Funkcialaj Ekvacioj, 36 (1993) 275–301

Multivalued Volterra Integral Equations in Banach Spaces

By

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1. Introduction

In this paper we examine Volterra integral inclusions driven by m-accretive operators and evolution inclusions driven by time dependent convex subdifferentials. Problems of this form appear in the optimal control of distributed parameter systems (see Ahmed [1] and Papageorgiou [34]), in mathematical physics (see Chang [11] and Shuzhong [42]) and in certain systems monitored by partial differential equations (see Glashoff-Sprekels [18] and Kiffe [23]). In fact in a companion paper to this [3], the authors will use the results obtained in this article, to study the optimal control and relaxation of systems governed by Volterra integral inclusions with time dependent control constraints.

In the first part of the paper (section 3), we consider nonlinear Volterra integral inclusions driven by an m-accretive operator. The interesting feature of our equation is that the possibly multivalued m-accretive operator $A(\cdot)$ appears behind the integral sign, which makes our results more general than those of Papageorgiou [38], where the unbounded m-accretive operator is outside the integral. Also our work extends those of Chuong [12], Lyapin [25], Papageorgiou [30], [31] and Ragimkhanov [39], who studied integral inclusions, but did not allow the presence of unbounded operators, which in concrete applications model partial differential terms. Our approach in dealing with these equations is to use a result of Crandall-Nohel [13], which under mild hypotheses on the data, shows that the Volterra integral inclusion under consideration is equivalent to a functional evolution inclusion. So our results in this first part of the paper can also be viewed as new results about functional evolution inclusions, extending earlier ones obtained by Aizicovici [2], Avgerinos-Papageorgiou [7], Gutman [19], Mitidieri-Vrabie [26], Vrabie [43], [44]. Finally our work extends to a multivalued setting that of Crandall-Nohel [13].

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** Research supported by NSF Grant 8802688.

In the second part of the paper (section 5), we turn our attention to inclusions where the maximal monotone operator is a convex subdifferential, but we allow it to be time varying. Again our goal is to study Volterra integral inclusions monitored by such operators, using as before an equivalent functional-evolution inclusion. This approach leads us to extensions of recent results by Papageorgiou [36], [37] (theorems 3.1). Also we extend an earlier result by Attouch-Damlamian [5] (theorem 3.2), where the function involved in the subdifferential is time independent and the multivalued perturbation is convex-valued. Furthermore our work extends that of Moreau [27], who studies the particular case where the function $\phi(t, x)$ involved in the subdifferential, is the indicator function of a time dependent closed convex set and there are no multivalued perturbations. Problems of this form are important in theoretical mechanics, mathematical economics and feedback control systems (see Aubin-Cellina [6], chapters 5, 6; in that book those evolution inclusions are called "differential variational inequalities").

The first to consider evolutions involving time dependent convex subdifferentials was Watanabe [46], who extended the work of Brezis [10] (chapter III). Subsequently the work of Watanabe [46], was extended further by Kenmochi [21], [22], Yamada [47], Yotsutani [48] and very recently by Kubo [24]. In section 5, we use these works as our starting point and we obtain results generalizing them. Finally our results in section 5 partially extend those of Otani [29] (convex case) and Papageorgiou [33] (nonconvex case), who didn't have a functional term in their evolution inclusion; their hypothesis on the integrand $\phi(t, x)$ is more restrictive, but on the other hand their growth hypothesis on the multivalued perturbation is more general.

2. Preliminaries

Let (Ω, Σ) be a measurable and $(X, \|\cdot\|)$ a real separable Banach space. Throughout this paper we will be using the following notations: $P_{f(c)}(X) = \{A \subseteq X$: nonempty, closed, (convex)$\}$ and $P_{(w)k(c)}(X) = \{A \subseteq X$: nonempty, ($w$-) compact, (convex)$\}$.

A multifunction $F: \Omega \to 2^X \setminus \{\emptyset\}$ is said to be "graph measurable" if and only if $\mathrm{Gr}\, F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. If $F(\cdot)$ has closed values in X (i.e. $F(\omega) \in P_f(X)$ for all $\omega \in \Omega$), then we say that $F(\cdot)$ is a measurable multifunction, if for all $x \in X$ $\omega \to d(x, F(\omega)) = \inf\{\|x - y\| : y \in F(\omega)\}$ is measurable. Note that measurability of closed valued multifunctions, implies graph measurability. The converse is true if there exists a complete σ-finite measure $\mu(\cdot)$ on (Ω, Σ). For further details, we refer to the survey paper of Wagner [45].

For a multifunction $F: \Omega \to P_f(X)$, we define $S_F^p (1 \leq p \leq \infty)$ to be the set of all $L^p(\Omega, X)$-selectors of $F(\cdot)$, i.e. $S_F^p = \{f \in L^p(\Omega, X): f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$. This set may be empty. It is nonempty if $F(\cdot)$ is measurable and $\omega \to |F(\omega)| = \sup\{\|x\|: x \in F(\omega)\}$ belongs to $L_+^p(\Omega)$. Recall that a subset $B \subseteq L^p(\Omega, X)$ is decomposable, if for every triple $(f, g, A) \in B \times B \times \Sigma$ we have that $\chi_A f + \chi_{A^c} g \in B$, with χ_A (resp. χ_{A^c}) being the characteristic function of A (resp. of A^c). Clearly S_F^p is decomposable.

On $P_f(X)$ we can define a generalized metric, known in the literature as the Hausdorff metric, by setting $h(A, B) = \max\left[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right]$ (recall that $d(a, B) = \inf\{\|a - b\|: b \in B\}$; similarly for $d(b, A)$). The metric space $(P_f(X), h)$ is complete. We say that a sequence $\{A_n\}_{n \geq 1} \subseteq P_f(X)$ converges in the Hausdorff metric (denoted by $A_n \xrightarrow{h} A$), if $h(A_n, A) \to 0$ as $n \to \infty$.

Another mode of set convergence that we will need in the sequel, is the so called "Kuratowski convergence of sets". So let $\{A_n\}_{n \geq 1}$ be a sequence of nonempty subsets of X. We define

$$\underline{\lim}\, A_n = \{x \in X: x = s - \lim x_n, x_n \in A_n, n \geq 1\}$$

and

$$\overline{\lim}\, A_n = \{x \in X: x = s - \lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \cdots < n_k < \cdots\},$$

(here s- indicates the strong topology on the Banach space X). It is clear from the above definitions that we always have $\underline{\lim}\, A_n \subseteq \overline{\lim}\, A_n$. We say that the A_n's converge to A in the Kuratowski sense (denoted by $A_n \xrightarrow{K} A$) if and only if $\underline{\lim}\, A_n = \overline{\lim}\, A_n = A$.

If instead of $\overline{\lim}\, A_n$, we define $w\text{-}\overline{\lim}\, A_n = \{x \in X: x = w\text{-}\overline{\lim}\, x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \cdots < n_k < \cdots\}$ (here w- denotes the weak topology on X), then we say that the A_n's converge to A in the Kuratowski-Mosco sense (denoted by $A_n \xrightarrow{K-M} A$) if and only if $\underline{\lim}\, A_n = w\text{-}\overline{\lim}\, A_n = A$. Using this mode of set convergence, we can define a convergence for $\overline{\boldsymbol{R}}$-valued functions, which is in general different from the pointwise convergence. So if $\{f_n, f\}_{n \geq 1} \subseteq \overline{\boldsymbol{R}}^X$, we say that $f_n \xrightarrow{e} f$, if and only if $\text{epi}\, f_n \xrightarrow{K-M} \text{epi}\, f$, where for a $g \in \overline{\boldsymbol{R}}^X$, epi g denotes its epigraph, i.e. $\text{epi}\, g = \{(x, \lambda) \in X \times \boldsymbol{R}: g(x) \leq \lambda\}$. For further details on this subject, we refer to the book of Attouch [4] and the original paper of Mosco [28].

Now let Y, Z be Hausdorff-topological spaces. A multifunction $G: Y \to 2^Z \backslash \{\emptyset\}$ is said to be upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)), if for all $U \subseteq Z$ open, $F^+(U) = \{y \in Y: F(y) \subseteq U\}$ (resp. $F^-(U) = \{y \in Y: F(y) \cap U \neq \emptyset\}$) is open in Y. If additional hypotheses are made on the spaces Y, Z and on the multifunction $G(\cdot)$, then we can have other equivalent definitions of upper and lower semicontinuity of $G(\cdot)$. For details we refer to Delahaye-Denel [14].

Let E be a real Banach space and let A be a set valued operator with domain $D(A)$. We say that A is "accretive", if $\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|$ for all $\lambda > 0$ and $y_i \in Ax_i$, $i = 1, 2$. If in addition $I + \lambda A$ is surjective for each $\lambda > 0$, where I stands for the identity on X, then A is called "m-accretive". It is well known (see for example Vrabie [44], p. 38), that if A is m-accretive, then $-A$ generates a semigroup of nonexpansive mappings $S(t)$, $t \geq 0$ on $\overline{D(A)}$. The semigroup $S(t)$ is said to be compact, if for each $t > 0$, $S(t)$ is a compact operator.

Now let H be a Hilbert space with inner product denoted by $(\cdot, \cdot)$. Let $\phi\colon \mathrm{H} \to \overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous (l.s.c.) function, with effective domain $\operatorname{dom} \phi = \{x \in H\colon \phi(x) < \infty\}$. The class of all such functions will be denoted by $\Gamma_0(H)$. A function $\phi \in \Gamma_0(H)$ is said to be of compact type, if the subset $\{x \in H\colon \|x\|^2 + \phi(x) \leq \lambda\}$ is compact for each $\lambda > 0$. An important example of an m-accretive operator on H is the subdifferential of a function in $\Gamma_0(H)$. So if $\phi \in \Gamma_0(H)$, its subdifferential is defined by $\partial\phi(x) = \{y \in H\colon \phi(z) - \phi(x) \geq (y, z - x)$ for all $z \in \operatorname{dom} \phi\}$. It can be shown (see Brezis [10]), that $\partial\phi(\cdot)$ is m-accretive (equivalently maximal monotone) on H.

If $\phi \in \Gamma_0(H)$, one can also define the "regularization" ϕ_λ of ϕ $(\lambda > 0)$ as follows: $\phi_\lambda(x) = \inf\left\{\phi(y) + \frac{\|y - x\|^2}{2\lambda}\colon y \in H\right\}$. It is easy to check that $\phi_\lambda(\cdot)$ $\lambda > 0$ is convex and Fréchet differentiable. Furthermore $\phi_\lambda(x) \leq \phi(x)$ for all $\lambda > 0$ and all $x \in H$ and $\lim_{\lambda \to 0} \phi_\lambda(x) = \phi(x)$ for all $x \in H$.

3. Existence results

Throughout the rest of this paper, X denotes a real separable Banach space of norm $\|\cdot\|$. Let $T = [0, b]$ and consider the Volterra integral inclusion

$$\text{(V)} \qquad x(t) + \int_0^t k(t - s)(Ax(s) + F(s, x(s)))ds \ni g(t) \qquad 0 \leq t \leq b$$

where $A\colon D(A) \subseteq X \to 2^X$ is an m-accretive operator, $F\colon T \times X \to 2^X \backslash \{\varnothing\}$ is a multivalued perturbation and $k(\cdot) \in L^1(T, \boldsymbol{R}) = L^1(T)$, $g \in L^1(T, X)$.

By a "strong solution" of (V), we understand a function $x(\cdot) \in L^1(T, X)$, for which there exist w, $f \in L^1(T, X)$ such that $w(t) \in Ax(t)$ a.e., $f(t) \in F(t, x(t))$ a.e. on T and $x(t) + k * (w + f)(t) = g(t)$ a.e., on T. Here $*$ denotes the operation of convolution.

As we already mentioned in the introduction, using a result of Crandall-Nohel [13], we will transform (V) into a functional evolution inclusion. So along with (V) we consider the following problem:

$$\text{(P)} \qquad \begin{cases} \dot{x}(t) + Ax(t) + F(t, x(t)) \ni G(x)(t) \text{ a.e. on } T \\ x(0) = x_0 \end{cases}$$

where $x_0 \in \overline{D(A)}$ and $G\colon C(T, \overline{D(A)}) \to L^1(T, X)$.

By a "strong solution" of (P), we mean a function $x(\cdot) \in W^{1,1}(T, X) \cap C(T, \overline{D(A)})$ satisfying $x(0) = x_0$ and $\dot{x}(t) + w(t) + f(t) = G(x)(t)$ a.e. on T, with $w, f \in L^1(T, X)$ as above. Also by an "integral solution" of (P), we mean a function $x(\cdot) \in C(T, X)$ such that there exists $f \in L^1(T, X)$, $f(t) \in F(t, x(t))$ a.e., with $x(\cdot)$ being an integral solution in the sense of Benilan [9] of $\dot{x}(t) + Ax(t) \ni G(x)(t) - f(t)$, $x(0) = x_0$. Recall that if $\dim X < \infty$, then the notions of integral and strong solution coincide. More generally, if X is a Hilbert space, $f \in L^2(T, X)$, $G(x)(\cdot) \in L^2(T, X)$ and $A = \partial\phi$ with $\phi \in \Gamma_0(H)$, then again integral and strong solutions coincide. For details we refer to Brezis [10] (p. 72 and p. 82).

The result of Crandall-Nohel [13], that we will use to solve (V), is the following:

Proposition 3.1. *If $k(\cdot) \in AC(T, \boldsymbol{R})$, $\dot{k} \in BV(T, \boldsymbol{R})$, $k(0) = 1$, $g \in W^{1,1}(T, X)$ with $g(0) \in \overline{D(A)}$ and $x(\cdot)$ is a strong solution of* (V), *then $x(\cdot)$ is a strong solution of* (P), *where*

(i) $G(x)(\cdot) = h(\cdot) + r * h(\cdot) - r(0)x(\cdot) + r(\cdot)x_0 - x * \dot{r}(\cdot)$ $\left(\textit{with } x * \dot{r}(t) = \int_0^t x(t-s)dr(s)\right)$

(ii) $h(t) = \dot{g}(t)$

(iii) $x_0 = g(0)$,

(iv) $r + a * r = -a$, $a = \dot{k}$.

Conversely if $r(\cdot) \in BV(T, \boldsymbol{R})$, $h \in L^1(T, X)$, $x_0 \in \overline{D(A)}$, $G(\cdot)$ is as in (i) *above and $x(\cdot)$ is a strong solution of* (P), *then $x(\cdot)$ is a strong solution of* (V), *where*

(i)′ $g(t) = x_0 + \int_0^t h(s)ds$,

(ii)′ $a + a * r = -r$

(iii)′ $k(t) = 1 + \int_0^t a(s)ds$.

We are primarily interested in the existence of integral solutions of (V). By an integral solution of (V), we mean a function $x(\cdot) \in C(T, X)$, which is an integral solution of (P), with x_0 and $G(\cdot)$ given by (i) → (iv) of Proposition 3.1.

We will start with a (local) existence theorem for the case where the multivalued perturbation is nonconvex valued. So we need the following hypotheses on the data:

$\underline{H(A)}$: $A: D(A) \subseteq X \to 2^X$ is an m-accretive operator such that $-A$ generates a compact semigroup $S(t)$ on $\overline{D(A)}$.

$\underline{H(F)}$: $F: T \times \overline{D(A)} \to P_f(X)$ is a multifunction such that

(1) $F(\cdot, \cdot)$ is graph measurable,

(2) for every $t \in T$, $F(t, \cdot)$ is l.s.c.,

(3) for every $B \subseteq \overline{D(A)}$ bounded, there exists $m_B(\cdot) \in L^1_+(T)$ such that

$$\sup_{u \in B} |F(t, u)| = \sup_{u \in B} \sup_{y \in F(t,u)} \|y\| \leq m_B(t) \qquad \text{a.e. on } T.$$

$\underline{H(k)}$: $k \in AC(T, \boldsymbol{R})$, $\dot{k} \in BV(T, \boldsymbol{R})$, $k(0) = 1$.

$\underline{H(g)}$: $g \in W^{1,1}(T, X)$, $g(0) \in \overline{D(A)}$

Theorem 3.2. *Let $H(A)$, $H(F)$, $H(k)$, $H(g)$ be satisfied. Then there exists $b_0 \in (0, b]$ such that* (V) *has an integral solution on $T_0 = [0, b_0]$. Moreover, if there exist α, $\beta \in L^1_+(T)$ such that $|F(t, x)| \leq \alpha(t) + \beta(t)\|x\|$ a.e. on T, then each integral solution of* (V) *can be defined on T.*

Proof. In view of the result of Crandall-Nohel above, we have to prove the existence of local integral solution of inclusion (P), where $x_0 = g(0)$ and $G(\cdot)$ is as in the first part of proposition 3.1. Our proof uses some ideas of Avgerinos-Papageorgiou [7], Mitidieri-Vrabie [26] and Papageorgiou [36].

Because of hypotheses $H(F)$, $H(k)$ and $H(g)$, we can find $\varepsilon > 0$, $\eta \in L^1_+(T)$ and $b_0 \in (0, b]$ such that

$$(1) \quad \sup\{|F(t, u)|: u \in \overline{D(A)}, \|u - x_0\| \leq \varepsilon\} + \|G(u)(t)\| \leq \eta(t) \qquad \text{a.e. on } (0, b_0)$$

and

$$(2) \qquad \|S(t)x_0 - x_0\| + \int_0^t \eta(s)ds \leq \varepsilon, \qquad t \in (0, b_0]$$

Indeed note that for the constant function $x(t) = u \in \overline{D(A)}$, we have

$$\|G(u)(t)\| \leq \|h(t)\| + \|r * h\|_{L^\infty} + \|r(0)u\| + \|r(t)\| \cdot \|x_0\| + (\|r(t)\| + \|r(0)\|)\|u\|.$$

So if $\|u - x_0\| \leq \varepsilon$, then $\|G(u)(t)\| \leq c_\varepsilon + \|h(t)\| = \hat{m}(t)$ a.e. with $\hat{m}(\cdot) \in L^1_+(T)$. Here c_ε denotes a positive constant depending on ε. Combining this with hypothesis $H(F)$ (3), we see that it suffices to choose $\eta(t) = m_{B(x_0,\varepsilon)}(t) + \hat{m}(t)$ to have (1) on T. Then choose $b_0 \in (0, b]$ so that (2) holds. This is possible since $S(t)x_0 \xrightarrow{s} x_0$ as $t \to 0^+$.

Let $T_0 = [0, b_0]$, $K = \{q \in L^1(T_0, X): \|q(t)\| \leq \eta(t) \text{ a.e. on } T_0\}$ and for $q \in K$ consider the following initial value problem

$$(3) \qquad \begin{aligned} \dot{x}(t) + Ax(t) &\ni q(t), \qquad \text{on } T_0 \\ x(0) &= x_0 \end{aligned}$$

By [9], we know that (3) has a unique integral solution $x(q)(\cdot) \in C(T_0, X)$ and we have

$$\|x(q)(t) - x_0\| \leq \|x(q)(t) - S(t)x_0\| + \|S(t)x_0 - x_0\| \leq \|S(t)x_0 - x_0\| + \int_0^t \|q(s)\|ds \leq \|S(t)x_0 - x_0\| + \int_0^t \eta(s)ds \tag{4}$$

Since K is uniformly integrable and by hypothesis $H(A)$, $-A$ generates a compact semigroup, from a result of Baras [8], it follows that

$$D = \overline{\{x(q)(\cdot): q \in K\}} \subseteq C(T_0, X)$$

is compact. Furthermore inequality (4) above implies

$$\|x(t) - x_0\| \leq \varepsilon \tag{5}$$

for all $x(\cdot) \in D$ and all $t \in T_0$.

Let $L: D \to P_f(L^1(T_0, X))$ be defined by $L(x) = S^1_{G(x)(\cdot)-F(\cdot, x(\cdot))}$. Since $G(\cdot)$ is continuous from $C(T_0, X)$ into $L^1(T_0, X)$ and $F(t, \cdot)$ is l.s.c. (see hypothesis $H(F)$ (2)), we may use theorem 4.1 of Papageorgiou [32], to conclude that $L(\cdot)$ is l.s.c. Furthermore it is obvious that $L(\cdot)$ has decomposable values. So we can apply Fryszkowski's continuous selection theorem [17], to get $\lambda: D \to L^1(T_0, X)$ continuous such that $\lambda(x) \in L(x)$ for every $x \in D$. Hence we have

$$-\lambda(x)(t) + G(x)(t) \in F(t, x(t)) \qquad \text{a.e. on } T_0 \tag{6}$$

From (1), (5) and (6) above, we easily deduce that $\lambda(D) \subseteq K$. Also the continuity of $\lambda(\cdot)$ together with Mazur's theorem yields that $Q = \overline{\text{conv}}\, \lambda(D)$ is a compact, convex subset of $L^1(T_0, X)$. Since K is closed, convex and $\lambda(D) \subseteq K$, we have that $Q \subseteq K$. Define $\gamma: Q \to L^1(T_0, X)$ by $\gamma(q) = \lambda(x(q))$, $q \in Q$, where we recall that $x(q)(\cdot)$ is the integral solution of (3). Recall that $q \to x(q)$ is continuous from $L^1(T_0, X)$ into $C(T_0, X)$ (see e.g. Vrabie [44], corollary 2.3.1, p. 67), while by construction $\lambda: D \subseteq C(T_0, X) \to L^1(T_0, X)$ is continuous. Combining these two facts with the remark that $\lambda(x(q)) \in Q$ for each $q \in Q$ (since $Q \subseteq K$ and consequently $x(q) \in D$), we see that γ maps Q continuously into itself. So we can apply Schauder's fixed point theorem, to get $q \in Q$ such that $\gamma(q) = q$, i.e. $\lambda(x(q)) = q$. By (6) this means that $q(t) = -f(t) + G(x(q))(t)$ a.e. on T_0, with $f \in S^1_{F(\cdot, x(q)(\cdot))}$. Thus $x(q)(\cdot)$ is the desired (local) integral solution of (V). If the sublinear growth condition of the second part of the theorem is satisfied, then it is easy to see using Benilan's definition of integral solutions (see for example Vrabie [44], p. 32) and the definition of

$G(\cdot)$ (see proposition 3.1), that $\|x(t)\|$ satisfies a Gronwall type inequality on T_0. Consequently $\|x(t)\|$ is bounded on T_0, so that necessarily $T_0 = T$. Q.E.D.

Remarks. (i) If X is a finite dimensional Hilbert space, then every integral solution of (V) is automatically a strong solution. This is a consequence of a result due to Benilan-Brezis, which can be found for example in Vrabie [44] (theorem 1.9.1, p. 41).

(ii) If X is a Hilbert space and $A = \partial\phi$ with $\phi \in \Gamma_0(X)$ and of compact type, $g \in W^{1,2}(T, X)$ and $|F(t, x)| \leq \alpha(t) + \beta(t)\|x\|$ a.e. with $\alpha(\cdot)$, $\beta(\cdot) \in L^2_+(T)$, then again every integral solution of (V) is a strong solution (see Brezis [10], theorem 3, p. 72).

(iii) An analogous existence theory can be developed for inclusions of the form (V), with $k(t - s)$ replaced by a general nonconvolution kernel $k(t, s)$ $0 \leq s \leq t \leq b$ (see Crandall-Nohel [13], pp. 326–327 and Rennolet [40]). The case where the kernel k is operator valued (i.e. takes values in $\mathscr{L}(X) =$ space of linear, bounded operators on X), can be treated similarly.

(iv) If there is no multivalued perturbation (i.e. $F \equiv 0$), Crandall-Nohel [13] proved the existence and uniqueness of an integral solution of (V) on T, for an arbitrary m-accretive operator.

Our proof of theorem 3.2, also extends to the following class of functional evolution inclusions

$$(\mathrm{P}') \qquad \begin{cases} -\dot{x}(t) \in Ax(t) + (Fx)(t)\,, & t \in T \\ x(0) = x_0 \end{cases}$$

We need the following hypothesis on $F(\cdot)$

$\underline{H(F)_1}$: $F: C(T, \overline{D(A)}) \to P_f(L^1(X))$ is a multifunction such that

(1) $F(\cdot)$ is graph measurable, l.s.c. and has decomposable values

(2) for each $B \subseteq C(T, \overline{D(A)})$ bounded, there is an $m_B(\cdot) \in L^1_+(T)$ such that

$$\sup_{x \in B} |F(x)(t)| = \sup_{x \in B}\ \sup_{y \in F(x)(t)} \|y\| \leq m_B(t) \qquad \text{a.e. on } T$$

(3) for every $0 < b' < b$ and every $x \in C(T, \overline{D(A)})$, $(Fx_{b'})(t) = (Fx)_{b'}(t)$ a.e. $t \in (0, b')$ where $x_{b'}$ denotes the restriction of $x(\cdot)$ on $[0, b']$.

Then by using the same ideas as in the case of theorem 3.2 we obtain:

Theorem 3.3. *Let hypotheses $H(A)$, $H(F)_1$ be satisfied and $x_0 \in \overline{D(A)}$. Then there exists $b_0 \in (0, b]$ such that* (P′) *has an integral solution on $T_0 = [0, b_0]$.*

If in addition, there are constants α, $\beta > 0$ such that

$$|F(x)| = \sup\{\|v\|_{L^1}: v \in F(x)\} \leq \alpha + \beta\|x\|_{L^1} \qquad \text{for all } x \in C(T, \overline{D(A)})\,,$$

then $b_0 = b$.

Next we turn our attention to the convex case, i.e. we consider the Volterra inclusion (V) with a convex perturbation $F(t, x)$. As before we use the result of Crandall-Nohel [13] (see proposition 3.1), to pass from (V) to the equivalent evolution inclusion (P). To simplify our presentation, we will restrict ourselves to a global existence result. The reader can easily furnish a local version, following the steps of theorem 3.2. We will need the following hypothesis on the multivalued perturbation $F(t, x)$.

$\underline{H(F)_2}$: $F: T \times X \to P_{fc}(X)$ is a multifunction such that

(1) $F(\cdot, \cdot)$ is graph measurable,

(2) for every $t \in T$, $x \to F(t, x)$ is u.s.c. from X into X_w (here X_w denote the space X endowed with the weak topology),

(3) $|F(t, x)| = \sup\{\|y\|: y \in F(t, x)\} \le \alpha(t) + \beta(t)\|x\|$ a.e. for some $\alpha(\cdot)$, $\beta(\cdot) \in L^1_+(T)$.

Theorem 3.4. *If X is reflexive and hypotheses $H(A)$, $H(F)_2$, $H(k)$, $H(g)$ hold, then* (V) *admits an integral solution on T.*

Proof. Thanks to proposition 3.1, we know that it suffices to establish the existence of an integral solution for the equivalent inclusion (P). First we will obtain an a priori bound for the trajectories of (P). So let $x(\cdot) \in C(T, X)$ be an integral solution of (P). Then we know that

$$\|x(t) - S(t)x_0\| \le \int_0^t \|G(x)(s) - f(s)\| ds, \qquad f \in S^1_{F(\cdot, x(\cdot))}$$

$$\Rightarrow \|x(t)\| \le \|S(t)x_0\| + \int_0^t (m_1(s) + m_2(s)\|x\|_\infty(s) + \alpha(s) + \beta(s)\|x(s)\|) ds$$

for some m_1, $m_2 \in L^1_+(T)$. Invoking Gronwall's inequality, we get $\|x(t)\| \le M$ for some $M > 0$. Set $\mu(t) = \alpha(t) + m_1(t) + (m_2(t) + \beta(t))M$ and define $K = \{q \in L^1(T, X): \|q(t)\| \le \mu(t) \text{ a.e.}\}$. Then for $q \in K$, consider the evolution equation

$$\dot{x}(t) + Ax(t) \ni q(t), \qquad t \in T$$

$$x(0) = x_0$$

where recall that $x_0 = g(0)$. This has a unique integral solution $x(q)(\cdot) \in C(T, X)$ and by $H(A)$ it follows (see Baras [8]) that the map $q \to x(q)(\cdot)$ is continuous from K endowed with the relative weak $L^1(T, X)$-topology into $C(T, X)$. Then let $R: K \to P_{fc}(K)$ be the multifunction defined by $R(q) = S^1_{\hat{G}(x(q)(\cdot)) - \hat{F}(\cdot, x(q)(\cdot))}$, where $G(\cdot)$ is given by proposition 3.1, and $\hat{F}(t, x) = F(t, p_M(x))$ with $p_M(\cdot)$ being the M-radial retraction and $\hat{G} = G(p_M)$. We claim that $R(\cdot)$ is u.s.c. on K with the relative weak $L^1(T, X)$-topology, henceforth denoted by K_w. Note that K_w is a compact metric space, as a weakly compact subset (cf. Diestel-Uhl [15], theorem 1, p. 101) of the separable Banach space $L^1(T, X)$ (see Dunford-Schwartz [16], theorem 3, p. 434). So to establish the upper semicontinuity of $R(\cdot)$, it suffices to show that $\mathrm{Gr}\, R$ is sequentially closed in $K_w \times K_w$. So let $(q_n, f_n) \in \mathrm{Gr}\, R$, $n \geq 1$ and assume that $(q_n, f_n) \to (q, f)$ as $n \to \infty$ in $K_w \times K_w$. Then we have $x(q)_n(\cdot) \to x(q)(\cdot)$ in $C(T, X)$ and $\hat{G}(x(q_n)(\cdot)) \xrightarrow{s} \hat{G}(x(q)(\cdot))$ in $L^1(T, X)$, since $\hat{G}(\cdot)$ is continuous. Also invoking theorem 3.1 of [32] we have

$$f(t) \in \overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\, \{f_n(t)\}_{n\geq 1} \subseteq \overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\, [\hat{G}(x(q_n))(t) - \hat{F}(t, x(q_n)(t))] \quad \text{a.e.}$$

$$\subseteq \hat{G}(x(q)(t)) - \overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\, \hat{F}(t, x(q_n)(t)) \quad \text{a.e.}$$

But from the definition of $\hat{F}(\cdot, \cdot)$ and hypothesis $H(F)_2$ (2), we have

$$w\text{-}\overline{\lim}\, \hat{F}(t, x(q_n)(t) \subseteq \hat{F}(t, x(q)(t)) \quad \text{a.e.}$$

$$\Rightarrow f(t) \in \hat{G}(x(q)(t)) - \hat{F}(t, x(q)(t)) \quad \text{a.e.}$$

By the Kakutani-KyFan fixed point theorem, we now get $\hat{q} \in R(\hat{q})$. Then $x(\hat{q})(\cdot)$ solves (P) (and hence (V) too) with $\hat{F}(t, x)$ and $\hat{G}$. An easy estimate as in the beginning of the proof, gives us that $\|x(\hat{q})(t)\| \leq M$ for all $t \in T \Rightarrow \hat{F}(t, x(\hat{q})(t)) = F(t, x(\hat{q})(t))$ and $\hat{G}(x(\hat{q}))(\cdot) = G(x(q))(\cdot) \Rightarrow x(\hat{q})(\cdot)$ is the desired integral solution of (V). Q.E.D.

4. Properties of the solution set

The first result of this section shows that every solution of a convexified Volterra inclusion can be approximated arbitrarily close in the L^∞-norm, by a solution of a nonconvex related problem, provided that we strengthen the conditions on the orientor field $F(t, x)$. Such a result is known in the literature as a "relaxation theorem". From the theory of differential inclusions, we know that lower semicontinuity or even continuity of $F(t, \cdot)$ is not enough to guarantee a relaxation (density) result. There is a nice two dimensional counterexample due to Pliss (see Aubin-Cellina [6]), illustrating this. So we will need a stronger hypothesis on $F(\cdot, \cdot)$.

Let $H(A)$, $H(k)$, $H(g)$ be satisfied. In addition assume:

<u>$H(F)_3$</u>: $F: T \times X \to P_f(X)$ is a multifunction satisfying

(1) for every $x \in X$, $t \to F(t, x)$ is measurable,

(2) $h(F(t, x), F(t, y)) \leq \theta(t)\|x - y\|$ a.e., with $\theta(\cdot) \in L^1_+(T)$

(3) $|F(t, x)| \leq \alpha(t) + \beta(t)\|x\|$ a.e. with $\alpha(\cdot)$, $\beta(\cdot) \in L^1_+(T)$.

Note that $H(F)_3$ is stronger than $H(F)_1$, so that by theorem 3.2, equation (V) has at least one global integral solution. Let S be the set of integral solutions of (V) and S_c the set of integral solutions of the same equation with $F(t, x)$ replaced by $\overline{\text{conv}}\, F(t, x)$.

Theorem 4.1. *If X is reflexive and hypotheses $H(A)$, $H(F)_3$, $H(k)$, $H(g)$ hold, then $S_c = \bar{S}$, where the closure is taken in $C(T, X)$.*

Proof. Again we exploit the equivalence between (V) and (P) established in Proposition 3.1. Let $x(\cdot) \in S_c$; then by definition, there exists $f(\cdot) \in S^1_{\overline{\text{conv}}\, F(\cdot, x(\cdot))}$ such that

$$\dot{x}(t) + Ax(t) \ni G(x)(t) - f(t)$$

$$x(0) = x_0 = g(0).$$

Let $W(y) = G(y)(\cdot) - S^1_{F(\cdot, y(\cdot))}$, $y(\cdot) \in C(T, X)$. Clearly $W(y)$ is closed and decomposable. Also $\overline{\text{conv}}\, W(y) = G(y)(\cdot) - S^1_{\overline{\text{conv}}\, F(\cdot, y(\cdot))}$ (see Hiai-Umegaki [20]). Furthermore recall that the map $u: L^1(T, X) \to C(T, X)$, which to each $p \in L^1(T, X)$ assigns the unique solution of $\dot{y}(t) + Ay(t) \ni p(t)$, $t \in T$, $y(0) = x_0$ is weakly-strongly continuous (cf. Baras [8]). So given $\varepsilon > 0$, there exists U, a symmetric weak neighborhood of the origin in $L^1(T, X)$ such that if $G(x)(\cdot) - f(\cdot) - p_1(\cdot) \in U$, then $\|x - x_1\|_\infty < \varepsilon$ with $x_1 = u(p_1)$. Invoking theorem 4.1 of [35], we know that we can choose $p_1 \in W(x) = G(x)(\cdot) - S^1_{F(\cdot, x(\cdot))}$. Then $p_1 = G(x) - f_1$ with $f_1 \in S^1_{F(\cdot, x(\cdot))}$. By a straightforward applicaton of Aumann's selection theorem (see Wagner [45]), we can find $f_2 \in S^1_{F(\cdot, x_1(\cdot))}$ such that $d(f_1(t), F(t, x_1(t))) = \|f_1(t) - f_2(t)\|$ a.e., where $x_1 = u(p_1)$. Set $p_2 = G(x_1) - f_2$ and $x_2 = u(p_2)$. From the properties of the integral solution, we have:

$$\begin{aligned}
\|x_1(t) - x_2(t)\| &\leq \int_0^t \|p_1(s) - p_2(s)\|\, ds \\
&\leq \int_0^t \|G(x)(s) - G(x_1)(s)\|\, ds + \int_0^t \|f_1(s) - f_2(s)\|\, ds \\
&\leq \int_0^t \gamma(s)\|x - x_1\|_\infty ds + \int_0^t h(F(s, x(s)), F(s, x_1(s)))ds \\
&\leq \int_0^t (\gamma(s) + \theta(s))\|x - x_1\|_\infty ds \\
&= \int_0^t \hat{\theta}(t)\|x - x_1\|_\infty ds
\end{aligned}$$

where $\gamma(\cdot) \in L^1_+(T)$ (in fact $\gamma(t) = |r(0)| + \mathrm{var}\,(r\colon [0, t])$) and hence $\hat{\theta} \in L^1_+(T)$ $\hat{\theta}(t) = |r(0)| + \mathrm{var}\,(r\colon [0, t]) + \theta(t)$. So we have:

$$\|x_1(t) - x_2(t)\| \leq \varepsilon \int_0^t \hat{\theta}(s)ds$$

This implies that $\|x_2(t) - x(t)\| \leq \|x_2(t) - x_1(t)\| + \|x_1(t) - x(t)\| \leq \varepsilon \int_0^t \hat{\theta}(s)ds + \varepsilon = \varepsilon\left(\int_0^t \hat{\theta}(s)ds + 1\right)$. Suppose we have chosen $p_1, \dots, p_n \in L^1(T, X)$ such that

$$(1) \qquad x_k = u(p_k)\,, \qquad p_{k+1} = G(x_k) - f_{k+1}\,, \qquad f_{k+1} \in S^1_{F(\cdot, x_k(\cdot))}$$

and for every $\tau \in T$

$$(2) \qquad \int_0^\tau \|p_{k+1}(t) - p_k(t)\| dt \leq \int_0^\tau \frac{\varepsilon\hat{\theta}(t)}{(k-1)!}\left(\int_0^t \hat{\theta}(s)ds\right)^{k-1} dt\,, \qquad k = 1, 2, \dots, n-1$$

(note that $p_1 = G(x) - f_1$ as above). Then once again from the properties of integral solutions, we have

$$\begin{aligned} \|x_{k+1}(t) - x_k(t)\| &\leq \int_0^t \|p_{k+1}(s) - p_k(s)\| ds \\ &\leq \frac{\varepsilon}{(k-1)!}\int_0^t \hat{\theta}(s)\left(\int_0^s \hat{\theta}(\tau)d\tau\right)^{k-1} ds \\ &= \frac{\varepsilon}{k!}\left(\int_0^t \hat{\theta}(s)ds\right)^k. \end{aligned}$$

Hence from the triangle inequality we get

$$(3) \qquad \|x_{k+1}(t) - x(t)\| \leq \varepsilon \sum_{i=0}^{k+1} \frac{1}{i!}\left(\int_0^t \hat{\theta}(s)ds\right)^i \leq \varepsilon \exp \|\hat{\theta}\|_1$$

As before, by Aumann's selection theorem, we can find $f_{n+1} \in S^1_{F(\cdot, x_n(\cdot))}$ such that

$$\begin{aligned} \|f_{n+1}(t) - f_n(t)\| &= d(f_n(t), F(t, x_n(t))) \qquad \text{a.e.} \\ &\leq h(F(t, x_{n-1}(t)), F(t, x_n(t))) \qquad \text{a.e.} \\ &\leq \theta(t)\|x_{n-1}(t) - x_n(t)\| \qquad \text{a.e.} \end{aligned}$$

Set $p_{n+1} = G(x_n) - f_{n+1}$. We have for every $\tau \in T$:

$$\int_0^\tau \|p_{n+1}(t) - p_n(t)\| dt \le \int_0^\tau \|G(x_n)(t) - G(x_{n-1})(t)\| dt + \int_0^\tau \theta(t) \|x_{n-1}(t) - x_n(t)\| dt$$

$$\le \int_0^\tau \hat{\theta}(t) \|x_{n-1}(t) - x_n(t)\|_\infty dt$$

$$\le \int_0^\tau \frac{\varepsilon}{(n-1)!} \hat{\theta}(t) \left(\int_0^t \hat{\theta}(s) ds \right)^{n-1} dt \,.$$

So we have obtained $p_{n+1} \in L^1(T, X)$ and thus $x_{n+1} = u(p_{n+1})$. Therefore we have completed the induction and defined a sequence $\{p_n\}_{n\ge1} \subseteq L^1(T, X)$ such that (1) and (2) are satisfied. From (2), we deduce that $\{p_n\}_{n\ge1}$ is a convergent sequence in $L^1(T, X)$. So $p_n \xrightarrow{s} \hat{p}$ in $L^1(T, X) \Rightarrow x_n = u(p_n) \xrightarrow{s} \hat{x} = u(\hat{p})$ in $C(T, X)$ and we also have $f_n(t) \xrightarrow{s} \hat{f}(t)$ a.e. in X. Because of hypothesis $H(F)_3$ (2), $\hat{f}(t) \in \lim F(t, x_n(t)) = F(t, \hat{x}(t))$ a.e. Thus $\hat{x} \in S$ and so by passing to the limit as $n \to \infty$ in (3), we get $\|\hat{x} - x\|_\infty \le \varepsilon \exp \|\hat{\theta}\|_1$. Since $\varepsilon > 0$ was arbitrary, we deduce that $S_c \subseteq \bar{S}$, the closure taken in $C(T, X)$. But S_c is closed (see the remark below). So $\bar{S} = S_c$. Q.E.D.

Remark. Under the more general hypotheses of theorem 3.4, it follows that the set of integral solutions for the convex problem is compact in $C(T, X)$. The proof of this useful fact goes as follows: Note that $S_c \subseteq \overline{u(K)}$, with K defined in the proof of theorem 3.2. The latter is compact in $C(T, X)$ (see Baras [8]). So it suffices to show that S_c is closed in $C(T, X)$. Let $\{x_n\}_{n\ge1} \subseteq S_c$, $x_n \to x$ in $C(T, X)$. Then by definition we have $x_n = u(p_n)$, $p_n = G(x_n) - f_n$, $f_n \in S^1_{F(\cdot, x_n(\cdot))}$. Note that since by hypothesis $H(F)_2$ (2), $F(t, \cdot)$ is strongly-weakly u.s.c., $E(t) = \overline{\text{conv}} \bigcup_{n\ge1} F(t, x_n(t)) \in P_{wkc}(X)$ and $E(\cdot)$ is measurable. Hence S^1_E is weakly compact in $L^1(T, X)$ and so $\{f_n\}_{n\ge1}$ is relatively sequentially weakly compact in $L^1(T, X)$. Thus we may assume that $f_n \xrightarrow{w} f$ in $L^1(T, X) \Rightarrow f(t) \in \overline{\text{conv}}\, w\text{-}\overline{\lim}\, \{f_n(t)\}_{n\ge1} \subseteq \overline{\text{conv}}\, w\text{-}\overline{\lim}\, F(t, x_n(t)) \subseteq F(t, x(t))$ a.e. (see theorem 3.1 of [32]). So $f \in S^1_{F(\cdot, x(\cdot))}$ and for $p = G(x) - f$, $u(p) = x \in S_c \Rightarrow S_c$ is closed, hence compact in $C(T, X)$.

In the next theorem, we examine the dependence of the solution set on the multivalued perturbation $F(t, x)$. We restrict our attention to the convex case. We will need the following auxiliary result:

Lemma 4.2. *Let X be a reflexive Banach space and let $F\colon T \times X \to P_{fc}(X)$ be measurable in t and Hausdorff continuous (h-continuous) in x. If $v\colon T \to X$ is measurable, then $(t, x) \to p(t, x) = \text{proj}\,(v(t); F(t, x))$ is a Caratheodory map.*

Proof. Fix $x \in X$. Then $\text{Gr}\, p(\cdot, x) = \{(t, y) \in T \times X\colon \|v(t) - y\| = d(v(t); F(t, x))\}$. Note that because of our hypothesis on $F(t, x)$, $t \to d(v(t); F(t, x))$ is measurable $\Rightarrow (t, y) \to \|v(t) - y\| - d(v(t), F(t, x))$ is a Caratheodory function, thus jointly measurable $\Rightarrow \mathrm{Gr}\, p(\cdot, x) \in B(T) \times B(X) \Rightarrow p(\cdot, x)$ is $\mathscr{L}(T) \times B(X)$-measurable, where $\mathscr{L}(T)$ is the Lebesgue completion of the Borel σ-field $B(T)$.

Also if $x_n \to x \Rightarrow F(t, x_n) \xrightarrow{h} F(t, x)$ and so from theorem 3.33, p. 322 in Attouch [4], it follows that $\mathrm{proj}\,(v(t); F(t, x_n)) \xrightarrow{s} \mathrm{proj}\,(v(t); F(t, x)) \Rightarrow p(t, x_n) \xrightarrow{s} p(t, x) \Rightarrow p(t, \cdot)$ is continuous. Thus $p(\cdot, \cdot)$ is a Caratheodory map, as claimed.

Q.E.D.

Consider the following sequence of Volterra integral inclusions:

$$(\mathrm{V})_n \qquad x(t) + \int_0^t k(t-s)(Ax(s) + F_n(s, x(s)))ds \ni g(t) \qquad t \in T$$

along with (V). We will need the following hypothesis on the sequence of orientor fields $F_n(t, x)$ $n \geq 1$.

$\underline{H(F)_4}$: F_n, $F\colon T \times X \to P_{fc}(X)$ are multifunctions such that for every $n \geq 1$

(1) for every $x \in X$, $t \to F_n(t, x)$ is measurable,

(2) $h(F_n(t, x), F_n(t, y)) \leq \theta(t)\|x - y\|$ a.e. with $\theta(\cdot) \in L^1_+(T)$,

(3) for almost all $t \in T$ and all $x \in X$, $F_n(t, x) \xrightarrow{h} F(t, x)$ as $n \to \infty$,

(4) $|F_n(t, x)| \leq \alpha(t) + \beta(t)\|x\|$ a.e. with $\alpha(\cdot)$, $\beta(\cdot) \in L^1_+(T)$.

Let S_n, S be the solution sets of $(\mathrm{V})_n$ and (V) respectively. Note that S_n, S are nonempty as soon as $H(A)$, $H(k)$, $H(g)$ are satisfied (cf. Theorem 3.2).

Theorem 4.3. *If X is reflexive and hypotheses $H(A)$, $H(F)_4$, $H(k)$, $H(g)$ hold, then S_n, $S \in P_k(C(T, X))$ and $S_n \xrightarrow{h} S$ as $n \to \infty$.*

Proof. Let $x(\cdot) \in S$. Then passing to the equivalent functional-evolution inclusion (see proposition 3.1), we have

$$\dot{x}(t) + Ax(t) \ni G(x)(t) - f(t)$$

$$x(0) = x_0 = g(0)$$

with $f \in S^1_{F(\cdot, x(\cdot))}$. Set $m_n(t) = \mathrm{proj}\,(f(t); F(t, x_n(t)))$ and $v_n(t, z) = \mathrm{proj}\,(m_n(t); F_n(t, z))$. Note that $m_n(\cdot)$ is measurable and $v_n(\cdot, \cdot)$ is a Caratheodory function (cf. Lemma 4.2). Also observe that $m_n(t) = v_n(t, x(t))$.

Next consider the following functional-evolution inclusion

$$\dot{x}_n(t) + Ax_n(t) \ni G(x_n)(t) - v_n(t, x_n(t)) \qquad t \in T$$

$$x_n(0) = x_0$$

From theorem 3.2, we know that the above multivalued Cauchy problem has an integral solution $x_n(\cdot) \in C(T, X)$. Clearly $x_n(\cdot) \in S_n$, $n \geq 1$. Using the properties of integral solutions, we have:

$$\|x_n(t) - x(t)\| \leq \int_0^t \|G(x_n)(s) - G(x)(s)\| ds + \int_0^t \|v_n(s, x_n(s)) - f(s)\| ds$$

$$\leq \int_0^t \gamma(s) \cdot \|x_n(s) - x(s)\|_\infty ds + \int_0^t \|v_n(s, x_n(s)) - v_n(s, x(s))\| ds$$

$$+ \int_0^t \|v_n(s, x(s)) - f(s)\| ds$$

where $\gamma(t) = |r(0)| + \mathrm{var}\,(r\colon [0, t])$. Note that

$$\|v_n(s, x(s)) - v_n(s, x_n(s))\| = d(m_n(s), F_n(s, x_n(s)))$$

$$\leq h(F_n(s, x(s)), F_n(s, x_n(s)))$$

$$\leq \theta(s)\, \|x_n(s) - x(s)\| \qquad \text{a.e.}$$

and $\|v_n(s, x(s)) - f(s)\| \leq h(F_n(s, x(s)), F(s, x(s))) \to 0$ as $n \to \infty$ (cf. hypothesis $H(F)_4$ (3)). Thus given $\varepsilon > 0$, we can find $n_0(\varepsilon) \geq 1$ such that for $n \geq n_0$, we have

$$\|x_n(t) - x(t)\| \leq \varepsilon + \int_0^t \hat{\theta}(s)\, \|x_n(s) - x(s)\|_\infty ds\,, \qquad \hat{\theta}(\cdot) = \gamma(\cdot) + \theta(\cdot)\,.$$

Applying Gronwall's inequality, we get

$$\|x_n(t) - x(t)\| \leq \varepsilon \exp \|\hat{\theta}\|_1 \qquad n \geq n_0 \Rightarrow x_n \to x \qquad \text{in } C(T, X)\,.$$

Therefore we have established that

$$S \subseteq \underline{\lim}\, S_n \tag{4}$$

Next let $x_n \in S_n$ $n \geq 1$ and assume $x_n \to x$ in $C(T, X)$. We have:

$$\dot{x}_n(t) + Ax_n(t) \ni G(x_n)(t) - f_n(t)\,, \qquad t \in T$$

$$x_n(0) = x_0$$

with $f_n \in S^1_{F_n(\cdot, x_n(\cdot))}$. Because of hypotheses $H(F)_4$ ((2) and (3)), we have $F_n(t, x_n(t)) \xrightarrow{h} F(t, x(t))$, so that by theorem 4.4 of [32], $S^1_{F_n(\cdot, x_n(\cdot))} \xrightarrow{h} S^1_{F(\cdot, x(\cdot))}$. Also in view of $H(F)_4$ (4), $\{f_n\}_{n \geq 1}$ is relatively sequentially weakly compact. So, by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(T, X)$. For every $q \in L^\infty(T, X^*) = L^1(T, X)^*$ we have

$$\langle q, f_n \rangle = \sup\{\langle q, v \rangle\colon v \in S^1_{F_n(\cdot, x_n(\cdot))}\} = \sigma_{S^{1(v)}_{F_n(\cdot, x_n(\cdot))}} \Rightarrow \langle q, f \rangle \leq \sigma_{S^{1(v)}_{F(\cdot, x(\cdot))}}$$

since $S^1_{F_n(\cdot, x_n(\cdot))} \xrightarrow{h} S^1_{F(\cdot, x(\cdot))}$ (here $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair $(L^1(T, X), L^\infty(T, X^*))$). So $f \in S^1_{F(\cdot, x(\cdot))} \Rightarrow f(t) \in F(t, x(t))$ a.e., $f \in L^1(T, X)$. Now note that $x_n = u(p_n)$, $p_n = G(x_n) - f_n$ and $p_n \xrightarrow{w} p = G(x) - f$ in $L^1(T, X)$.

Since $u(\cdot)$ is weakly-strongly continuous, $u(p_n) \to u(p)$ in $C(T, X) \Rightarrow x = u(p) \Rightarrow x \in S$. So we have proved that

$$\overline{\lim}\, S_n \subseteq S \tag{5}$$

Combining (4) and (5), we deduce that $S_n \xrightarrow{K} S$. But by $H(A)$, $H(F)_4$ and Baras [8], S_n and S are included in a compact subset of $C(T, X)$ and S_n, $S \in P_k(C(T, X))$ (see the remark following theorem 4.1). Now recall that on compact metric spaces, Kuratowski and Hausdorff convergence of sets coincide. Hence $S_n \xrightarrow{h} S$ in $C(T, X)$. Q.E.D.

5. Time-dependent subdifferentials

In this section, we turn our attention to the second class of problems that we will be considering in this paper. These are Volterra integral inclusions monitored by time varying convex subdifferentials. As before our approach is based on the study of an equivalent functional-evolution inclusion. This way we obtain two new existence results, extending earlier ones existing in the literature, as we indicated in the introduction. Also we present a relaxation theorem and a convergence result analogous to theorem 4.3. Our starting point is the very recent work of Kubo [24], who extended earlier important works on the subject by Attouch-Damlamian [5], Watanabe [46], Kenmochi [21], [22], Yamada [47] and Yotsutani [48]. It should be mentioned that the work of Yamada [47] is the first important contribution in the theory of evolution equations with subdifferentials, after the work of Watanabe [46]. In that paper Yamada [47] gave an interesting application to P.D.E's. The works of Yotsutani [48] and Kubo [24] are improvements of this fundamental work of Yamada [47].

So let $T = [0, b]$ and H be a separable Hilbert space. Let $\phi(t, \cdot)$ $t \in T$ be a family of proper, convex, l.s.c. functions on H, with values in $\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{+\infty\}$. Assume that (see Kubo [24]):

<u>$H(\phi)$</u>:

(1) There is a constant $c \geq 0$ such that $\phi(t, z) + c\|z\| + c \geq 0$ for all $z \in H$ and $t \in T$,

(2) There is a function $j \in W^{1,1}(T; H)$ such that $t \to \phi(t, j(t))$ belongs to $L^1(T)$,

(3) For each $z \in H$ and $0 < \lambda \leq 1$, the function $t \to \phi_\lambda(t, z)$ is of bounded variation on T and satisfies $\phi_\lambda(t, z) - \phi_\lambda(s, z) \leq \int_s^t \frac{d}{d\tau}\phi_\lambda(\tau, z)d\tau$, for all $0 \leq s \leq t \leq b$,

(4) For each $m > 0$, there are constants $c_m^1 \in [0, 1)$, $c_m^2 \geq 0$ and two

functions μ_m^1, $\mu_m^2 \in L^1(T)$ such that for all $z \in H$ with $\|z\| \le m$ and all $0 < \lambda \le 1$

$$\frac{d}{dt}\phi_\lambda(t, z) \le c_m^1 \|\partial\phi_\lambda(t, z)\|^2 + \mu_m^1(t) + \mu_m^2(t)\phi_\lambda(t, z)$$

$$+ c_m^2(\phi_\lambda(t, z))^2 \qquad \text{a.e. on } T.$$

Note (cf. Kubo [24], section 5) that $H(\phi)$ contains as special cases the assumptions used by Watanabe [46], Kenmochi [21], [22] and Yotsutani [47].

Consider the initial-value problem

$$\text{(E)} \qquad \begin{aligned} &\dot{x}(t) + \partial\phi(t, x(t)) \ni p(t) \qquad \text{a.e. on } T \\ &x(0) = x_0 \end{aligned}$$

where $x_0 \in H$ and $p \in L^2(T, H)$.

A strong solution of (E) is a function $x \in W^{1,2}(T, H)$, with $x(0) = x_0$ and such that $x(t) \in \operatorname{dom} \phi(t, \cdot)$ a.e. on T and $p(t) - \dot{x}(t) \in \partial\phi(t, x(t))$ a.e. Exploiting the monotonicity of the subdifferential, it is easily verified that if $x_1(\cdot)$, $x_2(\cdot)$ denote strong solutions of (E) corresponding to p_1, p_2 respectively, then

$$\text{(1)} \qquad \|x_1(t) - x_2(t)\| \le \int_0^t \|p_1(s) - p_2(s)\|\, ds\,, \qquad t \in T$$

In particular inequality (1) above implies that (E) has at most one strong solution. The following existence result for (E) has recently been established by Kubo [24] (theorem 2).

Theorem 5.1. *Let assumption $H(\phi)$ be satisfied. Then for each $x_0 \in \operatorname{dom} \phi(0, \cdot)$ and $p \in L^2(T, H)$, problem* (E) *has a unique strong solution $x(\cdot)$ such that $t \to \phi(t, x(t))$ is of bounded variation on T.*

Remark. From Kubo [24] (see subsections 3.2 and 3.3), it follows that under the hypotheses of theorem 5.1, the solution $x(\cdot)$ of (E) satisfies

$$\text{(2)} \qquad \|\dot{x}\|_{L^2(T,H)} \le M\,; \qquad |\phi(t, x(t))| \le M$$

where $M = M(\|x_0\| + |\phi(0, x_0)| + \|p\|_{L^2(H)})$ is a locally bounded function.

Now we are ready to consider the following multivalued Cauchy problem:

$$\text{(E)}_1 \qquad \begin{cases} \dot{x}(t) + \partial\phi(t, x(t)) + F(t, x(t)) \ni 0 \qquad \text{a.e. on } T \\ x(0) = x_0 \end{cases}$$

We will need the following hypothesis on $F(t, x)$:

$\underline{H(F)_5}$: $F: T \times H \to P_f(H)$ is a multifunction such that

(1) $F(\cdot, \cdot)$ is graph measurable,

(2) for every $t \in T$, $x \to F(t, x)$ is l.s.c.,

(3) for every $B \subseteq H$ bounded, there exists $\psi_B(\cdot) \in L^2_+$ such that

$$\sup\{|F(t, x)|: x \in B\} \leq \psi_B(t) \qquad \text{a.e.}\,.$$

By a "strong solution" of $(E)_1$, we mean a function $x \in W^{1,2}(T, H)$ satisfying $x(0) = x_0$ and such that there exist η, $f \in L^2(T, H)$ with $x(t) \in \operatorname{dom} \partial\phi(t, \cdot)$, $\eta(t) \in \partial\phi(t, x(t))$ a.e., $f(t) \in F(t, x(t))$ a.e. and $\dot{x}(t) + \eta(t) + f(t) = 0$ a.e.

We are now in a position to prove an existence theorem for $(E)_1$, which extends theorem 3.1 of Papageorgiou [36] and theorem 3.1 in Papageorgiou [37].

Theorem 5.2. *If hypotheses $H(\phi)$ and $H(F)_5$ hold, $\phi(t, \cdot)$ is of compact type and $x_0 \in \operatorname{dom} \phi(0, \cdot)$, then there exists $b_0 \in (0, b]$ such that $(E)_1$ has a strong solution $x(\cdot)$ on $T_0 = [0, b_0]$ with the property that $t \to \phi(t, x(t))$ is of bounded variation on T_0.*

In addition, if instead of $H(F)_5$ (3) we have the following sublinear growth condition,

$$(3') \qquad |F(t, x)| \leq \alpha(t) + \beta(t)\|x\| \qquad \text{a.e. with } \alpha(\cdot), \beta(\cdot) \in L^2_+(T)$$

then $b_0 = b$ (i.e. the strong solution exists globally).

Proof. Let $\varepsilon > 0$ be fixed and set $B = \{y \in H: \|y - x_0\| \leq \varepsilon\}$. Then from hypothesis $H(F)_3$ (3), we know that there exists $\psi_B = \psi \in L^2_+(T)$ such that

$$\sup\{|F(t, y)|: y \in B\} \leq \psi(t) \qquad \text{a.e. on } T$$

Let $\bar{x}(\cdot)$ be the unique solution of

$$\dot{\bar{x}}(t) + \partial\phi(t, \bar{x}(t)) \ni 0 \qquad \text{a.e.}$$
$$\bar{x}(0) = x_0$$

(cf. theorem 5.1). Choose $b_0 \in (0, b]$ such that

$$\|\bar{x}(t) - x_0\| + \int_0^t \psi(s)ds \leq \varepsilon \qquad t \in T_0 = [0, b_0]\,.$$

This is always possible, inasmuch as $x(\cdot)$ is continuous and $\psi(\cdot) \in L^2_+(T) \subseteq L^1_+(T)$.

Let $W = \{p \in L^2(T, H): \|p(t)\| \leq \psi(t) \text{ a.e.}\}$. For $p \in W$ consider the following Cauchy problem:

$$\dot{x}(t) + \partial\phi(t, x(t)) \ni p(t) \qquad \text{a.e. on } T_0$$
$$x(0) = x_0$$

By theorem 5.1, the above initial value problem has a unique strong solution $x(p)(\cdot)$. We then have:

$$\begin{aligned}\|x(p)(t) - \bar{x}(t)\| \leq \int_0^t \|p(s)\|\,ds \Rightarrow \|x(p)(t) - x_0\| &\leq \|x(p)(t) - \bar{x}(t)\| + \|\bar{x}(t) - x_0\| \\ &\leq \|\bar{x}(t) - x_0\| + \int_0^t \|p(s)\|\,ds \\ &\leq \|\bar{x}(t) - x_0\| + \int_0^t \psi(s)ds \\ &\leq \varepsilon \qquad t \in [0, b_0]\,.\end{aligned}$$

So if $K = \{x(p)(\cdot): p \in W\}$, then from the above inequality, we have that for all $x(\cdot) \in K$ and all $t \in T_0$, $\|x(t) - x_0\| \leq \varepsilon$.

Next we will show that K is an equicontinuous subset of $C(T_0, H)$. Indeed for any $x(\cdot) \in K$, we have

$$\|x(t') - x(t)\| \leq \int_t^{t'} \|\dot{x}(s)\|\,ds \leq \|\dot{x}\|_2 (t' - t)^{1/2} \qquad 0 \leq t \leq t' \leq b_0\,.$$

But since W is $L^2(T, H)$-bounded, from inequalities (2) in this section, we know that $\|\dot{x}\|_2 \leq \overline{M}$, with $\overline{M} > 0$ independent of $x(\cdot) \in W$. Hence K is equicontinuous.

Now we will show that for each $t \in T_0$, the set $\{y(t): y \in K\}$ is relatively compact in H. Note that for every $y \in K$ and every $t \in T_0$

$$\|y(t)\| \leq \|\bar{x}\|_\infty + \|\psi\|_1 = M_1$$

while from (2) in this section, $\phi(t, y(t)) \leq \overline{M}$ with $\overline{M}$ independent of $y \in K$. So finally we have

$$\|y(t)\|^2 + \phi(t, y(t)) \leq M_1^2 + \overline{M} = M_2\,.$$

Since $\phi(t, \cdot)$ is of compact type (cf. hypothesis $H(\phi)$), we conclude that $\{y(t): y \in K\}$ is relatively compact in H for every $t \in T_0$. Invoking the Arzela-Ascoli theorem, we deduce that K is relatively compact in $C(T_0, H)$. So by Mazur's theorem $\hat{K} = \overline{\text{conv}}\, K$ is compact too. Define $R: \hat{K} \to P_f(L^1(T_0, H))$ by $R(x) = S^1_{F(\cdot, x(\cdot))}$. Clearly $R(\hat{K}) \subseteq W$. As in Papageorgiou [36] (p. 292) (see also the proof of theorem 3.2) we can show that $R(\cdot)$ is l.s.c. Apply Fryszkowski's selection theorem [17], to get $\mu: \hat{K} \to W$ continuous such that $\mu(x) \in R(x)$ for all $x \in \hat{K}$. For each $y \in \hat{K}$ consider the following Cauchy problem

$$\dot{x}(t) + \partial\phi(t, x(t)) \ni -\mu(y)(t) \qquad \text{a.e. on } T_0$$

$$x(0) = x_0$$

Since $\mu(y) \in W$, this has a unique solution $x(y)(\cdot)$ belonging to K. So the map $s(y) = x(y)(\cdot)$ maps $\hat{K}$ into itself. Since $\mu(\cdot)$ is continuous, it is clear that $s(\cdot)$ is continuous too. So applying Schauder's fixed point theorem, we get $x \in K$ such that $s(x) = x$. Let $\mu(x)(\cdot) = f(\cdot) \in L^2(T, H)$ and $f(t) \in F(t, x(t))$ a.e. i.e. $f \in S^2_{F(\cdot, x(\cdot))}$. Also $\dot{x}(t) + \partial\phi(t, x(t)) + f(t) \ni 0$ a.e. on T_0, $x(0) = x_0$; i.e. $x(\cdot)$ solves $(E)_1$.

If we assume the sublinear growth condition $H(F)_5$ (3'), we have for a solution $y(\cdot)$ of $(E)_1$ (recall $\bar{x}(\cdot)$ denotes the solution of $\dot{\bar{x}}(t) + \partial\phi(t, \bar{x}(t)) \ni 0$ a.e. on T_0; $\bar{x}(0) = x_0$)

$$\begin{aligned} \|y(t) - x_0\| &\leq \|y(t) - \bar{x}(t)\| + \|\bar{x}(t) - x_0\| \\ &\leq \|\bar{x}(t) - x_0\| + \int_0^t \|f(s)\| \, ds\,, \qquad f \in S^2_{F(\cdot, y(\cdot))}\,, \\ &\leq \|\bar{x}(t) - x_0\| + \int_0^t (\alpha(s) + b(s)\|y(s)\|) ds\,. \end{aligned}$$

Invoking Gronwall's inequality, we get $\|y(t)\| \leq M_3$, $M_3 > 0$, for all $t \in T$, with M_3 depending on $\|\bar{x}\|_\infty$, $\|x_0\|$, $\|\alpha\|_1$, $\|\beta\|_1$ only. Set $\hat{F}(t, x) = F(t, p_{M_3}(x))$, with $p_{M_3}(\cdot)$ being the M_3-radial retraction. Clearly $\hat{F}(\cdot, \cdot)$ has the same measurability and continuity properties as $F(\cdot, \cdot)$. In addition $|\hat{F}(t, x)| \leq \alpha(t) + \beta(t)M_3 = \hat{\psi}(t)$ a.e. with $\hat{\psi}(\cdot) \in L^2_+(T)$. Replace $\psi(\cdot)$ by $\hat{\psi}(\cdot)$ in the definition of W and consider the sets K and $\hat{K}$ as in the first part of the proof. Reasoning as before with $\hat{F}$ instead of F, we get a solution $y(\cdot)$ for $(E)_1$ with $\hat{F}(\cdot, \cdot)$ instead of $F(\cdot, \cdot)$. It is easy to check that $\|y(t)\| \leq M_3$ $t \in T \Rightarrow \hat{F}(t, y(t)) = F(t, y(t)) \Rightarrow y(\cdot)$ is a strong solution of $(E)_1$. Q.E.D.

Remark. A comparison of theorem 5.2, with theorem 3.1 of Papageorgiou [36], [37], shows that our hypotheses are weaker. Instead of Watanabe's [46] and Yotsutani's [47] hypotheses on $\phi(t, x)$, which were used in [36] and [37] respectively, we have used the more general one by Kubo [24]. Also our restrictions on $F(\cdot, \cdot)$ are weaker than the corresponding ones in [36] and [37].

We can have a "convex" version of this existence result extending this way the corresponding "convex" results of Papageorgiou [36], [37]. The hypothesis on $F(\cdot, \cdot)$ is now the following:

$\underline{H(F)_6}$: $F: T \times H \to P_{fc}(H)$ is a multifunction such that
(1) for every $x \in H$, $t \to F(t, x)$ is measurable,

(2) for every $t \in T$, $x \to F(t, x)$ is u.s.c. from H into H_w,
(3) for every $B \subseteq H$ bounded, there exists $\psi_B(\cdot) \in L_+^2$ such that

$$\sup \{|F(t, x)|: x \in B\} \le \psi_B(t) \qquad \text{a.e.}$$

Theorem 5.3. *If hypotheses $H(\phi)$ and $H(F)_6$ hold, $\phi(t, \cdot)$ is of compact type ($t \in T$) and $x_0 \in \operatorname{dom} \phi(0, \cdot)$, then there exists $b_0 \in T_0$ such that* (E)$_1$ *has a strong solution $x(\cdot)$ on $T_0 = [0, b_0]$ with the property that $t \to \phi(t, x(t))$ is of bounded variation on T_0.*

In addition, if $|F(t, x)| \le \alpha(t) + \beta(t)\|x\|$ a.e. with $\alpha(\cdot), \beta(\cdot) \in L_+^2$, then $b_0 = b$.

Proof. Since the proof is analogous to that of theorem 5.2, we will only present a sketch of it. So let $B \subseteq H$ be as in the proof of theorem 5.2 and let $\bar{x}(\cdot)$ be the unique strong solution of the evolution equation $\dot{\bar{x}}(t) + \partial\phi(t, x(t)) \ni 0$ a.e., $\bar{x}(0) = x_0$. As in the proof of theorem 5.2 choose $b_0 < b$ so that $\|\bar{x}(t) - x_0\| + \int_0^t \psi_B(s)ds \le \varepsilon$, $t \in T_0$. Let $W \subseteq L^2(T, H)$ be defined as before and for $q \in W$ consider $\dot{y}(t) + \partial\phi(t, y(t)) \ni q(t)$ a.e., $y(0) = x_0$. Let $\theta(q)(\cdot)$ be unique solution of this initial value problem. Consider the multifunction $L: W \to 2^{L^2(T,H)}$ defined by $L(q) = S^2_{F(\cdot,\theta(q)(\cdot))}$. Let $s_n(\cdot)$ be simple functions such that $\|s_n(t) - x_0\| \le \varepsilon$ and $s_n(t) \to \theta(q)(t)$ a.e. Then because of hypothesis $H(F)_6$ (1), $t \to F(t, s_n(t))$ $n \ge 1$ has an $L^2(T, H)$-selector $f_n(\cdot)$. By passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^2(T, H)$. Because of hypothesis $H(F)_6$ (2), we have $w\text{-}\overline{\lim}\, F(t, s_n(t)) \subseteq F(t, \theta(q)(t))$ a.e. $\Rightarrow f(t) \in F(t, \theta(q)(t))$ a.e. $\Rightarrow L(q) \neq \varnothing$ for all $q \in W$. In fact it is easy to see that $L(q) \in P_{fc}(W)$. Also exploiting the weak to strong continuity of the map $\theta(\cdot)$ (see for example Papageorgiou [37]) and using theorem 3.1 of [32], we can see that $L(\cdot)$ is u.s.c. from W_w into itself (here W_w denotes the set W with the relative weak $L^2(T, H)$-topology). Apply the Kakutani-KyFan fixed point theorem to get $q_0 \in L(q_0)$. Then $\theta(q_0)(\cdot)$ is the desired solution of (E)$_1$ on T_0. As before it becomes global, if $|F(t, x)|$ satisfies the sublinear growth condition. Q.E.D.

Now we will use theorems 5.2 and 5.3 to study the following Volterra integral inclusion:

$$(\mathrm{V})_1 \qquad x(t) + \int_0^t k(t-s)(\partial\phi(s, x(s)) + F(s, x(s)))ds \ni g(t)\,, \qquad t \in T$$

with $k \in L^1(T)$, $g \in C(T, H)$, $\phi: T \times H \to \overline{\boldsymbol{R}}$ and $F: T \times H \to 2^H$.

By a "strong solution" of (V)$_1$, we mean a function $x(\cdot) \in W^{1,2}(T, H)$, for which there exist w, $f \in L^2(T, H)$, $w(t) \in \partial\phi(t, x(t))$ a.e., $f(t) \in F(t, x(t))$ a.e. and $u(t) + b * (w + f)(t) = g(t)$ for all $t \in T$ (here $*$ stands for the operation of convolution).

As we did with problem (V), we relate problem $(V)_1$ to the following functional-evolution inclusion:

$$(E)_2 \qquad \begin{cases} \dot{x}(t) + \partial\phi(t, x(t)) + F(t, x(t)) \ni G(x)(t) & \text{a.e. on } T \\ x(0) = x_0 \end{cases}$$

where $x_0 \in H$ and $G\colon C(T, H) \to L^2(T, H)$.

By a "strong solution" of $(E)_2$, we mean a function $x(\cdot) \in W^{1,2}(T, H)$ satisfying $x(0) = x_0$ and $\dot{x}(t) + w(t) + f(t) = G(x)(t)$ a.e., where $w, f \in L^2(T, H)$ and $w(t) \in \partial\phi(t, x(t))$ a.e., $f(t) \in F(t, x(t))$ a.e.

The following extension of proposition 3.1 is obvious.

Proposition 5.4. *If $k \in AC(T)$, $\dot{k} \in BV(T)$, $k(0) = 1$, $g \in W^{1,2}(T, H)$ and $x(\cdot)$ is a strong solution of $(V)_1$, then $x(\cdot)$ is a strong solution of $(E)_2$, where*

$$(3) \qquad \begin{cases} \text{(i)} \quad G(x)(t) = h(t) + r * h(t) - r(0)x(t) + r(t)x_0 - x * \dot{r}(t) \\ \qquad\qquad \left(x * \dot{r}(t) = \int_0^t x(t-s)dr(s)\right) \\ \text{(ii)} \quad h(t) = \dot{g}(t)\,, \qquad \text{(iii)} \quad x_0 = g(0)\,, \qquad \text{(iv)} \quad r + a * r = -a\,, \qquad a = \dot{k} \end{cases}$$

Conversely, if $r \in BV(T)$, $h \in L^2(T, H)$, $x_0 \in H$, $G(\cdot)$ is as above and $x(\cdot)$ is a strong solution of $(E)_2$, then $x(\cdot)$ is also a strong solution of $(V)_1$, where

$$(4) \qquad \begin{cases} \text{(i)} \quad g(t) = x_0 + \int_0^t h(s)ds\,, \qquad \text{(ii)} \quad a + a * r = -r \\ \text{(iii)} \quad k(t) = 1 + \int_0^t a(s)ds \end{cases}$$

Now we can state our first existence result for the Volterra integral inclusion $(V)_1$.

Theorem 5.5. *If hypotheses $H(\phi)$ and $H(F)_5$ hold, $\phi(t, \cdot)$ is of compact type $(t \in T)$ and $k \in AC(T)$, $k(0) = 1$, $\dot{k} \in BV(T)$, $g \in W^{1,2}(T, H)$, $g(0) \in \operatorname{dom} \phi(0, \cdot)$ then there exists $b_0 \in (0, b]$ such that $(V)_1$ admits a strong solution on $T_0 = [0, b_0]$.*

In addition, if $|F(t, x)| \leq \alpha(t) + \beta(t)\|x\|$ a.e. with $\alpha(\cdot), \beta(\cdot) \in L^2_+(T)$, then $b_0 = b$.

Proof. In view of proposition 5.4, we have that $(V)_1$ is equivalent to $(E)_2$, with x_0 and $G(\cdot)$ given by (3) (i) → (iv). Then combine the proof of theorem 5.2, with the proof of theorem 3.2, to deduce the existence of a strong solution for $(V)_1$. Note that $G(\cdot)$ takes values in $L^2(T, H)$, since $h = \dot{g} \in L^2(T, H)$ and $G\colon C(T, H) \to L^2(T, H)$ is continuous. So when $|F(t, x)|$ satisfies the sublinear growth condition, we can still use Gronwall's inequality. Remark only that $\|G(x)(t) - G(y)(t)\| \le (|r(0)| + \text{var}(r: [0, t]))\|x - y\|_{L^\infty([0,t],H)}$, for all $x, y \in C(T, H)$ and all $t \in T$. Q.E.D.

Remark. Theorem 5.2 is a special case of theorem 5.5, corresponding to $k \equiv 1$, $g(t) = x_0$ in $(\mathrm{V})_1$.

In a completely analogous manner, using theorems 5.3 and 3.3 we can prove a "convex" version of the above result.

Theorem 5.6. *If hypotheses $H(\phi)$ and $H(F)_6$ hold, $\phi(t, \cdot)$ is of compact type and in addition $k \in AC(T)$, $k(0) = 1$, $\dot{k} \in BV(T)$, $g \in W^{1,2}(T, H)$ and $g(0) \in$* dom $\phi(0, \cdot)$, *then there exists $b_0 \in (0, b]$ such that equation $(\mathrm{V})_1$ has a strong solution on T_0.*

If in addition $|F(t, x)| \le \alpha(t) + \beta(t)\|x\|$ a.e. with $\alpha(\cdot), \beta(\cdot) \in L^2_+$, then $b_0 = b$.

Consider next the "convexified" version of (V_1)

$$(\mathrm{V})_1^c \qquad x(t) + \int_0^t k(t-s)(\partial\phi(s, x(s)) + \overline{\text{conv}}\, F(s, x(s)))ds \ni g(t), \qquad t \in T$$

Denote the solution set of $(\mathrm{V})_1$ by S and that of $(\mathrm{V})_1^c$ by S_c. Employing the technique of the proof of theorem 4.1, we get the following relaxation result.

Theorem 5.7. *If hypotheses $H(\phi)$ and $H(F)_3$ hold, $\phi(t, \cdot)$ is of compact type and in addition $k \in AC(T)$, $k(0) = 1$, $\dot{k} \in BV(T)$, $g \in W^{1,2}(T, H)$, $g(0) \in$* dom $\phi(0, \cdot)$, *then S_c is nonempty, compact in $C(T, H)$ and $S_c = \bar{S}$, the closure taken in $C(T, H)$.*

The last result of this section is a stability theorem for equation $(\mathrm{V})_1$. Consider the following sequence of Volterra integral inclusions

$$(\mathrm{V})_1^n \qquad x_n(t) + \int_0^t k(t-s)(\partial\phi_n(s, x_n(s)) + F_n(s, x_n(s)))ds \ni g(t), \qquad t \in T$$

along with $(\mathrm{V})_1$. We assume that the hypotheses of theorem 5.6 are in force. In addition we need the following hypotheses:

$\underline{H(\phi)_1}$: $\phi_n: T \times H \to \bar{\boldsymbol{R}} = \boldsymbol{R} \cup \{+\infty\}$, $n \ge 1$ are functions such that

(1) for every $t \in T$, $\{\phi_n(t, \cdot)\}_{n\ge1} \subseteq \Gamma_0(H)$ and is equi-compact, i.e. for each $\lambda \in \boldsymbol{R}$, $\bigcup_{n\ge1} \{x \in H: \|x\|^2 + \phi_n(t, x) \le \lambda\}$ is relatively compact in H,

(2) hypotheses $H(\phi_n)$ (1) and (4) hold uniformly in $n \ge 1$, hypothesis $H(\phi_n)$ (3) holds and $r_1(t) + \lambda_1\|x\|^2 \le \phi_n(t, x) \le r_2(t) + \lambda_2\|x\|^2$ a.e. $n \ge 1$ with $r_1, r_2 \in L^2(T)$, $\lambda_1, \lambda_2 \ge 0$,

(3) $\phi_n(t, \cdot) \xrightarrow{e} \phi(t, \cdot)$ a.e.

$\underline{H(F)_7}$: F_n, $F: T \times H \to P_{fc}(H)$, $n \geq 1$ are multifunctions such that
(1) for every $x \in H$, $t \to F_n(t, x)$ is measurable
(2) for every $t \in T$, $x \to F_n(t, x)$ is u.s.c. from H into H_w,
(3) if $x_n \xrightarrow{s} x$, $w\text{-}\overline{\lim}\, F_n(t, x_n) \subseteq F(t, x)$,
(4) $|F_n(t, x)| \leq \alpha(t) + \beta(t)\|x\|$ a.e. with $\alpha(\cdot)$, $\beta(\cdot) \in L^2_+(T)$.

Theorem 5.8. *Let $H(\phi)$, $H(\phi)_1$, $H(F)_7$ be satisfied. Also assume that $k \in AC(T)$, $k(0) = 1$ $\dot{k} \in B(V(T)$, $g \in W^{1,2}(T, H)$, $g(0) \in \operatorname{dom} \phi(0, \cdot)$. If $x_n(\cdot)$ $n \geq 1$ are solutions of $(\mathrm{V})_1^n$, then there exists a subsequence $x_{n_k}(\cdot)$ such that $x_{n_k} \to x$ in $C(T, H)$ as $k \to \infty$, where $x(\cdot)$ is a solution of $(\mathrm{V})_1$.*

Proof. Note that the existence of $x_n(\cdot) \in C(T, H)$, $n \geq 1$ is guaranteed by theorem 5.5. Again we consider the equivalent function-evolution inclusion. From the proof of theorem 5.2, we know that $\{x_n\}_{n \geq 1}$ is relatively compact in $C(T, H)$. So by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $C(T, H)$. By definition we have

$$\dot{x}_n(t) + \partial\phi_n(t, x_n(t)) + f_n(t) \ni G(x_n)(t) \qquad \text{a.e.}$$
$$x_n(0) = x_0$$

with G and x_0 given by proposition 5.4 and with $f_n \in S^1_{F_n(\cdot, x_n(\cdot))}$. Because of hypothesis $H(F)_7$ (4), we may assume that $f_n \xrightarrow{w} f$ in $L^2(T, H)$. So combining theorem 3.1 of [32] and hypothesis $H(F)_7$ (3), we get that $f(t) \in \overline{\operatorname{conv}}\, w\text{-}\overline{\lim}\, F_n(t, x_n(t)) \subseteq F(t, x(t))$ a.e. $\Rightarrow f \in S^1_{F(\cdot, x(\cdot))}$. Also $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^2(T, H)$ and $G(x_n) \xrightarrow{s} G(x)$ in $L^1(T, H)$. Note that $-\dot{x}_n - f_n + G(x_n) \in \partial\Phi_n(x_n)$ $\left(\Phi_n(x) = \int_0^b \phi_n(t, x(t))dt \text{ if } \phi_n(\cdot, x(\cdot)) \in L^1(T), \ +\infty \text{ otherwise}\right)$. But from hypothesis $H(\phi)_1$ and theorem 3.1 of Salvadori [41], we have $\Phi_n \xrightarrow{e} \Phi$ (where Φ is defined in the same way as Φ_n with $\phi_n(t, x)$ replaced by $\phi(t, x)$). So $\partial\Phi_n \xrightarrow{K-M} \partial\Phi \Rightarrow -\dot{x} - f + G(x) \in \partial\Phi(x) \Rightarrow \dot{x}(t) + \partial\phi(t, x(t)) + f(t) \ni G(x)(t)$ a.e., $x(0) = x_0 \Rightarrow x(\cdot)$ solves $(\mathrm{V})_1$. Q.E.D.

Remark. If S_n = the solution set of $(\mathrm{V})_1^n$ and S = the solution set of $(\mathrm{V})_1$, we have proved that $\overline{\lim}\, S_n \subseteq S$ and $\bigcup_{n \geq 1} S_n$ is compact in $C(T, H)$. It will be interesting to know if under appropriate additional hypotheses on the data, we can have $S_n \xrightarrow{h} S$.

Finally note that our work in this section covers the case where $\partial\phi(t, x) = N(x)_{K(t)}$ (the normal cone to $K(t)$ at x; see Aubin-Cellina [6]), with $t \to K(t)$ being Hausdorff Lipschitz. This problem has important applications in theoretical mechanics (see Moreau [27]) and in mathematical economics and control theory (see Aubin-Cellina [6], chapter 5; in [6] $K(t) = K$, i.e. is time independent).

Acknowledgment. The authors are indebted to the referee for his (her) constructive criticism.

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(Ricevita la 14-an de februaro, 1991)
(Reviziita la 20-an de julio, 1991)