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An Abstract Doubly Nonlinear Volterra Integral Equation

By

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1. Introduction

This paper is concerned with the existence of solutions to the Volterra integral equation

$$u(t)+\int_0^t a(t-s)A(s)(Bu(s))\,ds\ni f(t),\qquad 0\le t\le T, \tag{1.1}$$

in a real Hilbert space H. Here $a\colon[0,T]\to\boldsymbol{R}$ is a given kernel, $A(t)$ ($t\in[0,T]$) and B denote nonlinear maximal monotone (possibly multivalued) operators in H, f maps $[0,T]$ into H, and the integral is taken in the sense of Bochner.

In the special case $B=I$ (identity), the existence, uniqueness and asymptotic behavior (as $T\to\infty$) of solutions of (1.1) was discussed by the author [1–3]. Little is known, however, on the solvability of general nonlinear equations of the form (1.1). Let us remark that by letting $v(t)\in Bu(t)$, equation (1.1) formally reduces to

$$B_1v(t)+\int_0^t a(t-s)A(s)v(s)\,ds\ni f(t),\qquad 0\le t\le T, \tag{1.2}$$

where $B_1=B^{-1}$. Equation (1.2) has been studied by Barbu [6, 7] under the assumption that $A(\cdot)$ is independent of t, and recently by Hokkanen [17], who allows both A and B_1 to vary with t. As regards [17], we point out that the domain of $A(t)$ is required to be constant (for all t), and that a technical condition (that also appears in [6, 7]) relating $A(t)$ and B_1 is used. (In particular, if $A(t)$ and B_1 are linear, that condition is close to the monotonicity of the composite mapping $A(t)\circ B_1^{-1}$, $\forall t$.) Unlike [17], in the current paper we allow the domain of $A(t)$ to depend on t. No condition relating $A(t)$ and B is imposed. In addition, our approach is different from the one in [17] (or [6, 7]), in the sense that a distinct approximation procedure is employed.

It is also worth mentioning that in the case when $a = 1$, (1.1) is formally equivalent to a Cauchy problem for a doubly nonlinear evolution equation of the type

$$u'(t) + A(t)(Bu(t)) \ni f_1(t), \qquad 0 < t < T, \tag{1.3}$$

where $f_1 : [0, T] \to H$. This equation has recently been considered by Kenmochi [19] and Kenmochi and Pawlow [20]. It is easy to see that (1.3) can be rewritten as

$$\frac{d}{dt} B_1 v(t) + A(t)v(t) \ni f_1(t), \qquad 0 < t < T, \tag{1.4}$$

where $B_1 = B^{-1}$, $v(t) \in Bu(t)$. Equation (1.4) arises as a model for a variety of nonlinear diffusion problems (including free-boundary problems), and has attracted a lot of attention. (See, e.g., [4, 5, 8, 9, 11, 14, 16, 18, 21, 22].)

From an abstract viewpoint, the present study is a direct attempt to generalize to (1.1) the existence theory of Kenmochi and Pawlow [20]. Our approach combines some of the techniques for evolution equations with methods specific to the theory of abstract Volterra equations developed in recent years (cf. e.g., [12, 13].)

The plan of the paper is as follows. In section 2 we recall for easy reference some basic facts about maximal monotone operators and time-dependent subdifferentials. The main existence result is stated and proved in Sections 3 and 4, respectively. A uniqueness theorem is also included. Finally, Section 5 discusses a model problem of physical interest, to which our abstract theory can be applied.

2. Preliminaries

For further background and details of this section we refer the reader to [10, 19, 20]. Throughout this paper H denotes a real Hilbert space with scalar product $(\,,)$ and norm $|\cdot|$. $\mathscr{H}$ will stand for the space $L^2(0, T; H)$ endowed with its usual inner product $\langle\,,\rangle$ and norm $|\cdot|$. We will use the symbols "$\rightarrow$" and "$\rightharpoonup$" to indicate the strong, and respectively, weak convergence in H or $\mathscr{H}$. Let A be a set-valued operator in H with domain $D(A)$ and range $R(A)$. We say that A is monotone if $(y_2 - y_1, x_2 - x_1) \geq 0$, for all $y_i \in Ax_i$, $i = 1, 2$. A is called maximal monotone if it is monotone, and $R(I + \lambda A) = H$, $\forall \lambda > 0$, where I stands for the identity on H. For a maximal monotone A, one can define the resolvent $J_\lambda = J_\lambda^A = (I + \lambda A)^{-1}$, and the Yosida approximation $A_\lambda = \lambda^{-1}(I - J_\lambda)$, $\forall \lambda > 0$. Several properties of A, J_λ, and A_λ are presented next.

Proposition 2.1. *Let A be maximal monotone in H. Then*:

(i) *A is demiclosed, i.e., the graph of A is closed in $H \times H_w$, where H_w denotes H equipped with the weak topology.*

(ii) *J_λ is nonexpansive on H, while A_λ is maximal monotone and Lipschitz continous on H with λ^{-1} as Lipschitz constant. Moreover $A_\lambda x \in AJ_\lambda x$, $\forall x \in H$.*

(iii) *If $x_\lambda \to x$ and $A_\lambda x_\lambda \rightharpoonup y$ as $\lambda \to 0^+$, then $x \in D(A)$ and $y \in Ax$.*

(iv) *The operator $\mathscr{A}$ defined in $\mathscr{H}$ by $(\mathscr{A}v)(t) = Av(t)$, a.e. on $(0, T)$, $\forall v \in \mathscr{H}$ is maximal monotone, and $(\mathscr{A}_\lambda v)(t) = A_\lambda v(t)$, a.e. $t \in (0, T)$, $\forall \lambda > 0$, $\forall v \in \mathscr{H}$. ($\mathscr{A}$ is said to be the $\mathscr{H}$-extension of A.)*

An important subclass of maximal monotone operators is that of subdifferentials. Let $\psi: H \to]-\infty, +\infty]$ be a proper convex, lower semicontinous (l.s.c.) function, with effective domain $D(\psi) = \{x \in H: \psi(x) < +\infty\}$. Its subdifferential $\partial\psi$ is defined by:

$$\partial\psi(x) = \{y \in H: \psi(z) - \psi(x) \geq (y, z - x) \ \ \forall z \in D(\psi)\}.$$

Proposition 2.2. *Let $A = \partial\psi$, where $\psi: H \to]-\infty, +\infty]$ is proper, convex and l.s.c. Then A is maximal monotone in H, with $A_\lambda = \partial\psi_\lambda (\lambda > 0)$, where*

$$\psi_\lambda(x) = \inf_{y \in H} \{\psi(y) + |y - x|^2/(2\lambda)\}, \qquad x \in H.$$

In addition, the following properties hold:

(i)
$$\psi_\lambda(z) = \psi(J_\lambda z) + |z - J_\lambda z|^2/(2\lambda), \qquad \forall z \in H, \lambda > 0,$$

and

$$\psi_\lambda(z) \uparrow \psi(z) \quad as \quad \lambda \downarrow 0.$$

(ii) *If $\lambda_n \downarrow 0$, and $z_n \to z$ in H (as $n \to \infty$), then*

$$\psi(z) \leq \liminf \psi_{\lambda_n}(z_n).$$

If also $\{\partial\psi_{\lambda_n}(z_n)\}$ is bounded in H, then

$$\psi_{\lambda_n}(z_n) \longrightarrow \psi(z).$$

(iii) *If $v \in W^{1,2}(0, T: H)$ and $g \in L^2(0, T; H)$ satisfy $v(t) \in D(A)$, $g(t) \in Av(t)$, a.e. on $(0, T)$, then $t \to \psi(v(t))$ is absolutely continous on $[0, T]$ and $(d/dt)\psi(v(t)) = (v'(t), g(t))$, for a.a. $t \in (0, T)$.*

We now consider a family $\{\varphi^t; 0 \leq t \leq T\}$ of proper, convex, l.s.c. functions from H into $[0, +\infty]$ satisfying:

(φ_1) There exist constants $c_1 > 0$, $\bar{c}_1 \geq 0$ such that

$$\varphi^t(z) \geq c_1 |z|^2 - \bar{c}_1, \qquad \forall z \in H, \ \forall t \in [0, T].$$

(φ_2) There exist functions $\alpha \in W^{1,2}(0, T)$ and $\beta \in W^{1,1}(0, T)$ such that for all $s, t \in [0, T]$, $s \leq t$, and $z \in D(\varphi^s)$, there is $\bar{z} \in D(\varphi^t)$ satisfying:

$$|\bar{z} - z| \leq |\alpha(t) - \alpha(s)| (1 + (\varphi^s(z))^{1/2})$$

$$\varphi^t(\bar{z}) - \varphi^s(z) \leq |\beta(t) - \beta(s)| (1 + \varphi^s(z)).$$

The following is a direct consequence of [19, Lemmas 1.2.1, 1.2.2, 1.5.2, 1.5.3] and [20, Lemmas 2.2, 2.3].

Proposition 2.3. *Let conditions (φ_1), (φ_2) be satisfied. Then*

(i) *The function $t \to \partial\varphi_\lambda^t(v(t))$ is measurable for each $\lambda \in (0, 1]$ and $v \in L^1(0, T; H)$.*

(ii) *There is a constant $c_1' \geq 0$ such that:*

$$|J_\lambda^t z| \leq |z| + c_1', \qquad |\partial\varphi_\lambda^t(z)| \leq \lambda^{-1}(2|z| + c_1')$$

for all $z \in H$, $t \in [0, T]$ and $\lambda \in (0, 1]$. (Here $J_\lambda^t = (I + \lambda\partial\varphi^t)^{-1}$).

(iii) $\varphi_\lambda^t(z) \geq (c_1/2)|z|^2 - \bar{c}_1$, *for all $z \in H$, and all $0 < \lambda \leq 1/(2c_1)$.*

(iv) *If $z_n \in D(\varphi^{t_n})$, $z_n \rightharpoonup z$ in H and $t_n \uparrow t$, as $n \to \infty$, then*

$$\varphi^t(z) \leq \liminf_{n \to \infty} \varphi^{t_n}(z_n).$$

(v) *The functional $\Phi\colon \mathscr{H} \to [0, +\infty]$, defined by*

$$\Phi(v) = \int_0^T \varphi^t(v(t))\, dt, \qquad \forall v \in \mathscr{H}$$

is proper, convex, and l.s.c. Its regularization $\Phi_\lambda (\lambda > 0)$ is given by:

$$\Phi_\lambda(v) = \int_0^T \varphi_\lambda^t(v(t))\, dt, \qquad \forall v \in \mathscr{H}, \ \lambda > 0.$$

In addition $(\partial\Phi_\lambda)(v)(t) = \partial\varphi_\lambda^t(v(t))$, a.e. on $(0, T)$, $\forall v \in \mathscr{H}$, and $v^* \in \partial\Phi(v)$ $(v, v^* \in \mathscr{H})$ iff $v^*(t) \in \partial\varphi^t(v(t))$, for a.a. $t \in (0, T)$.

(vi) *If $v \in W^{1,1}(0, T; H)$ and $\lambda > 0$, then $t \to \varphi_\lambda^t(v(t))$ is of bounded variation on $[0, T]$ and:*

$$\varphi_\lambda^t(v(t)) - \varphi_\lambda^s(v(s)) \leq \int_s^t \frac{d}{dr} \varphi_\lambda^r(v(r))\, dr, \qquad 0 \leq s \leq t \leq T. \tag{2.1}$$

In addition one has

$$\begin{aligned}(2.2)\qquad &\frac{d}{dt}\varphi_\lambda^t(v(t)) - (v'(t)),\ \partial\varphi_\lambda^t(v(t)))\\ &\leq |\alpha'(t)|\,|\partial\varphi_\lambda^t(v(t))|\,(1 + (\varphi_\lambda^t(v(t)))^{1/2})\\ &\quad + |\beta'(t)|\,(1 + \varphi_\lambda^t(v(t))),\ \textit{a.e. on } (0, T).\end{aligned}$$

3. Main result

We are primarily concerned with the existence of solutions to equation (1.1) under the key assumption that $A(t) = \partial\varphi^t$, $0 \leq t \leq T$, where $\varphi^t\colon H \to [0, +\infty]$ is proper, convex, and l.s.c. Besides (φ_1) and (φ_2) (see Section 2), the following condition will be used:

(φ_3) For each $t \in [0, T]$ and $r \geq 0$, the level set $\{z \in H\colon \varphi^t(z) \leq r\}$ is compact in H.

As regards the operator B in (1.1) we assume that:

(b_1) $B = \partial j$, where $j\colon H \to]-\infty, +\infty]$ is proper, convex and l.s.c.

(b_2) There exists $c_2 > 0$ such that

$$(y_2 - y_1, x_2 - x_1) \geq c_2|x_2 - x_1|^2, \qquad \forall y_i \in Bx_i,\ i = 1, 2.$$

We now consider the Volterra integral equation (cf. (1.1))

$$(3.1)\qquad u(t) + \int_0^t a(t-s)\partial\varphi^s(Bu(s))\,ds \ni f(t), \qquad 0 \leq t \leq T,$$

where φ^t and B satisfy conditions (φ_1)-(φ_3), and (b_1), (b_2), respectively. The following restrictions on a and f will be imposed throughout:

$$\text{(a)}\qquad a \in W^{2,1}(0, T),\quad a(0) = 1,$$

$$\text{(f)}\qquad f \in W^{2,1}(0, T; H),\quad f(0) \in D(B),\quad Bf(0) \cap D(\varphi^0) \neq \emptyset.$$

Remark 3.1. The assumption $a(0) = 1$ in (a) is made only for simplicity. What we actually need is $a(0) > 0$. (Obviously, when $a(0) > 0$ one may replace a by $\tilde{a} = a(0)^{-1}a$ and φ^t by $a(0)\varphi^t$ to obtain $\tilde{a}(0) = 1$.)

Definition 3.2. A function $u\colon [0, T] \to H$ is called a solution to (3.1) if $u \in W^{1,2}(0, T; H)$, $u(t) \in D(B)$, a.e. on $(0, T)$, and there exist $v \in L^\infty(0, T; H)$, $w \in L^2(0, T; H)$ such that $v(t) \in Bu(t) \cap D(\partial\varphi^t)$, and $w(t) \in \partial\varphi^t(v(t))$, for a.a. $t \in (0, T)$, and $u(t) + a * w(t) = f(t)$ for all $t \in [0, T]$, where $*$ denotes the convolution $\left(a * w(t) = \int_0^t a(t-s)w(s)\,ds\right)$.

The main existence result is:

Theorem 3.3. *Let conditions* (φ_1)-(φ_3), (b_1), (b_2), (a) *and* (f) *hold. Then equation* (3.1) *has at least one solution.*

Remark 3.4. (i) Theorem 3.3 extends (with no essential change in the proof) to equations of the form

$$u(t) + a*\partial\varphi^t(Bu(t)) + b*u(t) \ni f(t), \qquad 0 \leq t \leq T,$$

where $b \in W^{1,1}(0, T)$.

(ii) Since T in Theorem 3.3 is arbitrary, it is easy to formulate an existence result for equation (3.1) on $[0, +\infty]$. In such a case, it would be of interest to examine the asymptotic properties of solutions. We are going to address this question in a forthcoming paper.

(iii) Theorem 3.3 can be regarded as a generalization of [20, Theorem 1.1].

We conclude this section with some remarks on the uniqueness of solutions of (3.1). As is well-known, nonuniqueness may occur even in the case when $a = 1$ (and (3.1) formally reduces to an equation of the type (1.4).) See [14, Section 5]. However, the uniqueness of solutions to (3.1) can easily be derived at the expense of additional restrictions on φ^t and B. A sample result is the following.

Theorem 3.5. *Let the conditions of Theorem* 3.3 *be satisfied. In addition assume that there exists* $\eta \in L^1(0, T; \boldsymbol{R}_+)$ *such that*

$$(z_2^t - z_1^t, x_2^t - x_1^t) \geq -\eta(t)|x_1^t - x_2^t|^2, \qquad t \in [0, T], \tag{3.2}$$

for all $x_i^t \in D(B)$ *with* $Bx_i^t \cap D(\partial\varphi^t) \neq \emptyset$, *and* $z_i^t \in \partial\varphi^t(y_i^t)$, $y_i^t \in Bx_i^t (i = 1, 2)$. *Then the solution* u *of equation* (3.1) *is uniquely determined.*

Remark 3.6. Comparable uniqueness results for equation (1.2) may be found in [17].

4. Proofs

We first summarize some properties of B that are going to play an important role in the sequel.

Lemma 4.1 [20]. *Let* (b_1), (b_2) *hold. Then*

(i) B^{-1} *is everywhere defined and Lipschitz continous on* H, *and satisfies*

$$(B^{-1}z_1 - B^{-1}z_2, z_1 - z_2) \geq c_2|B^{-1}z_1 - B^{-1}z_2|^2, \qquad \forall z_1, z_2 \in H.$$

(ii) $(B_\mu z_1 - B_\mu z_2, z_1 - z_2) \geq c_2(1 + \mu c_2)^{-1}|z_1 - z_2|^2,$

for all $z_1, z_2 \in H$ *and* $\mu > 0$.

(iii) $(B_\mu)^{-1}$ *is everywhere defined on* H, *and*

$$(B_\mu)^{-1}z = B^{-1}z + \mu z, \qquad \forall z \in H,\ \mu > 0.$$

One of the key tools in the proof of our existence result is described next. Let (a) be satisfied, and define the resolvent kernel k of a' by:

$$k(t) + a' * k(t) = -a'(t), \qquad t \in [0, T]. \tag{4.1}$$

It is easily seen that $k \in W^{1,1}(0, T)$. Let $u \in W^{1,2}(0, T; H)$, $w \in L^2(0, T; H)$ and $f \in W^{1,2}(0, T; H)$ satisfy

$$u(t) + a * w(t) = f(t), \qquad 0 \le t \le T. \tag{4.2}$$

Applying the method of [12, Proposition 1] we are led to

Lemma 4.2. *Let* (*a*) *hold, k be defined by* (4.1), *and* $f \in W^{1,2}(0, T; H)$. *Then* $u \in W^{1,2}(0, T; H)$ *and* $w \in L^2(0, T; H)$ *satisfy* (4.2) *if and only if*

$$\text{(4.3)}\quad \begin{aligned} &\text{(i)}\ \ u'(t) + w(t) + k * u'(t) = F(t), \qquad \textit{a.e. on } (0, T),\\ &\text{(ii)}\ \ u(0) = u_0, \end{aligned}$$

where

$$F(t) = f'(t) + k * f'(t), \qquad u_0 = f(0). \tag{4.4}$$

Proof of Theorem 3.3. By assumption (f) and Lemma 4.1 (iii) we may define

$$u_{0,\mu} = (B_\mu)^{-1} u_0^*, \tag{4.5}$$

where $u_0 = f(0)$, and $u_0^* \in Bu_0 \cap D(\varphi^0)$. For each $\lambda, \mu > 0$, we consider the approximating equation

$$u_{\mu,\lambda}(t) + \int_0^t a(t-s)\,\partial\varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))\,ds = f(t) + u_{0,\mu} - u_0, \qquad 0 \le t \le T. \tag{4.6}$$

Inasmuch as $\partial\varphi_\lambda^t \circ B_\mu$ is Lipschitz continous on H for each fixed t with Lipschitz constant independent of t (cf. Proposition 2.1 (ii)), and the mapping $t \to \partial\varphi_\lambda^t(B_\mu z)$ is measurable on $[0, T]$ for each fixed $z \in H$ (cf. Proposition 2.3 (i)), a usual contraction mapping argument shows that (4.6) has a unique solution $u_{\mu,\lambda} \in W^{1,2}(0, T; H)$.

Our aim is to obtain a number of a priori estimates involving $u_{\mu,\lambda}$, which will enable us to pass to the limit as $\lambda \downarrow 0$ in (4.6) (with μ fixed), and subsequently let $\mu \downarrow 0$. Instead of using (4.6) we apply Lemma 4.2 and consider the equivalent initial-value problem:

$$(4.7)\qquad \begin{aligned}&\text{(i)}\quad u'_{\mu,\lambda}(t) + \partial\varphi^t_\lambda(B_\mu u_{\mu,\lambda}(t)) + k * u'_{\mu,\lambda}(t) = F(t), \qquad \text{a.e. on } (0, T),\\ &\text{(ii)}\quad u_{\mu,\lambda}(0) = u_{0,\mu}\end{aligned}$$

where F is given by (4.4). Note that by (*a*), (*f*), (4.1), (4.4), one has $F \in W^{1,1}(0, T; H)$.

In what follows we will use $c_3, c_4, \cdots$, etc. to denote various positive constants that are independent of μ and λ. We will also use inequalities of the following sort frequently and without comment:

$$(x + y)^{1/2} \leq x^{1/2} + y^{1/2},\ (x + y)^2 \leq 2(x^2 + y^2),\ xy \leq (1/(2\varepsilon))x^2 + (\varepsilon/2)y^2,$$

for $x, y, \varepsilon \in (0, \infty)$. All estimates below are for $0 \leq t \leq T$, $0 < \lambda \leq \min\{1, 1/(2c_1)\}$, $0 < \mu \leq c_2^{-1}$.

First form the inner-product of (4.7) (i) with $(d/dt)B_\mu u_{\mu,\lambda}$ and integrate over $(0, t)$. Noting (see Lemma 4.1 (ii)) that

$$\left(\frac{d}{dt}B_\mu u_{\mu,\lambda}(t),\ u'_{\mu,\lambda}(t)\right) \geq c_2(1 + \mu c_2)^{-1}|u'_{\mu,\lambda}(t)|^2,$$

and using (2.1), (2.2), (4.5), (4.7) (ii), and the definition of φ^0_λ (see Proposition 2.2), we obtain

$$(4.8)\qquad \begin{aligned}&(c_2/2)\int_0^t |u'_{\mu,\lambda}(s)|^2\,ds + \varphi^t_\lambda(B_\mu u_{\mu,\lambda}(t))\\ &\quad + \int_0^t \left(k * u'_{\mu,\lambda}(s),\ \frac{d}{ds}B_\mu u_{\mu,\lambda}(s)\right)ds\\ &\leq \varphi^0(u_0^*) + \int_0^t \left(F(s),\ \frac{d}{ds}B_\mu u_{\mu,\lambda}(s)\right)ds\\ &\quad + \int_0^t |\alpha'(s)|\,|\partial\varphi^s_\lambda(B_\mu u_{\mu,\lambda}(s))|\,(1 + [\varphi^s_\lambda(B_\mu u_{\mu,\lambda}(s))]^{1/2})\,ds\\ &\quad + \int_0^t |\beta'(s)|\,(1 + \varphi^s_\lambda(B_\mu u_{\mu,\lambda}(s)))\,ds.\end{aligned}$$

Remark that

$$(4.9)\qquad \begin{aligned}\int_0^t \left(k * u'_{\mu,\lambda}(s),\ \frac{d}{ds}B_\mu u_{\mu,\lambda}(s)\right)ds &= (k * u'_{\mu,\lambda}(t),\ B_\mu u_{\mu,\lambda}(t))\\ &\quad - \int_0^t (B_\mu u_{\mu,\lambda}(s),\ k(0)u'_{\mu,\lambda}(s) + k' * u'_{\mu,\lambda}(s))\,ds,\end{aligned}$$

while (cf. (4.4), (4.5))

$$\begin{aligned}\int_0^t \left(F(s), \frac{d}{ds} B_\mu u_{\mu,\lambda}(s)\right) ds &= (F(t), B_\mu u_{\mu,\lambda}(t)) \\ &\quad - (f'(0), u_0^*) - \int_0^t (F'(s), B_\mu u_{\mu,\lambda}(s))\, ds.\end{aligned} \tag{4.10}$$

Setting

$$X_{\mu,\lambda}(t) = \varphi_\lambda^t(B_\mu u_{\mu,\lambda}(t)) - (B_\mu u_{\mu,\lambda}(t), F(t)) \tag{4.11}$$

and using (4.9), (4.10) in (4.8) yields

$$\begin{aligned}(c_2/2) \int_0^t |u'_{\mu,\lambda}(s)|^2\, ds + X_{\mu,\lambda}(t) &\le c_3 + |k * u'_{\mu,\lambda}(t)|\,|B_\mu u_{\mu,\lambda}(t)| \\ &+ \int_0^t (B_\mu u_{\mu,\lambda}(s), k(0) u'_{\mu,\lambda}(s) + k' * u'_{\mu,\lambda}(s))\, ds - \int_0^t (F'(s), B_\mu u_{\mu,\lambda}(s))\, ds \\ &+ \int_0^t |\alpha'(s)|\,|\partial\varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))|(1 + [\varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))]^{1/2})\, ds \\ &+ \int_0^t |\beta'(s)|(1 + \varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s)))\, ds.\end{aligned} \tag{4.12}$$

Next observe that (4.11) and (iii) of Proposition 2.3 imply

$$(c_1/2)|B_\mu u_{\mu,\lambda}(t)|^2 - \bar{c}_1 \le \varphi_\lambda^t(B_\mu u_{\mu,\lambda}(t)) \le c_4 X_{\mu,\lambda}(t) + c_5 |F(t)|^2 + c_6. \tag{4.13}$$

We now seek to bound the right-hand side of (4.12), in terms of $X_{\mu,\lambda}(t)$ and $\int_0^t |u'_{\mu,\lambda}(s)|\, ds$. We first have

$$\begin{aligned}|k * u'_{\mu,\lambda}(t)|\,|B_\mu u_{\mu,\lambda}(t)| &\le \max_{[0,T]} |k| \left(\int_0^t |u'_{\mu,\lambda}(s)|\, ds\right) |B_\mu u_{\mu,\lambda}(t)| \\ &\le c_7 \left[\int_0^t |u'_{\mu,\lambda}(s)|\, ds\right]^2 + c_8 |B_\mu u_{\mu,\lambda}(t)|^2,\end{aligned} \tag{4.14}$$

where c_8 is to be sufficiently small, in a sense to be precised later. Invoking (4.13) in (4.14) implies

$$|k * u'_{\mu,\lambda}(t)|\,|B_\mu u_{\mu,\lambda}(t)| \le c_9 \left[\int_0^t |u'_{\mu,\lambda}(s)|\, ds\right]^2 + c_{10} X_{\mu,\lambda}(t) + c_{11}, \tag{4.15}$$

with a sufficiently small c_{10}.

In the following, we will continue to make use of (4.13) repeatedly, and without comment. We have

$$\begin{aligned} k(0)\int_0^t (B_\mu u_{\mu,\lambda}(s),\ u'_{\mu,\lambda}(s))\,ds &\le |k(0)|\int_0^t |B_\mu u_{\mu,\lambda}(s)|\,|u'_{\mu,\lambda}(s)|\,ds \\ &\le c_{12}\int_0^t |u'_{\mu,\lambda}(s)|^2\,ds + c_{13}\int_0^t X_{\mu,\lambda}(s)\,ds + c_{14}, \end{aligned} \tag{4.16}$$

where c_{12} is small enough. Similarly,

$$\begin{aligned} \int_0^t (B_\mu u_{\mu,\lambda}(s),\ k' * u'_{\mu,\lambda}(s))\,ds &\le \|k'\|_{L^1(0,T)}\,\|B_\mu u_{\mu,\lambda}\|_{L^2(0,t;H)}\,\|u'_{\mu,\lambda}\|_{L^2(0,t;H)} \\ &\le c_{15}\int_0^t |u'_{\mu,\lambda}(s)|^2\,ds + c_{16}\int_0^t X_{\mu,\lambda}(s)\,ds + c_{17}, \end{aligned} \tag{4.17}$$

with a sufficiently small c_{15}.

We now consider $\int_0^t (F'(s),\ B_\mu u_{\mu,\lambda}(s))\,ds$. By (*a*), (*f*) and (4.4), it follows that $F' \in L^1(0, T; H)$. Also note that (4.13) implies $|B_\mu u_{\mu,\lambda}(t)| \le c_{18} + (1/2)X_{\mu,\lambda}(t)$. We consequently deduce that:

$$\begin{aligned} -\int_0^t (F'(s),\ B_\mu u_{\mu,\lambda}(s))\,ds &\le \int_0^t (c_{18} + (1/2)X_{\mu,\lambda}(s))\,|F'(s)|\,ds \\ &\le c_{19} + \int_0^t \theta_1(s)X_{\mu,\lambda}(s)\,ds, \end{aligned} \tag{4.18}$$

where $\theta_1(t) = (1/2)|F'(t)|$, $\theta_1 \in L^1(0, T)$.

We next see that

$$\int_0^t |\beta'(s)|(1 + \varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))\,ds \le c_{20} + \int_0^t \theta_2(s)X_{\mu,\lambda}(s)\,ds, \tag{4.19}$$

where $\theta_2(t) = const\,|\beta'(t)|$, $\theta_2 \in L^1(0, T)$ (cf. (φ_2)).

It remains to estimate

$$\int_0^t |\alpha'(s)|\,|\partial\varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))|(1 + [\varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))]^{1/2})\,ds.$$

By (4.7) (i) one has

$$|\partial\varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))| \le |F(s)| + |u'_{\mu,\lambda}(s)| + |k * u'_{\mu,\lambda}(s)|, \qquad 0 \le s \le t.$$

We successively obtain

$$\int_0^t |\alpha'(s)|\,|F(s)|(1 + [\varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))]^{1/2})\,ds \tag{4.20}$$

$$\le c_{21} + \max_{[0,T]} |F| \, \|\alpha'\|_{L^2(0,T)} \left[\int_0^t \varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))\, ds \right]^{1/2}$$

$$\le c_{22} + c_{23} \int_0^t X_{\mu,\lambda}(s)\, ds,$$

and

$$\tag{4.21} \begin{aligned} &\int_0^t |\alpha'(s)|\,|u'_{\mu,\lambda}(s)|(1 + [\varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))]^{1/2})\, ds \\ &\quad \le c_{24} \int_0^t |u'_{\mu,\lambda}(s)|^2 \, ds + \int_0^t \theta_3(s) X_{\mu,\lambda}(s)\, ds + c_{25}, \end{aligned}$$

where c_{24} is small enough and $\theta_3(t) = const\, |\alpha'(t)|^2$, $\theta_3 \in L^1(0, T)$ (cf. (φ_2)). We finally have

$$\tag{4.22} \begin{aligned} &\int_0^t |\alpha'(s)|\,|k * u'_{\mu,\lambda}(s)|(1 + [\varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))]^{1/2})\, ds \\ &\quad \le c_{26} \int_0^t |\alpha'(s)|^2 \, ds + c_{27} \int_0^t |k * u'_{\mu,\lambda}(s)|^2 \, ds \\ &\qquad + c_{28} \int_0^t |\alpha'(s)|^2 \varphi_\lambda^s(B_\mu u_{\mu,\lambda}(s))\, ds \\ &\quad \le c_{29} \int_0^t |u'_{\mu,\lambda}(s)|^2 \, ds + \int_0^t \theta_4(s) X_{\mu,\lambda}(s)\, ds + c_{30}, \end{aligned}$$

where c_{29} is sufficiently small and $\theta_4(t) = const\, |\alpha'(t)|^2$, $\theta_4 \in L^1(0, T)$.

By combining (4.12), and (4.15)–(4.22), we get

$$\tag{4.23} \begin{aligned} (c_2/2) \int_0^t |u'_{\mu,\lambda}(s)|^2 \, ds + X_{\mu,\lambda}(t) &\le c_{31} + c_9 \left[\int_0^t |u'_{\mu,\lambda}(s)|\, ds \right]^2 \\ &+ c_{10} X_{\mu,\lambda}(t) + c_{32} \int_0^t |u'_{\mu,\lambda}(s)|^2 \, ds + \int_0^t \theta(s) X_{\mu,\lambda}(s)\, ds, \end{aligned}$$

where

$$c_{32} = c_{12} + c_{15} + c_{24} + c_{29},$$

$$\theta(t) = c_{13} + c_{16} + c_{23} + \sum_{i=1}^{4} \theta_i(t); \; \theta \in L^1(0, T; \boldsymbol{R}_+).$$

It is now clear that $c_{12}, c_{15}, c_{24}, c_{29}$ and c_{10} can be chosen so small that

$$\tag{4.24} c_{32} < c_2/2, \qquad c_{10} < 1.$$

Employing (4.24) in (4.23) yields

$$\begin{aligned} c_{33}\int_0^t |u'_{\mu,\lambda}(s)|^2\,ds + c_{34}X_{\mu,\lambda}(t) \\ \le c_{31} + c_9\left[\int_0^t |u'_{\mu,\lambda}(s)|\,ds\right]^2 + \int_0^t \theta(s)\,X_{\mu,\lambda}(s)\,ds. \end{aligned} \tag{4.25}$$

For fixed t, and $0 \le s \le t$, (4.25) implies

$$c_{34}X_{\mu,\lambda}(s) \le c_{31} + c_9\left[\int_0^t |u'_{\mu,\lambda}(\tau)|\,d\tau\right]^2 + \int_0^s \theta(\tau)\,X_{\mu,\lambda}(\tau)\,d\tau, \tag{4.26}$$

Applying Gronwall's inequality to (4.26) we infer that

$$X_{\mu,\lambda}(t) \le c_{35}\left[1 + \left[\int_0^t |u'_{\mu,\lambda}(\tau)|\,d\tau\right]^2\right], \qquad 0 \le t \le T. \tag{4.27}$$

From (4.25) and (4.27) it follows that

$$\int_0^t |u'_{\mu,\lambda}(s)|^2\,ds \le c_{36}\left[1 + \left[\int_0^t |u'_{\mu,\lambda}(s)|\,ds\right]^2\right]. \tag{4.28}$$

This inequality is completely similar to (2.29) in [13, p. 714]. Arguing as in [13], we see that (4.28) leads to

$$\|u'_{\mu,\lambda}\| \le c_{38}, \tag{4.29}$$

(where $\|\cdot\|$ denotes the $\mathscr{H}$-norm; $\mathscr{H} = L^2(0, T; H)$).

Together, (4.27) and (4.29) imply

$$X_{\mu,\lambda}(t) \le c_{38}. \tag{4.30}$$

Finally, from (4.13) and (4.30) we get

$$|B_\mu u_{\mu,\lambda}(t)| + \varphi^t_\lambda(B_\mu u_{\mu,\lambda}(t)) \le c_{39}. \tag{4.31}$$

On account of (4.29)–(4.31) we can now pass to the limit in (4.7) as $\lambda \downarrow 0$ (with μ fixed) and then let $\mu \downarrow 0$. Since the procedure follows that of [20, pp. 1195–96], we only sketch it.

Using the compactness assumption (φ_3), Proposition 2.1 (ii), Proposition 2.2 (i), (ii), and Lemma 4.1 (ii), in conjunction with (4.29)–(4.31), we conclude that there exist a sequence $\lambda_n \downarrow 0$ (as $n \to \infty$), and a function $u_\mu \in W^{1,2}(0, T; H)$ such that

$$\begin{aligned} &\text{(i)} && u_{\mu,\lambda_n} \longrightarrow u_\mu, && \text{in } C([0, T]; H), \\ &\text{(ii)} && u'_{\mu,\lambda_n} \rightharpoonup u'_\mu, && \text{in } \mathscr{H}, \end{aligned} \tag{4.32}$$

$$\text{(iii)} \qquad B_\mu u_{\mu,\lambda_n} \longrightarrow B_\mu u_\mu, \qquad \text{in } C([0, T]; H).$$

In addition, one has

$$(4.33) \qquad |u_\mu(t)| + \|u'_\mu\| + \varphi^t(B_\mu u_\mu(t)) \le c_{40}, \qquad 0 \le t \le T,$$

where c_{40} is independent of μ.

Invoking (4.32) (ii) in (4.7) (i) implies

$$(4.34) \qquad \partial\varphi^t_{\lambda_n}(B_\mu u_{\mu,\lambda_n}(t)) \rightharpoonup F(t) - u'_\mu(t) - k * u'_\mu(t), \qquad \text{in } \mathscr{H}.$$

In view of Proposition (2.1) (iii) and Proposition (2.3) (v), it follows from (4.32) (iii) and (4.34) that

$$(4.35) \qquad F(t) - u'_\mu(t) - k * u'_\mu(t) \in \partial\varphi^t(B_\mu u_\mu(t)), \qquad \text{a.e. on } (0, T).$$

Writing (4.7) with $\lambda = \lambda_n$ and letting $\lambda_n \downarrow 0$ now yields, by (4.32) (ii), (4.35)

$$(4.36) \qquad \begin{aligned} &\text{(i)} \quad u'_\mu(t) + \partial\varphi^t(B_\mu u_\mu(t)) + k * u'_\mu(t) \ni F(t), \qquad \text{a.e. on } (0, T),\\ &\text{(ii)} \quad u_\mu(0) = u_{0,\mu}, \end{aligned}$$

where u_μ satisfies (4.33).

It remains to let $\mu \downarrow 0$ in (4.36). By virtue of (4.33), (φ_3), Proposition 2.1 and Lemma 4.1, we can again find a sequence $\mu_n \downarrow 0$ (as $n \to \infty$) such that

$$(4.37) \qquad \begin{aligned} &\text{(i)} \quad u_{\mu_n} \longrightarrow u, && \text{in } C([0, T]; H),\\ &\text{(ii)} \quad u'_{\mu_n} \rightharpoonup u', && \text{in } \mathscr{H},\\ &\text{(iii)} \quad B_{\mu_n} u_{\mu_n} \longrightarrow v, && \text{weakly in } \mathscr{H} \text{ and weakly-star in } L^\infty(0, T; H), \end{aligned}$$

where $u \in W^{1,2}(0, T; H)$, $v \in L^\infty(0, T; H)$ and $v(t) \in Bu(t)$, a.e. on $(0, T)$. In particular, (4.36) (ii), (4.37) (i), and Lemma 4.1 (iii) yield

$$(4.38) \qquad u(0) = u_0.$$

Also, (4.37) (ii) leads to

$$(4.39) \qquad F(t) - k * u'_{\mu_n}(t) - u'_{\mu_n}(t) \rightharpoonup w, \qquad \text{in } \mathscr{H},$$

where $w(t) = F(t) - u'(t) - k * u'(t)$, a.e. $t \in (0, T)$. Calling on (4.36) (i), (4.37) (iii) and (4.39), we see that we only need to show that (cf. Proposition 2.3 (v))

$$(4.40) \qquad w \in \partial\Phi(v).$$

Let $q \in \mathscr{H}$ with $\varphi^t(q(t)) \in L^1(0, T)$, i.e., $q \in D(\Phi)$ (cf. Proposition (2.3) (v)). By (4.35), (4.36) (i), (4.37) (iii) and (4.39) it is easily verified that

$$(4.41) \qquad \Phi(q) - \Phi(v) \ge \lim_{n \to \infty} \langle F - u'_{\mu_n} - k * u'_{\mu_n}, q - B_{\mu_n} u_{\mu_n} \rangle,$$

where $\langle\,,\rangle$ denotes the inner product in $\mathscr{H}$. Employing (b$_1$), Proposition (2.2) (ii), (iii), and (4.37) we deduce that

$$\lim_{n\to\infty} \langle u'_{\mu_n}, B_{\mu_n}u_{\mu_n}\rangle = j(u(T)) - j(u(0)) = \langle u', v\rangle,$$

and consequently

$$\lim_{n\to\infty} \langle F - u'_{\mu_n}, q - B_{\mu_n}u_{\mu_n}\rangle = \langle F - u', q - v\rangle. \tag{4.42}$$

Let us now check that

$$\lim_{n\to\infty} \langle k*u'_{\mu_n}, q - B_{\mu_n}u_{\mu_n}\rangle = \langle k*u', q - v\rangle. \tag{4.43}$$

Since it is obvious that $\langle k*u'_{\mu_n}, q\rangle \to \langle k*u', q\rangle$, as $n\to\infty$, we only have to prove (cf. (4.43))

$$\lim_{n\to\infty} \langle k*u'_{\mu_n}, B_{\mu_n}u_{\mu_n}\rangle = \langle k*u', v\rangle. \tag{4.44}$$

Remarking that ($k\in W^{1,1}(0, T)$)

$$k*u'_{\mu_n}(t) = k(t)u_{0,\mu_n} - k(0)u_{\mu_n}(t) + k'*u_{\mu_n}(t),$$

we obtain

(4.45)
$$\langle k*u'_{\mu_n}, B_{\mu_n}u_{\mu_n}\rangle = \langle k(\cdot)u_{0,\mu_n}, B_{\mu_n}u_{\mu_n}\rangle - k(0)\langle u_{\mu_n}, B_{\mu_n}u_{\mu_n}\rangle + \langle k'*u_{\mu_n}, B_{\mu_n}u_{\mu_n}\rangle.$$

Clearly $k(t)u_{0,\mu_n} \to k(t)u(0)$, strongly in $\mathscr{H}$. This in conjunction with (4.37) (iii) and (4.38) implies

$$\lim_{n\to\infty} \langle k(\cdot)u_{0,\mu_n}, B_{\mu_n}u_{\mu_n}\rangle = \langle k(\cdot)u(0), v\rangle. \tag{4.46}$$

By (4.37) (i), (iii) we also have

$$k(0)\langle u_{\mu_n}, B_{\mu_n}u_{\mu_n}\rangle \longrightarrow k(0)\langle u, v\rangle \qquad (n \longrightarrow \infty). \tag{4.47}$$

Inasmuch as $k'*u_{\mu_n} \to k'*u$, strongly in $\mathscr{H}$, we finally infer that

$$\langle k'*u_{\mu_n}, B_{\mu_n}u_{\mu_n}\rangle \longrightarrow \langle k'*u, v\rangle \qquad (n \longrightarrow \infty). \tag{4.48}$$

On account of (4.45)–(4.48) we now arrive at (4.44). It follows that (4.43) also holds. Using (4.42), (4.43) in (4.41) yields

$$\Phi(q) - \Phi(v) \geq \langle w, q - v\rangle,$$

which, according to the definition of a subdifferential, is equivalent to (4.40).

The passage to the limit in (4.7) (equivalently, (4.6)) is now completed. We conclude that the functions $u \in W^{1,2}(0, T; H)$, $v \in L^\infty(0, T; H)$ with $v(t) \in Bu(t)$, a.e., and $w \in L^2(0, T; H)$, with $w(t) \in \partial\varphi^t(v(t))$, a.e., satisfy (4.3). By Lemma 4.2 this means that u is a solution of (3.1) in the sense of Definition 3.2. The proof is closed.

Proof of Theorem 3.5. Let u_1 and u_2 be two solutions of (3.1) (cf. Definition 3.2), and $p(t) = u_1(t) - u_2(t)$, $0 \le t \le T$. Invoking Lemma 4.2, it is easily seen that

$$\text{(4.49)} \quad \begin{aligned} &\text{(i)} \quad p'(t) + w_1(t) - w_2(t) + k * p'(t) = 0, \qquad \text{a.e. } t \in (0, T),\\ &\text{(ii)} \quad p(0) = 0, \end{aligned}$$

where $w_i(t) \in \partial\varphi^t(v_i(t))$, $v_i(t) \in Bu_i(t)$, a.e. on $(0, T)$, $i = 1, 2$. Multiplying (4.49) (i) by $p(t)$ and making use of condition (3.2) leads to:

$$(1/2)\frac{d}{dt}|p(t)|^2 \le \eta(t)|p(t)|^2 - (k(0)p(t) + k' * p(t), p(t)).$$

This implies

$$\text{(4.50)} \qquad (1/2)\frac{d}{dt}|p(t)|^2 \le (\eta(t) + |k(0)|)|p(t)|^2 + (|k'| * |p|(t))|p(t)|.$$

Integrate (4.50) over $(0, t)$ and use (4.49) (ii) to find

$$|p(t)|^2 \le 2\int_0^t (\eta(s) + |k(0)|)|p(s)|^2\,ds + \|k'\|_{L^1(0,T)}\int_0^t |p(s)|^2\,ds,$$

or equivalently

$$\text{(4.51)} \qquad |p(t)|^2 \le \int_0^t \eta_1(s)|p(s)|^2\,ds, \qquad 0 \le t \le T,$$

where $\eta_1(t) = 2(\eta(t) + |k(0)|) + \|k'\|_{L^1(0,T)}$, $\eta_1 \in L^1(0, T; \boldsymbol{R}_+)$. An application of Gronwall's inequality to (4.51) yields $p \equiv 0$, i.e., $u_1 \equiv u_2$, and the proof is complete.

5. An example

In this section we suggest a special heat flow model to which our previous theory applies. This can be regarded as a generalization of the discussion in [2, Section 5] and [15, Section 4]. See also [7, Section 4].

Consider a homogenous bar of unit length of a material with memory, and let $u(t, x)$, $e(t, x)$, $q(t, x)$ and $r(t, x)$ denote respectively the temperature, internal energy, heat flux, and external heat supply at time t and position $x (0 \leq t \leq T,\ 0 \leq x \leq 1)$. For simplicity let the history of u be prescribed as zero for $t < 0$, $0 \leq x \leq 1$. We assume that e and q are given by:

$$
\begin{aligned}
e(t, x) &= b_0 g(u(t, x)) + \int_0^t \delta(t-s) g(u(s, x))\, ds, \\
q(t, x) &= -a_0 u_x(t, x) + \int_0^t \gamma(t-s) u_x(s, x)\, ds,
\end{aligned}
\tag{5.1}
$$

where $0 \leq t \leq T$, $0 \leq x \leq 1$, $a_0 > 0$, $b_0 > 0$, while $\delta, \gamma\colon \boldsymbol{R}_+ \to \boldsymbol{R}$ and $g\colon \boldsymbol{R} \to \boldsymbol{R}$ are sufficiently smooth functions. The energy balance equation takes the form $e_t = -q_x + r$. If also $u(0, x) = u_0(x)$ $(0 \leq x \leq 1)$ is the initial temperature distribution, then (5.1) leads to:

$$
\begin{aligned}
&\text{(i)} \quad \frac{d}{dt}[b_0 g(u) + \delta * g(u)] = a_0 u_{xx} - \gamma * u_{xx} + r, \\
&\text{(ii)} \quad u(0, x) = u_0(x),
\end{aligned}
\tag{5.2}
$$

where $0 \leq t \leq T$, $0 \leq x \leq 1$.

To (5.2) we add the following boundary conditions at $x = 0$ and $x = 1$, expressing the presence of thermostatic controls at the ends of the rod:

$$
\begin{aligned}
& m_0(t) \leq u(t, 0) \leq n_0(t) && (0 \leq t \leq T) \\
& u_x(t, 0^+) \leq g_0(m_0(t)) && \text{if } u(t, 0) = m_0(t) \\
& u_x(t, 0^+) = g_0(u(t, 0)) && \text{if } m_0(t) < u(t, 0) < n_0(t) \\
& u_x(t, 0^+) \geq g_0(n_0(t)) && \text{if } u(t, 0) = n_0(t)
\end{aligned}
\tag{5.3}
$$

and

$$
\begin{aligned}
& m_1(t) \leq u(t, 1) \leq n_1(t) && (0 \leq t \leq T) \\
& -u_x(t, 1^-) \leq g_1(m_1(t)) && \text{if } u(t, 1) = m_1(t) \\
& -u_x(t, 1^-) = g_1(u(t, 1)) && \text{if } m_1(t) < u(t, 1) < n_1(t) \\
& -u_x(t, 1^-) \geq g_1(n_1(t)) && \text{if } u(t, 1) = n_1(t)
\end{aligned}
\tag{5.4}
$$

Here m_i, n_i $(i = 0, 1)$ are real functions on $[0, T]$ such that

$$
\begin{aligned}
& m_i, n_i \in W^{1,2}(0, T);\ m_i < n_i \quad \text{on } [0, T], \\
& 0 \in [m_i(0), n_i(0)],
\end{aligned}
\tag{5.5}
$$

while $g_i\colon \boldsymbol{R} \to \boldsymbol{R}$ $(i = 0, 1)$ satisfies

(5.6) $\qquad g_i$ is continous and nondecreasing on $\boldsymbol{R}$, with $g_i(0) = 0$.

Next let $\sigma\colon [0, T] \to \boldsymbol{R}$ be the solution of $b_0(\sigma(t) + (\delta * \sigma)(t) = -\delta(t)/b_0$, and define

$$a(t) = (K(t)/b_0) + \sigma * K(t), \; K(t) = a_0 - \int_0^t \gamma(s)\,ds, \tag{5.7}$$

$$\begin{aligned} f(t, \cdot\,) &= (\Gamma(t, \cdot\,)/b_0) + \sigma * \Gamma(t, \cdot\,), \\ \Gamma(t, x) &= b_0 g(u_0(x)) + \int_0^t r(s, x)\,ds. \end{aligned} \tag{5.8}$$

Then (5.2)–(5.4) is formally equivalent to

$$g(u) - a * u_{xx} = f \qquad (0 \le t \le T,\ 0 \le x \le 1), \tag{5.9}$$

where u satisfies (5.3), (5.4) and a, f are given by (5.7), (5.8) respectively. This can further be rewritten as a Volterra equation of the form (3.1) in the Hilbert space $H = L^2(0, 1)$. As in [2], let $G_i(t) = \int_0^t g_i(s)\,ds$ $(0 \le t \le T)$, $i = 0, 1$, and introduce the proper convex, l.s.c. functional $\varphi^t\colon H \to [0, +\infty]$ by

$$\varphi^t(z) = \begin{cases} (1/2)|z_x|_H^2 + G_0(z(0)) + G_1(z(1)), & \text{if } z \in \Delta(t), \\ +\infty & \text{otherwise} \end{cases} \tag{5.10}$$

where

$$\Delta(t) = \{z \in W^{1,2}(0, 1)\colon m_0(t) \le z(0) \le n_0(t),\ m_1(t) \le z(1) \le n_1(t)\}.$$

In view of (5.5), (5.6) it is easily seen that the functions φ^t given by (5.10) for all $t \in [0, T]$ satisfy (φ_1) and (φ_3). Moreover, by [15, Lemma 2], condition (φ_2) also holds. Finally, invoking Lemma 5 in [15], we conclude that $\partial\varphi^t(z) = -z_{xx}$, in the distributional sense on (0, 1), where $D(\partial\varphi^t)$ consists of all $z \in W^{2,2}(0, 1)$ satisfying (5.3), (5.4) with $z(0)$, $z_x(0^+)$, and $z(1)$, $z_x(1^-)$ in place of $u(t, 0)$, $u_x(t, 0^+)$, and $u(t, 1)$, $u_x(t, 1^-)$, respectively. The above remarks show that (5.9) together with the boundary conditions (5.3), (5.4) reduces to the following integral equation in H:

$$\tilde{g}(u(t)) + \int_0^t a(t - s)\partial\varphi^s(u(s))\,ds \ni \tilde{f}(t), \qquad t \in [0, T], \tag{5.11}$$

where $\tilde{g}(z)(x) = g(z(x))$, a.e. $x \in (0.1)$, $\forall z \in H$, and $\tilde{f}(t)(x) = f(t, x)(t \in [0, T]$, $x \in (0, 1))$, provided that appropiate restrictions are imposed on g and f. Equation (5.11) is essentially of the form (1.2) and can be rewritten as (3.1) with $B = \tilde{g}^{-1}$.

Our aim is to apply now Theorem 3.3 to obtain an exitence result for (5.11). We only need to make some technical assumptions on δ, γ, r, u_0 and g to ensure that (a), (f), (b_1) and (b_2) are satisfied. Specifically, we assume

$$\delta \in W^{2,1}(0, T), \qquad \gamma \in W^{1,1}(0, T), \tag{5.12}$$

$$r \in W^{1,1}(0, T; L^2(0, 1)), \tag{5.13}$$

$$u_0 \in W_0^{1,2}(0, 1), \tag{5.14}$$

$$\begin{aligned} &g: \boldsymbol{R} \longrightarrow \boldsymbol{R} \text{ is continous, nondecreasing, } g(0) = 0 \text{ and} \\ &(g(x_2) - g(x_1))(x_2 - x_1) \geq c\,|g(x_2) - g(x_1)|^2, \qquad \forall x_1, x_2 \in \boldsymbol{R}, \\ &\text{for some } c > 0. \end{aligned} \tag{5.15}$$

If $B = \tilde{g}^{-1}$ and a, f, φ^t are defined by (5.7), (5.8) and (5.10), respectively, it is easily verified that conditions (5.12)–(5.15) in conjunction with (5.5), (5.6) imply (a), (f), (b_1) and (b_2). Note that although $a(0) = a_0/b_0$, which may be different from 1, this case can be reduced to $a(0) = 1$; see Remark 3.1.

The following is a direct consequence of Theorem 3.3.

Theorem 5.1. *Let* (5.5), (5.6), (5.12)–(5.15) *be satisfied. Then the Volterra equation* (5.11) (*or equivalently the heat flow problem* (5.2)–(5.4)) *has at least one solution* $u \in L^\infty(0, T; L^2(0, 1))$, *such that* $g(u) \in W^{1,2}(0, T; L^2(0, 1))$, $u(t, \cdot) \in W^{2,2}(0, 1)$, *a.e. on* $(0, T)$, *and* $u_{xx} \in L^2(0, T; L^2(0, 1))$.

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