

Almost Periodically Forced Pendulum

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1. Introduction

In the present work, our aim is to study the a.p. (almost periodic) solutions of the pendulum equation in presence of an a.p. forcing term:

$$(\mathcal{P}) \quad \ddot{x} + \sin x(t) = e(t).$$

In assuming that e is the second derivative of an a.p. function E , the equation (\mathcal{P}) is equivalent to the following:

$$(\mathcal{P}') \quad \ddot{u}(t) + \sin(u(t) + E(t)) = 0.$$

When e is periodic, there exist numerous classical and recent works on this equation ([14]), and among these, there exist variational view-points to prove the existence of periodic solutions ([14]; [15] paragraph 1.6).

When e and E are a.p., we have also a variational view-point about (\mathcal{P}') ; the a.p. solutions of (\mathcal{P}') are exactly the critical points of the functional in mean time:

$$J_0(u) := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[\frac{1}{2} (\dot{u}(t))^2 + \cos(u(t) + E(t)) \right] dt.$$

More generally, the a.p. solutions of Euler-Lagrange equations or hamiltonian systems can be considered as the critical points of some appropriate functionals in mean time. The study of such functionals is called the calculus of variations in mean time. This view-point has permitted to discover new results on the lagrangian systems when the lagrangian is convex ([3]; [4]; [5]; [6]).

The domain of the functional J_0 is a space of a.p. functions of class C^1 . The compactness in such a space is very difficult to exhibit, because of the analogous of the Ascoli-Arzelà theorem for the a.p. functions, so-called the Lusternik theorem ([13], p. 7), contains a condition of equi-almost periodicity that is practically unverifiable. To use some weak compactness, in the

spirit of the direct methods of the Calculus of Variations, we shall work on a Hilbert space, whose definition needs a notion of generalized derivative (paragraph 2), and we shall extend J_0 to this Hilbert space. This generalized derivative, special to the almost periodicity, induces a notion of weak a.p. solution of ordinary differential equations. This method has been used previously for the convex lagrangians in [7]. In this Hilbert space, inspired by the Sobolev spaces, it is not possible to expect to have an analogous of the Rellich-Kondratchov theorem of the usual Sobolev spaces ([8], p. 162). It is another absence of compactness that induces some difficulties to build functionals that are weakly lower semicontinuous. To avoid this problem of semicontinuity, we shall reduce the problem of the research of weak solutions of (\mathcal{P}') to resolve a convex problem of minimization (paragraph 3).

In paragraph 4, Theorem 1 is a result of existence of weak a.p. solution of (\mathcal{P}) , Theorem 2 concerns the usual a.p. solutions of (\mathcal{P}) , Theorem 3 is a result of uniqueness for usual a.p. solutions of (\mathcal{P}) , and Theorem 4 is a result of existence and uniqueness for the periodic case.

2. Notations

For each $k \in \mathbb{N}$, AP^k denotes the space of the functions of class C^k from R into R which are, as also all their derivatives until order k , Bohr a.p. The norm of AP^0 is $\|f\|_\infty := \sup \{|f(t)|; t \in R\}$, and the norm of AP^k , when $k \geq 1$, is $\|f\|_{C^k} := \sum_{\alpha=0}^k \left\| \frac{d^\alpha f}{dt^\alpha} \right\|_\infty$.

If $T > 0$, C_T^k denotes the space of the T -periodic functions of class C^k from R into R . If $p > 0$, L_{loc}^p is the space of the (classes of) functions f from R into R such that $|f|^p$ is locally Lebesgue-integrable. H_{loc}^1 is the Sobolev space of the functions from R into R which are locally A.C. (absolutely continuous) with a derivative in L_{loc}^2 . H_T^1 is the Sobolev space of the elements of H_{loc}^1 which are T -periodic.

Let $f: R \rightarrow R$, its means value (when it exists) (resp. upper mean value, resp. lower mean value) is denoted by $\mathcal{M}\{f(t)\}_t$ (resp. $\overline{\mathcal{M}}\{f(t)\}_t$, $\underline{\mathcal{M}}\{f(t)\}_t$); it is the limit (resp. lim sup, resp. lim inf) of the $\frac{1}{2T} \int_{-T}^T f(t) dt$, when $T \rightarrow \infty$.

Now, we introduce the spaces of Besicovitch-a.p. functions ([2] chapter II). \mathcal{B}^p is the completion of AP^0 for the semi-norm $f \rightarrow (\overline{\mathcal{M}}\{|f(t)|^p\}_t)^{1/p}$; when $f, g \in \mathcal{B}^p$, $f \sim_p g$ means that $\mathcal{M}\{|f - g|^p\} = 0$, and the class of f relatively to \sim_p is denoted by $[f]_p$; $B^p := \mathcal{B}^p / \sim_p$. In practice, we do not distinguish between f and $[f]_p$. The norm of B^p is $\|f\|_p := (\mathcal{M}\{|f(t)|^p\}_t)^{1/p}$. Endowed with the inner product $(f|g) := \mathcal{M}\{f(t)g(t)\}_t$, B^2 is a Hilbert space. If f is

Bohr-a.p. or Besicovitch-a.p., and if $\lambda \in R$, the complex numbers $a(f; \lambda) := \mathcal{M}\{f(t)e^{-i\lambda t}\}_t$ are the Fourier-Bohr coefficients of f .

When $f: R \rightarrow R$ and $s \in R$, the translation operator is denoted by $\tau_s f(t) := f(t + s)$.

In [7] we have described the generalized derivative of Vo-Khac in \mathcal{B}^2 : if $f \in \mathcal{B}^2$, $\nabla f := \lim_{s \rightarrow 0} \frac{1}{s}(\tau_s f - f)$ in \mathcal{B}^2 when it exists. $\mathcal{B}^{1,2} := \{f \in \mathcal{B}^2; \nabla f \text{ exists into } \mathcal{B}^2\}$. Inductively, $\nabla^2 f := \nabla(\nabla f)$ and $\mathcal{B}^{2,2} := \{f \in \mathcal{B}^2; \nabla f \text{ and } \nabla^2 f \text{ exist into } \mathcal{B}^2\}$. Endowed with the inner product $\langle f|g \rangle := (f|g) + (\nabla f|\nabla g)$ (resp. $\langle\langle f|g \rangle\rangle := (f|g) + (\nabla f|\nabla g) + (\nabla^2 f|\nabla^2 g)$), $B^{1,2}$ (resp. $B^{2,2}$) is a Hilbert space and the associated enclidean norm is denoted by $\|f\|_{1,2}$ (resp. $\|f\|_{2,2}$).

A function $u \in \mathcal{B}^{2,2}$, such that $\nabla^2 u + \sin u \sim_2 e$, is called a weak a.p. solution of (\mathcal{P}) .

If $n \in N, n \neq 0, K_n(t) := \sum_{|v| < n} \left(1 - \frac{|v|}{n}\right) e^{-ivt}$, and if $I = (n_1, \dots, n_p; \beta_1, \dots, \beta_p)$ where the $n_j \in N, n_j \neq 0$, and the β_j are \mathcal{L} -linearly independent real numbers, the kernel of Bochner-Fejer is $K_I(t) := \prod_{j=1}^p K_{n_j}(\beta_j t)$. If $f \in AP^0$, the polynomial of Bochner-Fejer associated to f and to the multi-index I is $\sigma_I(t) := \mathcal{M}\{f(t + s)K_I(s)\}_s$, ([2] paragraph 9).

Finally, if X is a subset of a vector space V , $\text{lin } X$ (resp. $\text{aff. } X$) will denote the linear hull (resp. affine hull) of X into V .

3. Some preliminary results

In this section we fix $e \in AP^0$ such that there exists $E \in AP^2$ satisfying $\ddot{E} = e$, and we consider the equation of the pendulum forced by e : $\ddot{x}(t) + \sin x(t) = e(t)$. In setting $u := x - E$, this equation is equivalent to: $\ddot{u}(t) + \sin(u(t) + E(t)) = 0$. In order to treat this last equation by a variational way, we introduce the following functional on $\mathcal{B}^{1,2}$: $J(u) := \mathcal{M}\{\frac{1}{2}(\nabla u)^2 + \cos(u + E)\}$, and J_0 denotes the restriction of J at AP^1 .

Proposition 1.

- i) J is Fréchet- C^1 on $\mathcal{B}^{1,2}$ and, for each $u, h \in \mathcal{B}^{1,2}$, $J'(u) \cdot h = \mathcal{M}\{\nabla u \cdot \nabla h - \sin(u + E)h\}$.
- ii) Let $u \in \mathcal{B}^{1,2}$; we have $J'(u) = 0$ if and only if $u \in \mathcal{B}^{2,2}$ and $\nabla^2 u + \sin(u + E) \sim_2 0$.
- iii) J_0 is Fréchet- C^2 on AP^1 and for each $u, h, k \in AP^1$, $J'_0(u) \cdot h = \mathcal{M}\{\dot{u}\dot{h} - \sin(u + E)h\}$, $J''_0(u) \cdot (h, k) = \mathcal{M}\{\dot{h}\dot{k} - \cos(u + E)hk\}$.
- iv) Let $u \in AP^1$, we have $J'_0(u) = 0$ if and only if $u \in AP^2$ and $\ddot{u} + \sin(u + E) = 0$.

Proof.

- i) We split J into two functionals: $K(u) := \mathcal{M}\{\frac{1}{2}(\nabla u)^2\}$ and $L(u) := \mathcal{M}\{\cos(u + E)\}$, $J = K + L$. The operator of derivation $\mathcal{V}: \mathcal{B}^{1,2} \rightarrow \mathcal{B}^2$ is linear continuous, hence it is Fréchet- C^∞ , and $Q(v) = \mathcal{M}\{\frac{1}{2}(v)^2\}$ is quadratic on \mathcal{B}^2 , hence it is Fréchet- C^∞ . And so $K = Q \circ \mathcal{V}$ is Fréchet- C^∞ on $\mathcal{B}^{1,2}$ and its differentials are: $K'(u) \cdot h = Q'(\mathcal{V}u) \cdot \mathcal{V}(u) \cdot h = Q'(\mathcal{V}u) \cdot \mathcal{V}h = \mathcal{M}\{\mathcal{V}u \cdot \mathcal{V}h\}$ and $K''(u)(h, k) = \mathcal{M}\{\mathcal{V}h \cdot \mathcal{V}k\}$. We shall show that L is Fréchet- C^1 on \mathcal{B}^2 that will imply that it is Fréchet- C^1 on $\mathcal{B}^{1,2}$ since the inclusion of $\mathcal{B}^{1,2}$ into \mathcal{B}^2 is linear continuous. The inequality $|\cos(u + E) - \cos(v + E)| \leq |u - v|$ and the fact that $\cos(v + E) \in AP^0$ when $v \in AP^0$ ([2], p. 3) ensure that $\cos(u + E) \in \mathcal{B}^2$ when $u \in \mathcal{B}^2$. Moreover, $u \sim_2 v$ implies $\cos(u + E) \sim_2 \cos(v + E)$, and $|L(u) - L(v)| \leq \mathcal{M}\{|u - v|\} \leq \|u - v\|_2$, and so L is 1-lipschitzian on \mathcal{B}^2 , hence its continuity on \mathcal{B}^2 . By the same reasoning, we see that $\sin(u + E) \in \mathcal{B}^2$ when $u \in \mathcal{B}^2$ and $\sin(u + E) \sim_2 \sin(v + E)$ when $u \sim_2 v$.

By the mean value theorem ([12] p. 103) we have: $|\cos(u(t) + \theta h(t) + E(t)) - \cos(u(t) + E(t)) + \sin(u(t) + E(t))\theta h(t)| \leq |\theta h(t)| \cdot \sup\{|\sin(\xi + E(t)) - \sin(u(t) + E(t))|; \xi \in [u(t), u(t) + \theta h(t)]\} \leq |\theta h(t)| \cdot \sup\{|\xi - u(t)|; \xi \in [u(t), u(t) + \theta h(t)]\} = |\theta h(t)|^2$, therefore $\left| \mathcal{M}\left\{\frac{1}{\theta}(\cos(u + \theta h + E) - (\cos(u + E) + \sin(u + E)h))\right\} \right| \leq |\theta| \|h\|_2^2$, and $\lim_{\theta \rightarrow 0} \frac{1}{\theta}(L(u + \theta h) - L(u)) = \mathcal{M}\{-\sin(u + E)h\}$. And so the directional derivative of L at u , in the direction h , exists, it is $\vec{D}_h L(u) = -\mathcal{M}\{\sin(u + E) \cdot h\}$. $h \rightarrow \vec{D}_h L(u)$ is clearly linear, and by the Cauchy-Schwarz inequality we have $|\vec{D}_h L(u)| \leq \|\sin(u + E)\|_2 \cdot \|h\|_2$, hence its continuity. Then ([10] p. 255), we can assert that L is Gâteaux-differentiable at u , and its Gâteaux-differential is $D_G L(u) \cdot h = -\mathcal{M}\{\sin(u + E) \cdot h\}$. As the function \sin is 1-lipschitzian by Cauchy-Schwarz, we obtain: $|D_G L(u) \cdot h - D_G L(v) \cdot h| \leq \|\sin(u + E) - \sin(v + E)\|_2 \cdot \|h\|_2 \leq \|u - v\|_2 \cdot \|h\|_2$, and so, in norm of operators, we have $\|D_G L(u) - D_G L(v)\| \leq \|u - v\|_2$. And so, $D_G L$ is continuous from \mathcal{B}^2 into the dual space $(\mathcal{B}^2)^*$, and therefore, L is Fréchet- C^1 ([10] p. 257) and its differential is $L'(u) \cdot h = -\mathcal{M}\{\sin(u + E) \cdot h\}$. Consequently L is also Fréchet- C^1 on $\mathcal{B}^{1,2}$, and therefore J is Fréchet- C^1 on $\mathcal{B}^{1,2}$ and $J'(u) \cdot h = K'(u) \cdot h + L'(u) \cdot h = \mathcal{M}\{\mathcal{V}u \cdot \mathcal{V}h\} - \mathcal{M}\{\sin(u + E) \cdot h\}$.

- ii) If we assume that $u \in \mathcal{B}^{2,2}$ and $\mathcal{V}^2 u + \sin(u + E) \sim_2 0$, because of $\mathcal{M}\{\mathcal{V}^2 u \cdot h\} = -\mathcal{M}\{\mathcal{V}u \cdot \mathcal{V}h\}$ ([7] Prop. 9) and (i), we have: $J'(u) \cdot h = \mathcal{M}\{-\mathcal{V}^2 u \cdot h - \sin(u + E) \cdot h\} = \mathcal{M}\{-\mathcal{V}^2 u + \sin(u + E)\} \cdot h = \mathcal{M}\{0\} = 0$. Conversely, we assume that $J'(u) = 0$, and so, for each $h \in \mathcal{B}^{1,2}$, by (i), we have $0 = J'(u)h = \mathcal{M}\{\mathcal{V}u \cdot \mathcal{V}h - \sin(u + E)h\}$. Therefore for every $h \in AP^1$ we have $\mathcal{M}\{\sin(u + E) \cdot h\} = \mathcal{M}\{\mathcal{V}u \cdot h\}$; then, by Proposition 10 of [7], we know that $\mathcal{V}u \in \mathcal{B}^{1,2}$, i.e. $u \in \mathcal{B}^{2,2}$, and $\mathcal{V}^2 u \sim_2 -\sin(u + E)$.

iii) We denote $k_0(u) := \mathcal{M}\{\frac{1}{2}(\dot{u})^2\}$, $L_0(u) := \mathcal{M}\{\cos(u + E)\}$; $J_0 = K_0 + L_0$. It is clear that K_0 is of class C^∞ on AP^1 since it is a continuous quadratic functional on AP^1 . As the inclusion from AP^1 into AP^0 is linear continuous, if we prove that L_0 is of class C^2 on AP^0 , then necessarily L_0 will be of class C^2 on AP^1 . Since $E \in AP^2$, if we prove that $M(u) := \mathcal{M}\{\cos(u)\}$ is of class C^2 on AP^0 then $L_0(u) = M(u + E)$ will be also of class C^2 because the translation $u \rightarrow u + E$ is of class C^∞ ; moreover $L'_0(u) \cdot h = M'(u + E) \cdot h$, and $L''_0(u)(h, k) = M''(u + E)(h, k)$.

We consider the Nemytski operator built on the function \cos , $N: AP^0 \rightarrow AP^0$, $N(u)(t) := \cos u(t)$. By the mean value theorem ([12] p. 103) we have:

$|\cos(u(t) + h(t)) - \cos(u(t)) + \sin(u(t)) \cdot h(t)| \leq |h(t)| \cdot \sup\{|\sin \xi - \sin u(t)|; \xi \in [u(t), u(t) + h(t)]\} \leq |h(t)| \cdot \sup\{|\xi - u(t)|; \xi \in [u(t), u(t) + h(t)]\} = |h(t)|^2$, therefore, in taking the supremums over R , it comes: $\|N(u + h) - N(u) + \sin(u) \cdot h\|_\infty \leq \|h\|_\infty^2$, and since $\|h\|_\infty^2 = 0(\|h\|_\infty)$ we can assert that N is Fréchet-differentiable at u and that $N'(u) \cdot h = -\sin(u) \cdot h$. In norm of operators, $\|N'(u) - N'(v)\| = \sup\{\|N'(u) \cdot h - N'(v) \cdot h\|_\infty; \|h\|_\infty \leq 1\} = \sup\{\|N'(u) - N'(v)\|_\infty \cdot \|h\|_\infty; \|h\|_\infty \leq 1\} = \|N'(u) - N'(v)\|_\infty = \|\sin u - \sin v\|_\infty \leq \|u - v\|_\infty$, therefore N' is continuous from AP^0 into the dual space $(AP^0)^*$, and N is of class C^1 on AP^0 .

We fix $u, h, k \in AP^0$, and we use the mean value theorem to write:

$|-\sin(u(t) + h(t)) \cdot k(t) + \sin(u(t)) \cdot k(t) + \cos(u(t))k(t)h(t)| \leq |-\sin(u(t) + h(t)) + \sin u(t) + \cos(u(t)) \cdot h(t)| \cdot \|k\|_\infty \leq |h(t)| \cdot \sup\{|\cos \xi - \cos u(t)|; \xi \in [u(t), u(t) + h(t)]\} \cdot \|k\|_\infty \leq |h(t)|^2 \cdot \|k\|_\infty$, therefore, in taking the supremums over R , we have: $\|N'(u + h)k - N'(u)k + \cos(u)kh\|_\infty \leq \|h\|_\infty^2 \|k\|_\infty$; then, in norm of operators: $\|N'(u + h) - N'(u) + \cos(u)h\| \leq \|h\|_\infty^2$, and as $\|h\|_\infty^2 = 0(\|h\|_\infty)$, it comes that N' is Fréchet-différentiable at u and that $N''(u)(h, k) = -\cos(u)hk$. Now, we remark that $M = \mathcal{M} \circ N$, and that \mathcal{M} is linear continuous, hence of class C^∞ , therefore M is of class C^2 , and $M'(u) \cdot h = \mathcal{M}'(N(u)) \cdot N'(u) \cdot h = \mathcal{M}\{N'(u)h\} = \mathcal{M}\{-\sin(u) \cdot h\}$, $M''(u) \cdot (h, k) = \mathcal{M}\{N''(u)(h, k)\} = \mathcal{M}\{-\cos(u)hk\}$. After that its is easy to deduce (iii).

iv) If $u \in AP^2$ verifies $\ddot{u} + \sin(u + E) = 0$, in using the formula: $\mathcal{M}\{\ddot{u} \cdot h\} = -\mathcal{M}\{\dot{u} \cdot \dot{h}\}$; for $h \in AP^1$, and in using (iii) we have $J'_0(u) \cdot h = \mathcal{M}\{\dot{u}\dot{h} - \sin(u + E)h\} = \mathcal{M}\{-(\ddot{u} + \sin(u + E)) \cdot h\} = \mathcal{M}\{0\} = 0$. Conversely, we assume that $J'_0(u) = 0$, therefore $J'_0(u) \cdot h = \mathcal{M}\{\dot{u}\dot{h} - \sin(u + E) \cdot h\} = 0$. Then, in using the almost periodic distributions of Schwartz as in the proof of Theorem 1 of [3], we can conclude that $u \in AP^2$ and that $\ddot{u} + \sin(u + E) = 0$. ■

Now, we introduce some particular subsets of the considered functional spaces:

$$\left\{ \begin{array}{l} \mathcal{U}_1 := \left\{ u \in AP^1; \forall t \in R \quad \frac{\pi}{2} \leq u(t) + E(t) \leq \frac{3\pi}{2} \right\} \\ \mathcal{U}_2 := \left\{ u \in AP^1; \forall t \in R \quad -\frac{\pi}{2} \leq u(t) + E(t) \leq \frac{5\pi}{2} \right\} \\ \mathcal{C}_1 := \text{the closure of } \mathcal{U}_1 \text{ into } \mathcal{B}^{1,2} \\ \mathcal{C}_2 := \text{the closure of } \mathcal{U}_2 \text{ into } \mathcal{B}^{1,2} \end{array} \right.$$

It is easy to verify that these four sets are convex sets.

Proposition 2. *We assume that $\text{Osc}(E) < \frac{\pi}{2}$; then we have*

$$\text{Inf } J(\mathcal{U}_1) = \text{Inf } J(\mathcal{U}_2).$$

Proof. We fix $u \in \mathcal{U}_2$ and a real number ε such that $0 < \varepsilon \leq \frac{1}{2} \left(\frac{\pi}{2} - \text{Osc}(E) \right)$. The sketch of the proof consists in the building of an element $w \in \mathcal{U}_1$ such that $J(w) \leq J(u) + \varepsilon$.

We introduce $S_\varepsilon := -\sup E(R) + \frac{3\pi}{2} - \varepsilon$ and $I_\varepsilon := -\text{Inf } E(R) + \frac{\pi}{2} + \varepsilon$.

With the constraint on ε , we have $S_\varepsilon \geq I_\varepsilon$. We define a function $v: R \rightarrow R$ in setting:

$$(1) \quad v(t) := \max \{ \min \{ u(t), S_\varepsilon \}, I_\varepsilon \}.$$

In fact, $v(t) = u(t)$ when $I_\varepsilon \leq u(t) \leq S_\varepsilon$, $v(t) = S_\varepsilon$ when $u(t) \geq S_\varepsilon$, and $v(t) = I_\varepsilon$ when $u(t) \leq I_\varepsilon$. We obtain the following relations:

$$(2) \quad \left\{ \begin{array}{l} \frac{\pi}{2} + \varepsilon \leq v(t) + E(t) \leq \frac{3\pi}{2} - \varepsilon, \quad \text{for all } t \in R \\ v(t) + E(t) \geq \pi + \varepsilon, \quad \text{when } u(t) \geq S_\varepsilon \\ v(t) + E(t) \leq \pi - \varepsilon, \quad \text{when } u(t) \leq I_\varepsilon \end{array} \right.$$

In order to justify these relations, we remark that, when $u(t) \geq S_\varepsilon$, we have $v(t) + E(t) = S_\varepsilon + E(t) = E(t) - \sup E(R) + \frac{3\pi}{2} - \varepsilon \geq -\text{Osc}(E) + \frac{3\pi}{2} - \varepsilon \geq \left(2\varepsilon - \frac{\pi}{2} \right) + \frac{3\pi}{2} - \varepsilon = \pi + \varepsilon$, and when $u(t) \leq I_\varepsilon$, we have $v(t) + E(t) = I_\varepsilon + E(t) = E(t) - \text{Inf } E(R) + \frac{\pi}{2} + \varepsilon \leq \text{Osc}(E) + \frac{\pi}{2} + \varepsilon \leq \left(\frac{\pi}{2} - 2\varepsilon \right) + \frac{\pi}{2} + \varepsilon = \pi - \varepsilon$. The other inequalities are easy to verify.

The following step consists in the verification of the assertion:

$$(3) \quad \text{for all } t \in R, \quad \cos(u(t) + E(t)) \geq \cos(v(t) + E(t)).$$

When $I_\varepsilon \leq u(t) \leq S_\varepsilon$ this inequality is evident, since $u(t) = v(t)$. When $u(t) \geq S_\varepsilon$, we have $u(t) + E(t) \geq S_\varepsilon + E(t) = v(t) + E(t)$; then we distinguish between two cases: $u(t) + E(t) \geq \frac{3\pi}{2}$ and $u(t) + E(t) \leq \frac{3\pi}{2}$. In the first case, because of $u(t) + E(t) \leq \frac{5\pi}{2}$ and (2) we have $\cos(u(t) + E(t)) \geq 0 \geq \cos(v(t) + E(t))$.

In the second case, after (2), we have $\pi + \varepsilon \leq v(t) + E(t) \leq u(t) + E(t) \leq \frac{3\pi}{2}$ and the function \cos is increasing on $\left[\pi, \frac{3\pi}{2}\right]$, then $\cos(v(t) + E(t)) \leq \cos(u(t) + E(t))$.

When $u(t) \leq I_\varepsilon$, we have $u(t) + E(t) \leq I_\varepsilon + E(t) = v(t) + E(t)$; and then, as above, we consider two different cases: $u(t) + E(t) \leq \frac{\pi}{2}$ and $u(t) + E(t) \geq \frac{\pi}{2}$. In the first case, since $u(t) + E(t) \geq -\frac{\pi}{2}$, and using (2), we obtain: $\cos(u(t) + E(t)) \geq 0 \geq \cos(v(t) + E(t))$. In the second case, after (2), we have: $\frac{\pi}{2} \leq u(t) + E(t) \leq v(t) + E(t) \leq \pi - \varepsilon$, and the function \cos is decreasing on $\left[\frac{\pi}{2}, \pi\right]$, then $\cos(u(t) + E(t)) \geq \cos(v(t) + E(t))$. That justifies (3).

In considering the definition of the almost periodicity of H. Bohr ([2] chapter I), and in using the triangular inequality, it is clear that: $f \in AP^0 \Rightarrow |f| \in AP^0$. Therefore, because of the classical formulas: $\max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$ and $\min\{f, g\} = \frac{1}{2}(f + g - |f - g|)$, we see that the vector space AP^0 is a vector lattice. Since u , S_ε , and I_ε are elements of AP^0 , after (1), we can say that:

$$(4) \quad v \in AP^0$$

Consequently, $t \rightarrow \cos(v(t) + E(t))$ is almost periodic and so possesses a mean value; and so, (3) and the monotonicity of the mean value imply:

$$(5) \quad \mathcal{M}\{\cos(u(t) + E(t))\}_t \geq \mathcal{M}\{\cos(v(t) + E(t))\}_t.$$

Now, let's pass on to the study of the derivative of v . In considering the definition of the A.C. (absolutely continuous) functions ([16] p. 145) and the triangular inequality, it is clear that: f is A.C $\Rightarrow |f|$ is A.C.

Therefore the vector space of the locally $A.C$ functions from R into R is a vector lattice. Since u , S_e , I_e are of class C^1 on R , they are locally $A.C$, and after (1), we can say that v is locally $A.C$, and so $\dot{v} \in L^1_{loc}(R)$. By an explicit calculation, we obtain that $\dot{v}(t) = \dot{u}(t)$ when $I_e < u(t) < S_e$, and $\dot{v}(t) = 0$ otherwise. And so $|\dot{v}|^2 \leq |\dot{u}|^2$, and since $\dot{u} \in L^2_{loc}(R)$, we can assert that:

$$(6) \quad \dot{v} \in L^2_{loc}(R) \quad \text{and} \quad v \in H^1_{loc}(R).$$

The inequality $|\dot{v}|^2 \leq |\dot{u}|^2$ and (6) imply:

$$(7) \quad \mathcal{M}\{|\dot{v}|^2\} \leq \mathcal{M}\{|\dot{u}|^2\}$$

We shall not try to endeeper the study of \dot{v} , for instance in order to know if \dot{v} is an element of \mathcal{B}^2 or not; we shall approximate v by an element of \mathcal{U}_1 in using the regularization by the convolution in mean.

Let us consider an arbitrary kernel of Bochner-Fejer K_I and the associated polynomial $\sigma_I(t) := \mathcal{M}\{K_I(s)v(t+s)\}_s = \mathcal{M}\{K_I(s-t)v(s)\}_s$. We use the Taylor formula in the following form:

$$K_I(s-t-h) = K_I(s-t) - \dot{K}_I(s-t)h + \left(\int_0^1 (1-\zeta)\ddot{K}_I(s-t-\zeta h)d\zeta \right) h^2$$

(cf. [12] p. 110).

That permits us to write:

$$\begin{aligned} & \left| \frac{\sigma_I(t+h) - \sigma_I(t)}{h} + \mathcal{M}\{\dot{K}_I(s-t)v(s)\}_s \right| \\ &= \left| \mathcal{M}\left\{ \left(\int_0^1 (1-\zeta)\ddot{K}_I(s-t-\zeta h)d\zeta \right) hv(s) \right\}_s \right| \\ &\leq \|\ddot{K}_I\|_\infty. \end{aligned}$$

$\mathcal{M}\{|v|\} \cdot |h|$, and so, when $h \rightarrow 0$, we obtain:

$$(8) \quad \dot{\sigma}_I(t) = -\mathcal{M}\{\dot{K}_I(s-t)v(s)\}_s \quad \text{for all } t \in R.$$

Since $v \in H^1_{loc}(R) \cap AP^0$, the formula of the integration by parts gives us the relation:

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T K_I(s-t)\dot{v}(s)ds &= -\frac{1}{2T} \int_{-T}^T \dot{K}_I(s-t)v(s)ds \\ &+ \frac{1}{2T} (K_I(T-t)v(T) - K_I(-T-t)v(-T)) \end{aligned}$$

and since $K(\cdot - t)v \in AP^0$ it is bounded on R , therefore the last term of

this equation converges towards 0 when $T \rightarrow \infty$; since $\dot{K}(\cdot - t)v \in AP^0$, its mean value exists and so: $\mathcal{M}\{K_I(s - t)\dot{v}(s)\}_s$ exists and is equal to $-\mathcal{M}\{\dot{K}_I(s - t)v(s)\}_s$. With (8) we obtain:

$$(9) \quad \dot{\sigma}_I(t) = \mathcal{M}\{K_I(s - t)\dot{v}(s)\}_s \quad \text{for all } t \in R.$$

We can write this equation as $\dot{\sigma}_I(t) = \mathcal{M}\{K_I(s)\dot{v}(s + t)\}_s$.

The non-negative function $\frac{1}{2T}K_I$ can be considered as the density (Radon-Nycodim derivative) of a measure μ on R , and so $d\mu(s) = \frac{1}{2T}K_I(s)ds$. Then, by the Cauchy-Schwarz inequality, we have:

$$\left| \frac{1}{2T} \int_{-T}^T \dot{v}(s + t)K_I(s)ds \right|^2 = \left| \int_{-T}^T \dot{v}(s + t)d\mu(s) \right|^2 \leq \left(\int_{-T}^T |\dot{v}(s + t)|^2 d\mu(s) \right) \cdot \left(\int_{-T}^T 1^2 d\mu(s) \right) = \left(\frac{1}{2T} \int_{-T}^T |\dot{v}(s + t)|^2 K_I(s)ds \right) \left(\frac{1}{2T} \int_{-T}^T K_I(s)ds \right),$$

and when $T \rightarrow \infty$, we obtain:

$$|\mathcal{M}\{\dot{v}(s + t)K_I(s)\}_s|^2 \leq \underline{\mathcal{M}}\{|\dot{v}(s + t)|^2 K_I(s)\}_s \cdot \mathcal{M}\{K_I(s)\}_s,$$

therefore, in using (9),

$$(10) \quad |\dot{\sigma}_I(t)|^2 \leq \underline{\mathcal{M}}\{|\dot{v}(s + t)|^2 K_I(s)\}_s.$$

From this inequality and the monotonicity of $\overline{\mathcal{M}}$, we deduce:

$\overline{\mathcal{M}}\{|\dot{\sigma}_I|^2\} \leq \overline{\mathcal{M}}\{\underline{\mathcal{M}}\{|\dot{v}(s + t)|^2 K_I(s)\}_t\}$, but $\dot{\sigma}_I \in AP^0$, and so $|\dot{\sigma}_I|^2 \in AP^0$ and its mean value exists, and in realizing a change of variable in the second term we obtain:

$$(11) \quad \mathcal{M}\{|\dot{\sigma}_I|^2\} \leq \mathcal{M}\{\underline{\mathcal{M}}\{|\dot{v}(s)|^2 K_I(s - t)\}_s\}t.$$

By the definition of the lim sup, for each $\eta > 0$ and each $T' > 0$ exists a $T > T'$ such that:

$$(12) \quad \overline{\mathcal{M}}\{\underline{\mathcal{M}}\{|\dot{v}(s)|^2 K_I(s - t)\}_s\}_t < \frac{1}{2T} \int_{-T}^T \underline{\mathcal{M}}\{|\dot{v}(s)|^2 K_I(s - t)\}_s dt + \eta.$$

In using successively the theorems of Fatou and Fubini, we can say:

$$\begin{aligned} \int_{-T}^T \underline{\mathcal{M}}\{|\dot{v}(s)|^2 K_I(s - t)\}_s dt &= \int_{-T}^T \left(\liminf_{s \rightarrow \infty} \frac{1}{2S} \int_{-S}^S |\dot{v}(s)|^2 K_I(s - t) ds \right) dt \\ &\leq \liminf_{s \rightarrow \infty} \int_{-T}^T \left(\frac{1}{2S} \int_{-S}^S |\dot{v}(s)|^2 K_I(s - t) ds \right) dt \\ &= \liminf_{s \rightarrow \infty} \frac{1}{2S} \int_{-S}^S \left(|\dot{v}(s)|^2 \cdot \int_{-T}^T K_I(s - t) dt \right) ds \end{aligned}$$

$$\begin{aligned}
&= \underline{\mathcal{M}} \left\{ |\dot{v}(s)|^2 \int_{-T}^T K_I(s-t) dt \right\}_s \\
&\leq \overline{\mathcal{M}} \left\{ |\dot{v}(s)|^2 \int_{-T}^T K_I(s-t) dt \right\}_s
\end{aligned}$$

therefore,

$$\frac{1}{2T} \int_{-T}^T \underline{\mathcal{M}} \{ |\dot{v}(s)|^2 K_I(s-t) \}_s dt \leq \underline{\mathcal{M}} \left\{ |\dot{v}(s)|^2 \frac{1}{2T} \int_{-T}^T K_I(s-t) dt \right\}_s$$

if we take T big enough, we have:

$$0 \leq \frac{1}{2T} \int_{-T}^T K_I(s-t) dt \leq \underline{\mathcal{M}} \{ K_I(s-t) \}_t + \eta = 1 + \eta,$$

therefore, $\frac{1}{2T} \int_{-T}^T \underline{\mathcal{M}} \{ |\dot{v}(s)|^2 K_I(s-t) \}_s dt \leq \overline{\mathcal{M}} \{ |\dot{v}(s)|^2 \}_s (1 + \eta)$, then by (12), we have:

$\overline{\mathcal{M}} \{ \underline{\mathcal{M}} \{ |\dot{v}(s)|^2 K_I(s-t) \}_s \}_t \leq \overline{\mathcal{M}} \{ |\dot{v}(s)|^2 \}_s (1 + \eta) + \eta$, and by (11), we deduce that $\underline{\mathcal{M}} \{ |\dot{\sigma}_I|^2 \} \leq \overline{\mathcal{M}} \{ |\dot{v}(s)|^2 \} (1 + \eta) + \eta$, and as η is arbitrarily chosen, finally we obtain:

$$(13) \quad \underline{\mathcal{M}} \{ |\dot{\sigma}_I|^2 \} \leq \overline{\mathcal{M}} \{ |\dot{v}|^2 \} \quad \text{for all the multi-index } I.$$

Because of (7) we obtain:

$$(14) \quad \underline{\mathcal{M}} \{ |\dot{\sigma}_I|^2 \} \leq \underline{\mathcal{M}} \{ |\dot{u}|^2 \} \quad \text{for all the multi-indexes } I.$$

We know that the functional $x \rightarrow \underline{\mathcal{M}} \{ \cos(x + E) \}$ is 1-lipschitzian from $(AP^0, \|\cdot\|_\infty)$ into R , and there exists a multi-index J such that $\|\sigma_J - v\|_\infty \leq \varepsilon$, therefore $\underline{\mathcal{M}} \{ \cos(\sigma_J + E) \} \leq \underline{\mathcal{M}} \{ \cos(v + E) \} + \varepsilon$. We choose $w := \sigma_J$ and then, by this last inequality and by (5) we obtain $\underline{\mathcal{M}} \{ \cos(w + E) \} \leq \underline{\mathcal{M}} \{ \cos(u + E) \} + E$; and by (14) $\underline{\mathcal{M}} \{ |\dot{w}|^2 \} \leq \underline{\mathcal{M}} \{ |\dot{u}|^2 \}$, and so $J(w) \leq J(u) + \varepsilon$. To conclude, it is sufficient to verify that $W \in \mathcal{U}_1$; but we have $w(t) + E(t) \geq |v(t) + E(t)| - |v(t) - w(t)| \geq v(t) + E(t) - \varepsilon \geq \frac{\pi}{2} + \varepsilon - \varepsilon = \frac{\pi}{2}$ and $w(t) + E(t) \leq |w(t) - v(t)| + |v(t) + E(t)| \leq \varepsilon + \frac{3\pi}{2} - \varepsilon = \frac{3\pi}{2}$. ■

Remark. The part of the proof situated between the relations (9) and (13) is directly transferred of a proof of Besicovitch ([2] p. 108); but we ought to make this verification since Besicovitch works with a function that is in \mathcal{B}^2 , and here we don't know if $\dot{v} \in B^2$.

Proposition 3. For each $u_0 \in \mathcal{C}_1$, the closed linear subspace into $\mathcal{B}^{1,2}$, spanned by $\mathcal{C}_1 - u_0$ is equal to $\mathcal{B}^{1,2}$.

Proof. Fix $u_0 \in \mathcal{C}_1$. Since $0 \in \mathcal{C}_1 - u_0$, we have $\text{lin}(\mathcal{C}_1 - u_0) = \text{aff}(\mathcal{C}_1 - u_0)$. We remark that $\text{aff}(\mathcal{C}_1 - u_0) = \text{aff}(\mathcal{C}_1) - u_0 \supset \text{aff}(\mathcal{U}_1) - u_0$. For a positive number δ small enough, there exists $u_\delta \in \mathcal{U}_1$ such that $\frac{\pi}{2} + \delta < u_\delta(t) + E(t) < \frac{3\pi}{2} - \delta$ for all $t \in \mathbb{R}$. If $u \in AP^1$ satisfies $\|u - u_\delta\|_{\mathcal{C}_1} < \frac{\delta}{2}$, then $\|u - u_\delta\|_\infty < \frac{\delta}{2}$, and therefore $\frac{\pi}{2} + \frac{\delta}{2} < u(t) + E(t) < \frac{3\pi}{2} - \frac{\delta}{2}$ for all $t \in \mathbb{R}$, and so $u \in \mathcal{U}_1$. This reasoning shows that the topological interior of \mathcal{U}_1 into $(AP^1, \|\cdot\|_{\mathcal{C}_1})$ is non-void, and consequently we have $\text{aff}(\mathcal{U}_1) = AP^1$. Therefore we have established the inclusion: $\text{lin}(\mathcal{C}_1 - u_0) \supset AP^1 - u_0$, and taking the closed hulls into $\mathcal{B}^{1,2}$, we deduce that: $\text{cl. lin}(\mathcal{C}_1 - u_0) \supset \text{cl}(AP^1 - u_0) = \text{cl}(AP^1) - u_0 = \mathcal{B}^{1,2} - u_0 = \mathcal{B}^{1,2}$. ■

Proposition 4. The restriction of the functional J at \mathcal{C}_1 is convex.

Proof. First, we show that J_0 is convex on \mathcal{U}_1 . If $u \in \mathcal{U}_1$ then $\cos(u(t) + E(t)) \leq 0$ for all $t \in \mathbb{R}$, therefore, for each $h \in AP^1$, we have: $(h(t))^2 - \cos(u(t) + E(t))(h(t))^2 \geq 0$, for all $t \in \mathbb{R}$; and using Proposition 1, we obtain $J_0''(u)(h, h) \geq 0$. Then, using the convexity of \mathcal{U}_1 and the standard arguments, we see that J_0 is convex on \mathcal{U}_1 , hence the convexity of J on \mathcal{U}_1 . Because of the continuity of J and the definition of \mathcal{C}_1 , we have the convexity of J on \mathcal{C}_1 . ■

Proposition 5.

- i) Let $u \in \mathcal{C}_1$, then $J(u) = \text{Inf } J(\mathcal{C}_1)$ if and only if $J'(u) = 0$
- ii) Let $u \in \mathcal{U}_1$, then $J_0(u) = \text{Inf } J_0(\mathcal{U}_1)$ if and only if $J_0'(u) = 0$.

Proof.

- i) Because of the convexity of \mathcal{C}_1 , a necessary condition for $J(u) = \text{Inf } J(\mathcal{C}_1)$ is the following Euler inequation ([9] p. 47):

$$(15) \quad J'(u) \cdot (x - u) \geq 0 \quad \text{for all } x \in \mathcal{C}_1 .$$

This inequation is also a sufficient condition for $J(u) = \text{Inf } J(\mathcal{C}_1)$ because of the convexity of J on \mathcal{C}_1 . And so, if $J'(u) = 0$, then u satisfies (15), and $J(u) = \text{Inf } J(\mathcal{C}_1)$. Conversely, we assume that $J(u) = \text{Inf } J(\mathcal{C}_1)$. By Proposition 2, $\text{Inf } J(\mathcal{U}_1) = \text{Inf } J(\mathcal{U}_2)$, therefore by the continuity of J , $\text{Inf } J(\mathcal{C}_1) = \text{Inf } J(\mathcal{C}_2)$. Since $\mathcal{U}_1 \subset \mathcal{U}_2$, we have $\mathcal{C}_1 \subset \mathcal{C}_2$, and so $J(u) = \text{Inf } J(\mathcal{C}_2)$ and $u \in \mathcal{C}_2$. Because of the convexity of \mathcal{C}_2 , u satisfies the following Euler inequality:

$$(16) \quad J'(u) \cdot (y - u) \geq 0 \quad \text{for all } y \in \mathcal{C}_2 .$$

We fix an arbitrary $x \in \mathcal{U}_1$, and we choose a sequence $(Z_n)_n$ in \mathcal{U}_1 such that $\lim_{n \rightarrow \infty} z_n = u$. For each integer n , we set $y_n := 2z_n - x$, then $y_n + E = 2(z_n + E) - (x + E)$ and we have, for each $t \in R$, $y_n(t) + E(t) \geq 2\left(\frac{\pi}{2}\right) - \frac{3\pi}{2} = -\frac{\pi}{2}$, $y_n(t) + E(t) \leq 2\left(\frac{3\pi}{2}\right) - \frac{\pi}{2} = \frac{5\pi}{2}$. Consequently, for each n , $y_n \in \mathcal{U}_2$ and $y := \lim_{n \rightarrow \infty} y_n = 2u - x \in \mathcal{C}_2$. And so, $y - u = -(x - u)$. The inequation (16) implies:

$-J'(u) \cdot (x - u) = J'(u) \cdot (-(x - u)) = J'(u) \cdot (y - u) \geq 0$; therefore $J'(u) \cdot (x - u) \leq 0$, and (15) implies $J'(u) \cdot (x - u) \geq 0$, therefore $J'(u) \cdot (x - u) = 0$ for all $x \in \mathcal{U}_1$. Using the continuity of $J'(u)$, we obtain:

$$(17) \quad J'(u) \cdot (x - u) = 0 \text{ for all } x \in \mathcal{C}_1.$$

Since $J'(u)$ is linear continuous, from (17) we deduce that $J'(u) \cdot h = 0$, for all $h \in \text{cl lin } (\mathcal{C}_1 - u)$. Then, by Proposition 3, we obtain $J'(u) = 0$.

ii) If $J_0(u) = 0$, then, by Proposition 1, for all $h \in AP^1$, we have $0 = \mathcal{M}\{\dot{u}h - \sin(u + E)h\} = \mathcal{M}\{\nabla u \cdot \nabla h - \sin(u + E)h\}$, and since AP^1 is dense into $\mathcal{B}^{1,2}$, we obtain $J'(u) = 0$, with $u \in \mathcal{U}_1 \subset \mathcal{C}_1$. Then, by (i), $J(u) = \text{Inf } J(\mathcal{C}_1) = \text{Inf } J(\mathcal{U}_1)$, and $J_0(u) = J(u) = \text{Inf } J(\mathcal{U}_1) = \text{Inf } J_0(\mathcal{U}_1)$. Conversely, if $J_0(u) = \text{Inf } J_0(\mathcal{U}_1)$, then $J(u) = \text{Inf } j(\mathcal{C}_1)$ and, by (i), $J'(u) = 0$, therefore $J_0(u) = 0$. ■

4. Existence and uniqueness results

We introduce the subset of AP^1 : $\mathcal{X} := \left\{ u \in AP^1; \forall t \in R \frac{\pi}{2} \leq u(t) \leq \frac{3\pi}{2} \right\}$ and $\text{cl } (\mathcal{X})$ denotes the closure of \mathcal{X} into $\mathcal{B}^{1,2}$. These are two convex sets. We begin with a result of existence of a weak solution of the forced pendulum equaton.

Theorem 1. *Let $e \in AP^0$. We assume that there exists $E \in AP^0$ such that $\ddot{E} = e$ and $\text{Osc } (E) < \frac{\pi}{2}$. Then there exists $u \in \mathcal{B}^{2,2} \cap \text{cl } (\mathcal{X})$ such that $\nabla^2 u + \sin u \sim_2 e$.*

Proof. We seek to solve the following convex problem of minimization:

$$(MIN) \text{ minimize } J(u) \text{ when } u \in \mathcal{C}_1$$

J and \mathcal{C}_1 are defined in the previous section; in fact $\mathcal{C}_1 = \text{cl } (\mathcal{X}) - E$. Since $J \geq -1$ on $\mathcal{B}^{1,2}$, J is bounded from below on \mathcal{C}_1 . Since $\text{Inf } J(\mathcal{U}_1) = \text{Inf } J(\mathcal{C}_1)$, we can choose a minimizing sequence of (MIN), $(u_k)_k$, with values in \mathcal{U}_1 , such

that $J(u_k) \leq \text{Inf } J(\mathcal{C}_1) + \frac{1}{k}$. Considering the inequality: $\frac{1}{2} \|\dot{u}_k\|_2^2 = J(u_k) - \mathcal{M}\{\cos(u_k + E)\} \leq \text{Inf } J(\mathcal{C}_1) + 1 + \frac{1}{k}$ we can assert that $(\dot{u}_k)_k$ is a bounded sequence in \mathcal{B}^2 . By the definition of \mathcal{U}_1 , $(u_k)_k$ is bounded in AP^0 , and therefore $(u_k)_k$ is bounded in $\mathcal{B}^{1,2}$. Since $\mathcal{B}^{1,2}$ is hilbertian, it is reflexive, and there exists a subsequence $(v_k)_k$ of $(u_k)_k$ weakly convergent in $\mathcal{B}^{1,2}$ toward an element $u_* \in \mathcal{B}^{1,2}$. The convexity of \mathcal{C}_1 permits us to say that $u_* \in \mathcal{C}_1$.

Since $\mathcal{B}^{1,2}$ is hilbertian, $\mathcal{B}^{1,2}$ is uniformly convex ([8] p. 78) and consequently, after a theorem of Kakutani ([1] chapter II for classical references and generalizations), $\mathcal{B}^{1,2}$ possesses the Banach-Sacks property. And so, setting $w_k := \frac{1}{k} \sum_{j=1}^k v_j$, the sequence $(w_k)_k$ is strongly convergent toward u_* . Because of the convexity of \mathcal{U}_1 , we have $w_k \in \mathcal{U}_1$, for each k .

By the convexity of J on \mathcal{C}_1 (Proposition 4), we have $J(w_k) \leq \frac{1}{k} \sum_{j=1}^k J(v_j)$. By a classical theorem of Cauchy on the means of Cesaro ([11] p. 33) we have: $\lim_{k \rightarrow \infty} J(v_k) = \text{Inf } J(\mathcal{C}_1)$ implies $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k J(v_j) = \text{Inf } J(\mathcal{C}_1)$. And so, the sequence $(J(w_k))_k$ is inserted between $\text{Inf } J(\mathcal{C}_1)$ and a sequence that converges toward $\text{Inf } J(\mathcal{C}_1)$. Consequently, in using the continuity of J , we obtain $J(u_*) = J(\lim_{k \rightarrow \infty} w_k) = \lim_{k \rightarrow \infty} J(w_k) = \text{Inf } J(\mathcal{C}_1)$.

We have shown that u_* solves (MIN). Then, by Proposition 5, $J'(u_*) = 0$, and by Proposition 1, $u_* \in \mathcal{B}^{2,2}$ and $\nabla^2 u_* + \sin(u_* + E) \sim_2 0$. Let $u := u_* + E$, then $u \in \mathcal{B}^{2,2} \cap \text{cl}(\mathcal{X})$ and satisfies $\nabla^2 u = \nabla^2 u_* + \nabla^2 E \sim_2 \nabla^2 u_* + E \sim_2 \nabla^2 u_* + e$. Therefore, we have $\nabla^2 u + \sin u \sim_2 e$. ■

Now, from the previous theorem of existence of weak almost periodic solutions, we deduce informations on the existence of strong almost periodic solutions:

Theorem 2. *Let $e \in AP^0$ such that there exists $E \in AP^2$ satisfying: $\ddot{E} = e$ and $\text{Osc}(E) < \pi/2$. Then, for each $\varepsilon > 0$ there exists $e_\varepsilon \in AP^0$ such that $\mathcal{M}\{|e - e_\varepsilon|^2\}^{1/2} < \varepsilon$ and such that there exists $x_\varepsilon \in AP^2$ verifying $\ddot{x}_\varepsilon + \sin x_\varepsilon = e_\varepsilon$ and $\frac{\pi}{2} \leq x_\varepsilon(t) \leq \frac{3\pi}{2}$ for all $t \in R$.*

Proof. First, we introduce $\mathcal{X}_0 := \mathcal{X} \cap AP^2$. In using the polynomials of Bochner-Fejer, we can easily verify that \mathcal{X}_0 is dense into $\text{cl}(\mathcal{X}) \cap \mathcal{B}^{2,2}$. Secondly we introduce the non linear operator $\mathcal{T}: \mathcal{B}^{2,2} \rightarrow \mathcal{B}^2$, $\mathcal{T}(u) := \nabla_u^2 + \sin u$; \mathcal{T} is continuous. Let us fix e as in the statement, and then, by Theorem 1, there exists $u \in \mathcal{B}^{2,2}$ such that $\mathcal{T}(u) = e$, moreover this u belongs to the

closure of \mathcal{X}_0 into $\mathcal{B}^{2,2}$. Therefore there exists a sequence $(u_n)_n$ in \mathcal{X}_0 that converges toward this weak solution u . Because of the continuity of \mathcal{F} , $(\mathcal{F}(u_n))_n$ converges toward e in \mathcal{B}^2 . And so, fixing arbitrarily an $\varepsilon > 0$, there exists a so big integer n such that $x_\varepsilon := u_n$ verifies $\|\mathcal{F}(x_\varepsilon) - e\|_2 < \varepsilon$. To include, it is sufficient to take $e_\varepsilon := \mathcal{F}(x_\varepsilon)$. ■

We end in stating a result of uniqueness for the strong solutions.

Theorem 3. *Let $e \in AP^0$ such that there exists $E \in AP^2$ verifying $\ddot{E} = e$ and $\text{Osc}(E) < \frac{\pi}{2}$. Then there is at most one solution $u \in AP^2$ of $\ddot{u} + \sin u = e$ such that $\frac{\pi}{2} \leq u(t) \leq \frac{3\pi}{2}$ for all $t \in \mathbb{R}$.*

Proof. We assume that $u, v \in AP^2 \cap \mathcal{X}$ and satisfy $\ddot{u} + \sin u = e, \ddot{v} + \sin v = e$. In setting $x := u - E, y := v - E$, we have $x, y \in AP^2 \cap \mathcal{U}_1$ and $\ddot{x} + \sin(x + E) = 0, \ddot{y} + \sin(y + E) = 0$. Then, by Proposition 1, (iv), $J'_0(x) = J'_0(y) = 0$, and, by Proposition 3, $J_0(x) = J_0(y) = \text{Inf } J(\mathcal{U}_1)$. We set $j(\lambda) := J_0(x + \lambda(y - x))$, when $0 \leq \lambda \leq 1$. Because of the convexity of J_0 on \mathcal{U}_1 , j is constant equal to $\text{Inf } J(\mathcal{U}_1)$. Because of Proposition 1, (iii), we have $0 = j''(0) = J''_0(x)(y - x, y - x) = \mathcal{M}\{(\dot{y} - \dot{x})^2 - \cos(x + E)(y - x)^2\} = \mathcal{M}\{(\dot{y} - \dot{x})^2\} + \mathcal{M}\{-\cos(x + E)(y - x)^2\}$. This is a seem of two non negative terms that is equal to zero, therefore we have:

$$(18) \quad \mathcal{M}\{(\dot{y} - \dot{x})^2\} = 0 \quad \text{and} \quad \mathcal{M}\{\cos(x + E)(y - x)^2\} = 0$$

The Parseval equality ([2] p. 109) applied to the first equation of (18) gives us: $0 = a(\dot{y} - \dot{x}, \lambda) = i\lambda a(y - x, \lambda)$ for all non zero real λ , and so $a(y - x, \lambda) = 0$ for all non zero real λ , and consequently the Fourier series of $y - x$ is just limited at its mean value because of the Uniqueness theorem of H. Bohr ([2] p. 27) and so:

$$(19) \quad y - x = \mathcal{M}\{y - x\}$$

Then, the second equation of (18) gives us:

$$0 = \mathcal{M}\{\cos(x + E)(\mathcal{M}\{y - x\})^2\} = \mathcal{M}\{\cos(x + E)\}\mathcal{M}\{y - x\}^2$$

therefore:

$$(20) \quad \mathcal{M}\{\cos(x + E)\} = 0 \quad \text{or} \quad \mathcal{M}\{y - x\} = 0$$

If $\mathcal{M}\{\cos(x + E)\} = 0$, since the sign of $\cos(x(t) + E(t))$ is constant then, ([2] p. 20), $\cos(x(t) + E(t)) = 0$ for all $t \in \mathbb{R}$. Consequently $x(t) + E(t)$ is constantly equal to $\frac{\pi}{2}$ or constantly equal to $\frac{3\pi}{2}$, and $\sin(x(t) + E(t))$ is a constant

function equal to $+1$ or to -1 . And so, we have $\ddot{E}(t) = e(t) = \ddot{x}(t) \pm 1$. And ([3] Prop. 1), we obtain $0 = \mathcal{M}\{\dot{E}\} = \mathcal{M}\{\dot{x}\} \pm 1 = \pm 1$: that is impossible. Therefore, because of (20), we have: $\mathcal{M}\{y - x\} = 0$, and because of (19), we have $y - x = 0$. ■

Corollary. *The constant function equal to π is the unique almost periodic solution of the pendulum equation $\ddot{x} + \sin x = 0$ such that $\frac{\pi}{2} \leq x(t) \leq \frac{3\pi}{2}$ for all $t \in \mathbb{R}$.*

Remarks (on the quasi-periodic case). If $\omega = (\omega_1, \dots, \omega_s)$ is a list of \mathcal{L} -linearly independent real numbers, we can consider spaces of quasiperiodic functions. We define $QP^r(\omega)$ as the space of the $f \in AP^r$ such that every $\lambda \in \mathbb{R}$ satisfying $a(f, \lambda) \neq 0$ is a linear combination of the ω_j with integer coefficients. $\mathcal{B}^r(\omega)$ is the subspace of \mathcal{B}^2 generated by $QP^0(\omega)$, and $\mathcal{B}^{k,2}(\omega)$ is the subspace of $\mathcal{B}^{k,2}$ generated by $QP^k(\omega)$. We can restrict all the elements of the previous building at these subspaces, notably $\mathcal{U}_i(\omega) := \mathcal{U}_i \cap QP^1(\omega)$, $\mathcal{C}_i(\omega) := \mathcal{C}_i \cap \mathcal{B}^{1,2}(\omega)$ ($i = 1, 2$), and to restrict the functional J at $\mathcal{B}^{1,2}(\omega)$, and we can work in following the same process. And so, if we add the condition $e \in QP^0(\omega)$ in the hypotheses of Theorem 1, then, in the conclusion of Theorem 1, we can assert that the weak solution u belongs to $\mathcal{B}^{2,2}(\omega)$. In the same way, in Theorem 2, if we add the hypothesis $e \in QP^0(\omega)$, we can add, in the conclusion, that $e_\varepsilon \in QP^0(\omega)$ and $x_\varepsilon \in QP^2(\omega)$.

Remarks (on the periodic case). We fix an arbitrary positive period T , then an analogous process at this above can be realized for the T -periodic case. In using $\mathcal{U}_i \cap C_T^1$ instead of \mathcal{U}_i , the closure of $\mathcal{U}_i \cap C_T^1$ into H_T^1 instead of \mathcal{C}_i , and the functional $\int_0^T \left(\frac{1}{2}(\dot{u}(t))^2 + \cos(u(t)) + E(t) \right) dt$ defined on H_T^1 instead of J , we can establish all the analogous propositions at these of paragraph 3. Even, we can replace the condition $\text{Osc}(E) < \frac{\pi}{2}$ by $\text{Osc}(E) \leq \frac{\pi}{2}$. Our notion of weak solution becomes useless since instead of the \mathcal{V} -derivative we use the Sobolev (or distributional) derivative and the regularization of a solution in H_T^2 in a solution in C_T^2 is evident. And so, combining the specializations of Theorem 1 and Theorem 2 at the T -periodic case, we obtain:

Theorem 4. *Let $e \in C_T^0$. We assume that there exists $E \in C_T^2$ such that: $\ddot{E} = e$ and $\text{Osc}(E) \leq \frac{\pi}{2}$. Then there exists a unique $x \in C_T^2$ such that $\ddot{x} + \sin x = e$ and $\frac{\pi}{2} \leq x(t) \leq \frac{3\pi}{2}$ for all $t \in \mathbb{R}$.*

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