$\psi^+ = \varphi^+.$$

Furthermore $(3.8)_N$ is equivalent to the system of integral equations:

$$(3.9)_N \quad \psi(t_0, y^0) = \int_{\gamma} f_{[N]}(t, y(t), \psi(t, y(t))E(t))E(t)^{-1}t^{-1} dt,$$

where $y(t)$ is the solution to Σ' with $y^0 = y(t_0)$ and γ is a path of integration joining $t = 0$ to $t = t_0$ which will be specified later.

3.2. Fundamental lemma.

We now state the lemma which plays an important role in proving the convergence of $\sum u_k(t)y^k$.

Fundamental lemma. *Suppose that the assumptions given in Theorem hold. Then, for each $N \geq N_0$, the system $(3.8)_N$ possesses one and only one solution $\psi = \psi_{[N]}(t, y) = (\psi_{[N]}^+(t, y), \psi_{[N]}^*(t, y))$ holomorphic and bounded on $D(r_N, \delta_N)$ such that the i -th entry ψ_i of ψ satisfies*

$$(3.10) \quad \psi_i(t, y) = O(|y|^{N+1}|e_i(t)^{-1}|), \quad 1 \leq i \leq n,$$

provided that $r_N, \delta_N > 0$ are sufficiently small, N_0 being specified later, and $e_{p+j}(t) = e_j^*(t)$, $1 \leq j \leq q$.

Note that, if this lemma holds, $\varphi = (\varphi^+, \varphi^*)$ is also holomorphic and bounded on $D(r_N, \delta_N)$ with $\varphi = O(|y|^{N+1})$. By this lemma, we can prove the convergence of $\sum u_k(t)y^k$. Indeed, for each $N \geq N_0$, let $\psi_{[N]}(t, y) = (\psi_{[N]}^+(t, y), \psi_{[N]}^*(t, y))$ be the unique solution to $(3.8)_N$ and set

$$\begin{aligned} \varphi^{+N}(t, y) &= u_{[N]}^+(t, y) + \varphi_{[N]}^+(t, y), \\ \varphi^{*N}(t, y) &= u_{[N]}^*(t, y) + \varphi_{[N]}^*(t, y), \end{aligned}$$

where $\varphi_{[N]}^+ = \psi_{[N]}^+$ and $\varphi_{[N]}^* = \psi_{[N]}^* \text{diag}(e_1^*, \dots, e_q^*)$. Then, since $\varphi_{[N]} = (\varphi_{[N]}^+(t, y), \varphi_{[N]}^*(t, y))$ is a solution of $(3.3)_N^+$ and $(3.3)_N^*$, a change of coordinates $(t, x) \rightarrow (t, y)$ of the form

$$x^+ = y^+ \text{diag}(\varphi^{+N}(t, y)), \quad x^* = \varphi^{*N}$$

takes the system Σ into the system Σ' . We see that $\varphi^N = (\varphi^{+N}, \varphi^{*N})$ does not depend on N . We can assume that both the constants r_N and δ_N are monotone decreasing in N . For any N' and N with $N' > N > N_0$,

$$(u_{[N^1]} - u_{[N]}) + \varphi_{[N^1]},$$

where $u_{[N]} = (u_{[N]}^+, u_{[N]}^*)$, is a solution to the systems $(3.3)_N^+$ and $(3.3)_N^*$ on $D(r_N, \delta_N)$, of order $O(|y|^{N+1})$. From the uniqueness assertion in Fundamental lemma, it follows that

$$\varphi_{[N]} = (u_{[N^1]} - u_{[N]}) + \varphi_{[N^1]},$$

which implies that φ^N is independent of N . Now put

$$\varphi = \varphi^{N_0}, \quad r' = r_{N_0}, \quad \delta' = \delta_{N_0}.$$

Then φ is holomorphic and bounded on $D(r', \delta')$ and we see that

$$\varphi(t, y) - \sum_{|k| \leq N} u_k(t) y^k = O(|y|^{N+1}),$$

for each $N \geq N_0$, which yields the convergence of $\sum u_k(t) y^k$.

Therefore we have only to prove the fundamental lemma.

§4. Determination of sectorial domain \mathcal{S} and path of integration γ

To prove the fundamental lemma, we have to define a sectorial domain \mathcal{S} in the t -plane and a path of integration γ joining $t = 0$ to $t = t_0$ in \mathcal{S} . For this purpose, we begin by determining Ω with $\pi/4 < \Omega < \pi/2$.

Take κ so that (1.7) holds and fix it. For the κ thus determined, set

$$v_j = \lambda_{p+j} + \kappa \alpha_{p+j}, \quad 1 \leq j \leq q.$$

We choose an integer $N_0 \geq 2$ so that

$$N_0 > \max_{1 \leq j \leq q} \{v_j\} / \min_{1 \leq j \leq q} \{v_j\},$$

and we fix it.

Then define the quantities Ω' , Ω'' , Ω^+ , and Ω^* by

$$(4.1)' \quad \tan \Omega' = 2 \left[\max_{1 \leq i \leq p} \{|\xi_i^+|\} + \max_{1 \leq i \leq p} \{|\alpha_i^+|\} \max\{\kappa, 1\} + 1 \right. \\ \left. + \min_{1 \leq i \leq p} \{|\eta_i|\} \right] / \min_{1 \leq i \leq p} \{|\eta_i|\},$$

$$(4.1)'' \quad \tan \Omega'' = \left[\max_{1 \leq j \leq q} \{|\xi_j^*|\} + (3\kappa + 4) \max_{1 \leq j \leq q} \{|\alpha_j^*|\} \right. \\ \left. + \min_{1 \leq j \leq q} \{v_j\} \right] / \min_{1 \leq j \leq q} \{v_j\},$$

$$(4.1)^+ \quad \tan \Omega^+ = 2 \left[\max_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} \{|\zeta_i^+|, |\zeta_j^*|\} + \max_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} \{|\alpha_i^+|, |\alpha_j^*|\} \max \{\kappa, 1\} \right. \\ \left. + 1 + \min_{1 \leq i \leq p} \{\eta_i\} \right] / \min_{1 \leq i \leq p} \{\eta_i\},$$

$$(4.1)^* \quad \tan \Omega^* = 4 \left[\max_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} \{|\zeta_i^+|, |\zeta_j^*|\} + \max_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} \{|\alpha_i^+|, |\alpha_j^*|\} \right. \\ \left. + \max_{1 \leq j \leq q} \{v_j\} \right] / (\min_{1 \leq j \leq q} \{v_j\} - \max_{1 \leq j \leq q} \{v_j\} / N_0),$$

where

$$\zeta_i^+ = \operatorname{Re} \lambda_i, \quad \eta_i = \operatorname{Im} \lambda_i, \quad \alpha_i^+ = \alpha_i, \quad 1 \leq i \leq p, \\ \zeta_j^* = \operatorname{Re} \lambda_{p+j}, \quad \alpha_j^* = \alpha_{p+j}, \quad 1 \leq j \leq q.$$

We determine Ω by

$$(4.2) \quad \Omega = \max \{\Omega', \Omega'', \Omega^+, \Omega^*\}.$$

Note that

$$(4.3) \quad \tan \Omega > 1.$$

We pass to define a sectorial domain \mathcal{S} .

By the assumption (A_1) , $S(r)$ contains a half line $\arg t = \pi/2$.

Let $\underline{\ell}$ and $\bar{\ell}$ be half lines starting from the origin defined by $\{t; \arg t = \underline{\theta}\}$ and $\{t; \arg t = \bar{\theta}\}$, respectively. Then we denote by $\mathcal{S}(\underline{\theta}, \bar{\theta}; r)$ the domain in the t -plane bounded by $\underline{\ell}$, $\bar{\ell}$ and the curve

$$(4.4) \quad |t| = \begin{cases} r & \text{if } |\theta - \pi/2| < \pi/2 - \Omega \\ r|\cos \theta / \cos \Omega| & \text{if } \pi/2 - \Omega \leq |\theta - \pi/2| \leq \pi - \varepsilon, \end{cases}$$

where $\theta = \arg t$, $\varepsilon > 0$ being a small constant.

We divide \mathcal{S} into three parts:

$$\mathcal{S}_1 = \mathcal{S} \cap \{t; |\arg t - \pi/2| \leq \pi/2 - \Omega\},$$

$$\mathcal{S}_2 = \mathcal{S} \cap \{t; -\pi/2 < \arg t < \Omega\},$$

$$\mathcal{S}_3 = \mathcal{S} \cap \{t; \pi - \Omega < \arg t < \arg t < 3\pi/2\}.$$

For a given $t_0 \in \mathcal{S}$, we define, in \mathcal{S} , a path γ joining $t = 0$ to $t = t_0$ which generally consists of two paths γ' and γ'' .

(i) When t_0 is in \mathcal{S}_1 , γ consists of only γ' . The coordinate t , on γ' , is parametrized by τ as follows:

$$(4.5) \quad t(\tau)^{-1} = \tau + a - \sqrt{-1} b e^{k\tau}, \quad \tau \in [0, +\infty),$$

where

$$(4.6) \quad t_0^{-1} = a - \sqrt{-1} b, \quad t_0 = t(0).$$

(ii) When t_0 is in \mathcal{S}_2 (or \mathcal{S}_3), γ consists of two paths γ'' and γ' . The coordinate t , on γ'' , is parametrized by θ as follows:

$$(4.7) \quad t(\theta) = (|t_0|/\cos \theta_0) \cos \theta e^{\sqrt{-1}\theta},$$

for $\theta \in [\theta_0, \Omega]$ (or $\theta \in [\pi - \Omega, \theta_0]$) respectively, where

$$(4.8) \quad \theta_0 = \arg t_0.$$

Then $t(\Omega)$ (or $t(\pi - \Omega)$) is in \mathcal{S}_1 . The coordinate t on γ' is defined by the same way as (4.5), where the starting point is $t(\Omega)$ (or $t(\pi - \Omega)$).

Then we can verify

Lemma 4.1. *We can choose a sufficiently small positive constant r in such a way that*

- (i) *When $t_0 \in \mathcal{S}_1$, then $\gamma \subset \mathcal{S}_1$;*
- (ii) *When $t_0 \in \mathcal{S}$, then $\gamma \subset \mathcal{S}$.*

§5. The stability of the domain \mathcal{D} which is a deformation of D

We define a domain $\mathcal{D} = \mathcal{D}(r, \delta) = \mathcal{D}(\underline{\theta}, \bar{\theta}; r, \delta)$ in the (t, y) -space by

$$(5.1) \quad \mathcal{D} = \{(t, y) \in \mathbf{C} \times \mathbf{C}^n; t \in \mathcal{S}(r), |y_i| < \delta d_i(t)\},$$

where

$$(5.2) \quad d_i(t) = \begin{cases} (\cos \Omega)^{\alpha_i} & \text{on } \mathcal{S}_1 \\ |\cos \theta|^{\alpha_i} & \text{on } \mathcal{S}_2 \cup \mathcal{S}_3, \end{cases}$$

with $\theta = \arg t$.

We note that the domain $\mathcal{D}(r, \delta)$ is equivalent to the domain $D(r, \delta)$, namely, for given r and $\delta > 0$, we can choose constants r' and $\delta' > 0$ so that $\mathcal{D}(r', \delta') \subset D(r, \delta)$, and conversely, for given r' and $\delta' > 0$, we can take constants r and $\delta > 0$ so that $D(r, \delta) \subset \mathcal{D}(r', \delta')$.

We now give some properties for \mathcal{D} .

Lemma 5.1. *(Stability of \mathcal{D}). Let $(t_0, y^0) \in \mathcal{D}(r, \delta)$ and let $y(t)$ be the solution to Σ' with $y(t_0) = y^0$, then $(t, y(t)) \in \mathcal{D}(r, \delta)$ for every $t \in \gamma$, provided that r and $\delta > 0$ are sufficiently small.*

In order to prove lemma 5.1, we use the following lemma.

Lemma 5.2. *Let $y(t)$ be the function given in lemma 5.1. Then, for $t \in \gamma$ with $t_0 \in \mathcal{S}_1$, the pull back $y_i(\tau)$ of y_i by (4.5) satisfy*

$$(5.3)' \quad d \log |y_i^+(\tau)|^{N+1}/d\tau < -1,$$

$$(5.3)'' \quad d \log |y_j^*(\tau)|^{N+1}/d\tau < -3\nu_j/4,$$

and

$$(5.3)^+ \quad d \log |y_i^+(\tau)|^{N+1}|e_j^*(\tau)^{-1}|/d\tau < -1,$$

$$(5.3)^* \quad d \log |y_h^*(\tau)|^{N+1}|e_j^*(\tau)^{-1}|/d\tau < -\nu_h,$$

for all i, j, h with $1 \leq i \leq p$, $1 \leq j, h \leq q$, and for every $N \geq N_0$ where $y_i^+ = y_i$, $1 \leq i \leq p$, $y_j^* = y_{p+j}$, $1 \leq j \leq q$.

It should be remarked that, by (5.3)' and (5.3)'', $|y_i(\tau)|$, $1 \leq i \leq n$, are monotone decreasing functions with respect to τ .

Proof of lemma 5.1.

Let us suppose that $(t_0, y^0) = (t_0, y(t_0)) \in \mathcal{D}$.

(I) In case that $t_0 \in \mathcal{S}_1$; Clearly we have

$$|y_i(\tau)| < |y_i(0)| < \delta d_i(t), \quad 1 \leq i \leq n,$$

since $|y_i(\tau)|$ is monotone decreasing in τ .

(II) In case that $t_0 = \mathcal{S}_2$ (or \mathcal{S}_3); Noting that we have

$$\operatorname{Re} [\lambda_{it}(\theta)^{-1}] = \begin{cases} \xi_i^+ \cos \theta_0 |t_0|^{-1} + \eta_i \cos \theta_0 \tan \theta |t_0|^{-1}, & 1 \leq i \leq p, \\ \xi_{i-p}^* \cos \theta_0 |t_0|^{-1}, & p+1 \leq i \leq n, \end{cases}$$

on γ'' and

$$C_i = y_i(\theta_0) e^{\lambda_{it}(\theta_0)^{-1}} t(\theta_0)^{-\alpha_i}, \quad 1 \leq i \leq n,$$

where C_i , $1 \leq i \leq n$, are arbitrary constants in (1.10), we have the inequalities:

$$\begin{aligned} |y_i(t)| &= |y_i(\theta_0) e^{\lambda_{it}(\theta_0)^{-1}} t(\theta_0)^{-\alpha_i} e^{-\lambda_{it}(\theta)^{-1}} t(\theta)^{\alpha_i}| \\ &= \begin{cases} |y_i(\theta_0)| (|\cos \theta|/|\cos \theta_0|)^{\alpha_i} e^{\eta_i \cos \theta_0 |t_0| (\tan \theta_0 - \tan \theta)}, & 1 \leq i \leq p, \\ |y_i(\theta_0)| (|\cos \theta|/|\cos \theta_0|)^{\alpha_i}, & p+1 \leq i \leq n, \end{cases} \\ &< \delta d_i(t), \quad 1 \leq i \leq n. \end{aligned}$$

Thus we have proved Lemma 5.1. Q.E.D.

In order to prove Lemma 5.2, we make use of the following lemma.

Lemma 5.3. *We have*

$$(5.4) \quad |a/b| \leq 1/\tan \Omega < 1,$$

$$(5.5) \quad b \geq (\tan \Omega \sin \varepsilon)/r > \tan \Omega,$$

$$(5.6) \quad \kappa b \geq \tan \Omega$$

for any $t_0 \in \mathcal{S}_1$.

The proof of this Lemma is omitted (see Proposition 2 in [15]).

We now proceed to prove Lemma 5.2.

Proof of lemma 5.2. We prove only (5.3)⁺, because the others are shown in a similar way.

For any i and j with $1 \leq i \leq p$, $1 \leq j \leq q$, let

$$w^+(t) = y_i^+(t)^{N+1} e_j^*(t)^{-1}.$$

Then

$$d \log w^+(t)/dt = \{[(N+1)\lambda_i^+ - \lambda_j^*] + [(N+1)\alpha_i^+ - \alpha_j^*]t\}t^{-2}.$$

On the other hand, we have

$$-t^{-2} dt/d\tau = (1 - \sqrt{-1} \kappa b e^{\kappa\tau})$$

from (4.5). Consequently, we have

$$\begin{aligned} d \log |w^+(\tau)|/d\tau &= d \operatorname{Re} [\log w^+(\tau)]/d\tau \\ &= \operatorname{Re} [d \log w^+(t)/dt \cdot dt/d\tau] \\ &= -\operatorname{Re} \{ [((N+1)\lambda_i^+ - \lambda_j^*) + ((N+1)\alpha_i^+ - \alpha_j^*)(\tau + a - \sqrt{-1} b e^{\kappa\tau})^{-1}] \\ &\quad \times (1 - \sqrt{-1} b \kappa e^{\kappa\tau}) \}. \end{aligned}$$

So, it is sufficient to prove the following inequalities;

$$\begin{aligned} \operatorname{Re} \{ [((N+1)\lambda_i^+ - \lambda_j^*) + ((N+1)\alpha_i^+ - \alpha_j^*)(\tau + a - \sqrt{-1} b e^{\kappa\tau})^{-1}] \\ \times (1 - \sqrt{-1} b \kappa e^{\kappa\tau}) \} > 1 \end{aligned}$$

or equivalently,

$$(5.7) \quad \begin{aligned} (N+1)\eta_i \kappa b^3 e^{3\kappa\tau} + [(N+1)\xi_i^+ - \xi_j^* + ((N+1)\alpha_i^+ - \alpha_j^*)\kappa - 1] b^2 e^{2\kappa\tau} \\ + (N+1)\eta_i \kappa (\tau + a)^2 b e^{\kappa\tau} + ((N+1)\xi_i^+ - \xi_j^* - 1)(\tau + a)^2 \\ + ((N+1)\alpha_i^+ - \alpha_j^*)(\tau + a) > 0. \end{aligned}$$

Denote by $\mathfrak{A}(\tau)$ the function in τ

$$\begin{aligned} (N+1)\eta_i \kappa b^3 e^{3\kappa\tau} + [(N+1)\xi_i^+ - \xi_j^* + ((N+1)\alpha_i^+ - \alpha_j^*)\kappa - 1] b^2 e^{2\kappa\tau} \\ + ((N+1)\xi_i^+ - \xi_j^* - 1)(\tau + a)^2 + ((N+1)\alpha_i^+ - \alpha_j^*)(\tau + a). \end{aligned}$$

Since the left-hand side on (5.7) is equal to $\mathfrak{A}(\tau) + (N+1)\eta_i\kappa(\tau+a)^2be^{\kappa\tau}$, we have only to show $\mathfrak{A}(\tau) > 0$ for $\tau \geq 0$. Let

$$\mathfrak{A}(\tau) = \sum_{m \geq 0} \mathfrak{A}_m \tau^m$$

be the power series expansions in τ of $\mathfrak{A}(\tau)$. Then,

(5.8)

$$\begin{aligned} \mathfrak{A}_0 = & (N+1)\eta_i\kappa b^3 + [(N+1)\xi_i^+ - \xi_j^* + ((N+1)\alpha_i^+ - \alpha_j^*)\kappa - 1]b^2 \\ & + ((N+1)\xi_i^+ - \xi_j^* - 1)a^2 + ((N+1)\alpha_i^+ - \alpha_j^*)a, \end{aligned}$$

(5.9)

$$\begin{aligned} \mathfrak{A}_1 = & 3(N+1)\eta_i\kappa^2 b^3 + 2[(N+1)\xi_i^+ - \xi_j^* + ((N+1)\alpha_i^+ - \alpha_j^*)\kappa - 1]\kappa b^2 \\ & + 2((N+1)\xi_i^+ - \xi_j^* - 1)a + ((N+1)\alpha_i^+ - \alpha_j^*), \end{aligned}$$

(5.10)

$$\begin{aligned} \mathfrak{A}_2 = & \{9(N+1)\eta_i\kappa^3 b^3 + 4[(N+1)\xi_i^+ - \xi_j^* + ((N+1)\alpha_i^+ - \alpha_j^*)\kappa - 1]\kappa^2 b^2 \\ & + 2((N+1)\xi_i^+ - \xi_j^* - 1)\}/2!, \end{aligned}$$

and, for $m \geq 3$,

$$(5.11) \quad \begin{aligned} \mathfrak{A}_m = & \{3^m(N+1)\eta_i\kappa^{m+1} b^3 \\ & + 2^m[(N+1)\xi_i^+ - \xi_j^* + ((N+1)\alpha_i^+ - \alpha_j^*)\kappa - 1]\kappa^m b^2\}/m!. \end{aligned}$$

We can see that

$$(5.12) \quad \mathfrak{A}_m > 0, \quad m \geq 0.$$

In fact, we have (5.12) for $m = 0$:

$$\begin{aligned} \mathfrak{A}_0 & \geq \kappa b^3 \{(N+1)\eta_i - |(N+1)\xi_i^+ - \xi_j^* + ((N+1)\alpha_i^+ - \alpha_j^*)\kappa - 1|/(\kappa b) \\ & \quad - |(N+1)\xi_i^+ - \xi_j^* - 1|a^2/(\kappa b^3) - |(N+1)\alpha_i^+ - \alpha_j^*||a/(\kappa b^3)|\} \\ & \geq \kappa b^3 (\tan \Omega)^{-1} \{(N+1)\eta_i \tan \Omega - |(N+1)\xi_i^+ - \xi_j^* + ((N+1)\alpha_i^+ - \alpha_j^*)\kappa - 1| \\ & \quad - |(N+1)\xi_i^+ - \xi_j^* - 1| - |(N+1)\alpha_i^+ - \alpha_j^*|\} \\ & \geq \kappa b^3 (N+1) (\tan \Omega)^{-1} \{\eta_i \tan \Omega - (|\xi_i^+| + |\xi_j^*| + (|\alpha_i^+| + |\alpha_j^*|)\kappa + 1) \\ & \quad - (|\xi_i^+| + |\xi_j^*| + 1) - (|\alpha_i^+| + |\alpha_j^*|)\} \\ & > 0, \end{aligned}$$

by using (5.4), (5.5) and (5.6).

In a similar way we show (5.12) for $m = 1, 2$ and $m \geq 3$ as

$$\mathfrak{A}_1 = 3\kappa^2 b^3 (\tan \Omega)^{-1} \{(N+1)\eta_i \tan \Omega$$

$$\begin{aligned}
 & - |(N + 1)\xi_i^+ - \xi_j^* + ((N + 1)\alpha_i^+ - \alpha_j^*)\kappa - 1| \\
 & - |(N + 1)\xi_i^+ - \xi_j^* - 1| - |(N + 1)\alpha_i^+ - \alpha_j^*| \} \\
 \geq & (N + 1)3\kappa^2 b^3 (\tan \Omega)^{-1} \{ \eta_i \tan \Omega - (|\xi_i^+| + |\xi_j^*| + (|\alpha_i^+| + |\alpha_j^*|)\kappa + 1) \\
 & - (|\xi_i^+| + |\xi_j^*| + 1) - (|\alpha_i^+| + |\alpha_j^*|) \} \\
 > & 0, \\
 \mathfrak{A}_2 \geq & 9\kappa^3 b^3 (\tan \Omega)^{-1} \{ (N + 1)\eta_i \tan \Omega \\
 & - |(N + 1)\xi_i^+ - \xi_j^* + ((N + 1)\alpha_i^+ - \alpha_j^*)\kappa - 1| \\
 & - |(N + 1)\xi_i^+ - \xi_j^* - 1| \} / 2! \\
 \geq & 9(N + 1)\kappa^3 b^3 (\tan \Omega)^{-1} \{ \eta_i \tan \Omega - (|\xi_i^+| + |\xi_j^*| + (|\alpha_i^+| + |\alpha_j^*|)\kappa + 1) \\
 & - (|\xi_i^+| + |\xi_j^*| + 1) \} / 2! \\
 > & 0, \\
 \mathfrak{A}_m \geq & 3^m ((N + 1)\kappa^{m+1} b^3 (\tan \Omega)^{-1} \{ \eta_i \tan \Omega \\
 & - (|\xi_i^+| + |\xi_j^*| + (|\alpha_i^+| + |\alpha_j^*|)\kappa + 1) \} / m!. \\
 > & 0.
 \end{aligned}$$

Thus we have obtained (5.12), which yields (5.3)⁺. Q.E.D.

§6. Proof of Fundamental lemma

6.1. In this part, we show the following lemma.

Lemma 6.1. *Let $y(t)$ be the general solution to the reduced system Σ' . Then we can find a positive constant J_N depending on N so that we have the inequality;*

$$\begin{aligned}
 (6.1) \quad & \int_{\gamma} |y(t)|^{N+1} |e_j(t)^{-1}| |t|^{-1} |dt| \leq J_N |y(t_0)|^{N+1} |e_j(t_0)^{-1}|, \\
 & 1 \leq j \leq n,
 \end{aligned}$$

for any $t_0 \in \mathcal{S}(r)$, $r > 0$ being sufficiently small, and $N \geq N_0$.

Proof. Let $t_0 \in \mathcal{S}_1$ and let

$$w_i(t) = y_i(t)^{N+1} e_j(t)^{-1},$$

for any i with $1 \leq i \leq n$. Then we see that $w_i(t)$ satisfy

$$(6.2) \quad d \log |w_i(t)| > v / (\sqrt{2}\kappa)^{-1} |t|^{-1} |dt|,$$

where

$$v = \min_{1 \leq j \leq q} \{1, 3v_j/4\}.$$

Indeed, by the inequalities in Lemma 5.2, we have

$$\begin{aligned} d \log |w_i(t)|/|dt| &= d \log |w_i(\tau)|/d\tau \cdot d\tau/dt \\ &= d \log |w_i(\tau)|/d\tau (-|d\tau/dt|) \\ &> v|d\tau/dt|, \end{aligned}$$

and we have

$$(6.3) \quad |t||d\tau/dt| > (\sqrt{2} \kappa)^{-1},$$

because

$$\begin{aligned} &\text{the left hand side of (6.3)} \\ &= [(\tau + a)^2 + b^2 e^{2\kappa\tau}]^{1/2} (1 + \kappa^2 b^2 e^{2\kappa\tau})^{-1/2} \\ &> [b^2 e^{2\kappa\tau}]^{1/2} (2\kappa^2 b^2 e^{2\kappa\tau})^{-1/2} = (\sqrt{2} \kappa)^{-1}. \end{aligned}$$

Now, by (6.2), we have

$$\begin{aligned} &\int_{\gamma'} |w_i(t)| |t|^{-1} |dt| \\ &< \sqrt{2} \kappa v^{-1} \int_{\gamma'} |w_i(t)| d \log |w_i(t)| \\ &= \sqrt{2} \kappa v^{-1} \int_{\gamma'} d |w_i(t)| \\ &< \sqrt{2} \kappa v^{-1} |w_i(t_0)|, \end{aligned}$$

and hence we obtain (6.1).

Next let $t_0 \in \mathcal{S}_2$ or \mathcal{S}_3 . Then we have, on γ'' ,

$$t(\theta) = |t_0|/\cos \theta_0 \cos \theta e^{\sqrt{-1}\theta}.$$

Noting $|dt|$ is a decreasing or increasing function in $\theta = \arg t$ corresponding to $t_0 \in \mathcal{S}_2$ or \mathcal{S}_3 respectively, we have

$$\begin{aligned} |dt| &= \mp |dt/d\theta| d\theta \\ &= \mp |t_0/\cos \theta_0| d\theta \end{aligned}$$

and

$$|t|^{-1}|dt| = \frac{1}{\cos \Theta} d\Theta.$$

We have, hence,

$$\begin{aligned} \int_{\gamma''} |t|^{-1}|dt| &< \frac{1}{\cos \Theta} \int_{\gamma''} |\cos \Theta|^{-1} d\Theta \\ &= \begin{cases} \int_{\Omega}^{\theta_0} |\cos \Theta|^{-1} d\Theta < \pi / \min \{ \sin \varepsilon, \cos \Omega \} \\ \int_{\pi - \Omega}^{\theta_0} |\cos \Theta|^{-1} d\Theta < \pi / \min \{ \sin \varepsilon, \cos \Omega \} \end{cases} \end{aligned}$$

for $\Theta \in [\theta_0, \Omega]$ or $[\pi - \Omega, \theta_0]$ respectively. We have, then,

$$\int_{\gamma''} |t|^{-1}|dt| < L$$

for some positive constant L . Noting that, on γ'' ,

$$|y_i(t)| < |y_i(t_0)| |d_i(t)/d_i(t_0)| < L_1 |y_i(t_0)|,$$

for a suitable positive constant L_1 , we have

$$\begin{aligned} \int_{\gamma''} |w_i(t)| |t|^{-1}|dt| &\leq \int_{\gamma''} L_1^{N+1} |w_i(t_0)| |t|^{-1}|dt| \\ &\leq L_1^{N+1} L |w_i(t_0)| \end{aligned}$$

Therefore if we choose J_N such that

$$J_N > \sqrt{2} \kappa \nu^{-1} + L_1^{N+1} L,$$

then we have (6.1).

Q.E.D.

6.2. In this part, we complete the proof of fundamental lemma.

First of all, we define the family of functions \mathcal{F} by

$$\mathcal{F} = \{ \psi : \mathcal{D} \longrightarrow \mathbf{C}^n; \text{ the } i\text{-th entry } \psi_i \text{ of } \psi \text{ is holomorphic and bounded on } \mathcal{D} \text{ and } |\psi_i| \leq K_N |y|^{N+1} |e_i^{-1}|, \quad 1 \leq i \leq n \}$$

where $\mathcal{D} = \mathcal{D}(r_N, \delta_N)$ and K_N is a positive constant depending on N which will be specified later. It is evident that \mathcal{F} is non-empty and convex, and further closed and normal with respect to the topology of uniform convergence on every compact subset of \mathcal{D} . Next we define an operator \mathcal{T} acting on \mathcal{F} .

Let (t_0, y^0) be a point in \mathcal{D} and let $y(t)$ be the solution to Σ' satisfying the initial condition $y(t_0) = y^0$. We set

$$(6.4) \quad \Psi(t_0, y^0) = \int_{\gamma} f_{[N]}(t, y(t), \psi(t, y(t))E(t))E(t)^{-1}t^{-1}dt$$

and define an operator \mathcal{T} from \mathcal{F} into \mathcal{F} by

$$(6.5) \quad \psi(t, y) \longrightarrow \Psi(t, y).$$

This is well-defined. Indeed, first, we can take r_N, δ_N and ρ_N so small that $(t, y(t), \psi(t, y(t))E(t))$ is in the domain of definition of $f_{[N]}$ for every $t \in \mathcal{S}(r_N)$, where $y(t)$ is the solution to Σ' with $y(t_0) = y^0, t_0 \in \mathcal{S}(r_N), |y^0| < \delta_N$ and $\psi \in \mathcal{F}$.

Next let $\psi \in \mathcal{F}$, then, by (3.7) and (6.1), we have

$$\begin{aligned} |\Psi_i(t_0, y^0)| &\leq \int_{\gamma} |f_{i[N]}(t, y(t), \psi(t)E(t))| |e_i(t)^{-1}| |t|^{-1} |dt| \\ &\leq \int_{\gamma} [M(|y| + |\psi E|)|\psi E| + C_N|y|^{N+1}] |e_i^{-1}| |t|^{-1} |dt| \\ &\leq \int_{\gamma} [M(|y| + |\psi E|)K_N|y|^{N+1} + C_N|y|^{N+1}] |e_i^{-1}| |t|^{-1} |dt| \\ &\leq \int_{\gamma} [M(\delta_N + 2\rho_N)K_N + C_N]|y|^{N+1} |e_i^{-1}| |t|^{-1} |dt| \\ &\leq [M(\delta_N + 2\rho_N)K_N + C_N] J_N |y(t_0)|^{N+1} |e_i(t_0)^{-1}|, \end{aligned}$$

and hence we obtain

$$|\Psi_i(t_0, y^0)| \leq K_N |y(t_0)|^{N+1} |e_i(t_0)^{-1}|, \quad 1 \leq i \leq n,$$

provided that δ_N and ρ_N is chosen so small that

$$M(\delta_N + 2\rho_N)J_N < 1/2$$

and that we set $K_N = 2C_N J_N$. Furthermore we can verify that Ψ is holomorphic and bounded on \mathcal{D} . so, $\Psi(t_0, y^0) \in \mathcal{F}$, and hence $\mathcal{T}(\mathcal{F}) \subset \mathcal{F}$. It is evident, moreover, that \mathcal{T} is continuous on \mathcal{F} by (3.6). Thus we see that \mathcal{T} admits a fixed point by virtue of the Schauder-Tihonov fixed point theorem, which shows the existence of ψ_N in Fundamental lemma.

We now prove the uniqueness of ψ_N .

Let ψ^1 and ψ^2 be two solutions satisfying the lemma and put

$$\psi' = \psi^1 - \psi^2.$$

Then ψ' is holomorphic and bounded on \mathcal{D} and $|\psi'_i| = O(|y|^{N+1}|e_i^{-1}|), 1 \leq i \leq n$.

We define a positive constant H by

$$H = \inf \{H' \geq 0; |\psi'_i| \leq H'|y|^{N+1}|e_i^{-1}|, (t, y) \in \mathcal{D}, 1 \leq i \leq n\}.$$

Then we have

$$\begin{aligned} |\psi'_i(t_0, y^0)| &\leq \int_{\gamma} M(|y| + |(\psi^1 + \psi^2)E|)|\psi' E||e_i^{-1}||t|^{-1}|dt| \\ &\leq \int_{\gamma} M(\delta_N + 2\rho_N)H|y|^{N+1}|e_i^{-1}||t|^{-1}|dt| \\ &\leq M(\delta_N + 2\rho_N)HJ_N|y(t_0)|^{N+1}|e_i(t_0)^{-1}| \\ &< 2^{-1}H|y(t_0)|^{N+1}|e_i(t_0)^{-1}|, \quad 1 \leq i \leq n, \end{aligned}$$

by (3.6), and hence, by the definition of H , we have $H = 0$. Thus we have shown the lemma. Q.E.D.

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References

- [1] Hukuhara, M., Sur les points singuliers des équations différentielles linéaires, III. Mem. Fac. Sci., Kyūshū Univ., **2** (1942), 125–137.
- [2] Iwano, M., On a general solution of a nonlinear 2-system of the form $x^2 dw/dx = Aw + xh(x, w)$ with a constant diagonal matrix A of signature $(1, 1)$, Tôhoku Math. J., **32** (1980), 453–486.
- [3] Iwano, M., On an n -parameter family of solutions of a nonlinear n -system with an irregular type singularity, Ann. Mat. Pura Appl., **160** (1985), 57–132.
- [4] Iwano, M., On a 4-parameter family of a nonlinear 4-system with an irregular type singularity not satisfying Poincaré’s condition, Funkcial. Ekvac., **34** (1991), 39–84.
- [5] Iwano, M., Analytical simplification of a nonlinear autonomous 2-system not satisfying Poincaré’s condition, preprint, (1991).
- [6] Koitabashi, T., On a 2-parameter family of solutions to a nonlinear 2-system with an irregular singularity of signature $(1, 1)$, submitted to Kumamoto J. Math., (1991).
- [7] Malmquist, J., Sur l’étude analytique des solutions d’un système d’équations différentielles dans le voisinage d’un point singulier d’indétermination, I, Acta. Math., **73** (1940), 87–129.
- [8] Malmquist, J., Sur l’étude analytique des solutions d’un système d’équations différentielles dans le voisinage d’un point singulier d’indétermination, II, Ibid., **74** (1941), 1–64.
- [9] Malmquist, J., Sur l’étude analytique des solutions d’un système d’équations différentielles dans le voisinage d’un point singulier d’indétermination, III, Ibid., **74** (1941), 109–128.
- [10] Shimomura, S., Analytic integration of some nonlinear ordinary differential equations and the fifth Painlevé equation in the neighbourhood of an irregular singular point, Funkcial. Ekvac., **26** (1983), 301–338.
- [11] Takano, K., A 2-parameter family of solutions of Painlevé equation (V) near the point at

- infinity, Funkcial. Ekvac., **26** (1983), 79–113.
- [12] Takano, K., Reduction for Painlevé equations at the fixed singular points of the first kind, *Ibid.*, **29** (1986), 99–119.
- [13] Takano, K., Reduction for Painlevé equations at the fixed singular points of the second kind, *J. Math. Soc. Japan*, **42** (1990), 423–443.
- [14] Yoshida, S., A general solution of a nonlinear 2-system without Poincaré's condition at an irregular singular point, *Funkcial. Ekvac.*, **27** (1984), 367–391.
- [15] Yoshida, S., 2-parameter family of solutions for Painlevé equations (I) ~ (V) at an irregular singular point, *Ibid.*, **28** (1985), 233–248.

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