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Viscosity Solutions of Nonlinear Second Order Elliptic PDEs Associated with Impulse Control Problems

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§ 1. Introduction

In this paper we shall study the nonlinear second order elliptic partial differential equations (PDEs) with implicit obstacle.

Let $\Omega \subset \boldsymbol{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. For any function u on $\bar{\Omega}$, we define the nonlocal operator M as the following;

$$Mu(x) = \inf_{\substack{\xi \geqq 0 \\ x+\xi \in \bar{\Omega}}} \{k(\xi) + u(x+\xi)\},$$

where $k(\xi)$ is a nonnegative and continuous function on $(\boldsymbol{R}^+)^N$ and $\xi \geqq 0$ means $\xi \in (\boldsymbol{R}^+)^N$.

We consider the following nonlinear elliptic PDE with implicit boundary condition;

$$\max\{Lu - f, u - Mu\} = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\max\{u - g, u - Mu\} = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where L is a linear second order elliptic operator of the form;

$$Lu = -a_{ij}u_{x_i x_j} + b_i u_{x_i} + cu.$$

(Throughout this paper we use the usual summation convention on repeated indices.) The problem (1.1)–(1.2) is associated with optimal impulse control problems, whose state is governed by stochastic differential equations with impulsive jumps and whose value function has $k(\xi)$ as an impulsive cost. (For the details, see A. Bensoussan-J. L. Lions [3].)

In general the equation (1.1) with $u = g$ on $\partial\Omega$ has no solution because we don't know *a priori* whether $g \leqq Mu$ on $\partial\Omega$ or not. So we put the implicit obstacle in (1.2). (cf. B. Perthame [18] etc.)

From the view point of the impulse control, [3] treated the nondegenerate diffusions and J. L. Menaldi [15] did the degenerate case. They characterized the value function of impulse control problems as the maximum solution of the corresponding quasi-variational inequality (QVI) in some Sobolev spaces. Using the notion of viscosity solutins by M. G. Crandall-P. L. Lions [7], G. Barles [1] showed that the value function for deterministic impulse control problems is a unique viscosity solution of the corresponding first order Hamilton-Jacobi QVI in $\boldsymbol{R}^N$.

By an analytical treatment, B. Perthame [17] proved the existence and uniqueness of solutions in the class $W_{loc}^{2,\infty}(\Omega)\cap C(\bar{\Omega})$ under the assumption that (1.1) has a subsolution $\underline{u}$ satisfying $\underline{u}\leqq g\leqq M\underline{u}$ on $\partial\Omega$. Moreover, B. Perthame [18] remarked that the unique maximal subsolution of (1.1) with the usual Dirichlet condition is a unique viscosity solution of the problem (1.1)–(1.2). G. Barles [2] extended his results to the general Hamiltonian case and obtained the existence and uniqueness of viscosity solutions of Hamilton-Jacobi QVI as in M. G. Crandall-L. C. Evans-P. L. Lions [4] and M. G. Crandall-P. L. Lions [7] etc. J. Yong [19] treated a system of Hamilton-Jacobi QVI associated with switching and impulsive control problems in $\boldsymbol{R}^N$.

Our main purpose here is to obtain the comparison principle and existence of viscosity solutions of the problem (1.1)–(1.2) under the assumptioms weaker than [18]. Although (1.2) is not the usual Dirichlet condition, we see in §3 and §4 that the problem (1.1)–(1.2) can be treated similarly to that.

Our plan of this paper is the following. In Section 2 we state our assumptions and recall the definition of viscosity solutions to general PDEs and the properties of the operator M. In order to take the boundary condition (1.2) into account, we introduce the notion of *strong* viscosity solutions (see M. G. Crandall-H. Ishii-P. L. Lions [6] Section 7). Section 3 is devoted to the proof of the comparison principle of viscosity solutions. Our argument is based upon H. Ishii-P. L. Lions [12]. In Section 4 we construct a strong viscosity solution of the problem (1.1)–(1.2) by Perron's method. Because of the strongness, we observe easily that a strong viscosity solution satisfies (1.2) for each point of $\partial\Omega$ without the iterative process used in B. Hanouzet-J. L. Joly [9] and [18].

Finally we refer to H. Ishii-S. Koike [10] and S. M. Lenhart-N. Yamada [12], [13] for some problems and results related to ours.

§2. Preliminaries

In this section we shall state our assumptioms and shall recall the definition of viscosity solutions of nonlinear elliptic PDEs and the properties of the operator M. We make the following assumptions.

(A.1) $\Omega\subset\boldsymbol{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

(A.2) There exists $P:\bar{\Omega}\times(\boldsymbol{R}^+)^N\to(\boldsymbol{R}^+)^N$ satisfying

$$x + P(x, \xi) \in \bar{\Omega} \qquad \text{for any } x \in \bar{\Omega},\ \xi \in (\boldsymbol{R}^+)^N,$$
$$P(x, \xi) = \xi \qquad \text{if } x + \xi \in \bar{\Omega},$$
$$P(\cdot, \xi) \text{ is continuous on } \bar{\Omega} \text{ for each } \xi \in (\boldsymbol{R}^+)^N.$$

(A.3) For the matrix $(a_{ij}(x))$, there exists a nonnegative matrix $(\sigma_{ij}(x))$ such that

$$(a_{ij}) = {}^t(\sigma_{ij})(\sigma_{ij}) \text{ with } \sigma_{ij} \in W^{1,\infty}(\bar{\Omega}) \qquad (i, j = 1, \dots, N),$$

where ${}^tA$ is the transposed matrix of A.

(A.4) $b_i \in W^{1,\infty}(\bar{\Omega})$ $(i = 1, \dots, N)$.
(A.5) $c \in C(\bar{\Omega})$, $c \geqq \lambda_0$ on $\bar{\Omega}$ for some $\lambda_0 > 0$.
(A.6) $f \in C(\bar{\Omega})$.
(A.7) $k \in C((\boldsymbol{R}^+)^N)$, $k(\xi) \geqq k_0$ on $(\boldsymbol{R}^+)^N$ for some $k_0 > 0$.
(A.8) $g \in C(\partial\Omega)$.

We denote by ω_c and ω_f the modulus of continuity of c and f, respectively.

Remark. The assumption (A.2) does not hold if we only suppose the smoothness of $\partial\Omega$. When Ω is convex and regular, we can take $P(x, \xi)$ as the projection of ξ on $(\boldsymbol{R}^+)^N \cap (\bar{\Omega} - \{x\})$. (See A. Bensoussan-J. L. Lions [3] and J. L. Menaldi [15].)

For any function $u\colon \bar{\Omega} \to \boldsymbol{R}$, we define the function $u^*, u_*\colon \bar{\Omega} \to \boldsymbol{R} \cup \{-\infty, +\infty\}$ by

$$u^*(x) = \limsup_{r \to 0} \{u(y) \mid y \in \bar{\Omega},\ |y - x| < r\},$$

$$u_*(x) = \liminf_{r \to 0} \{u(y) \mid y \in \bar{\Omega},\ |y - x| < r\}.$$

It is easily seen that $u_* \leqq u \leqq u^*$ on $\bar{\Omega}$ and that u^* is upper semi-continuous (u.s.c.) on $\bar{\Omega}$ and that u_* is lower semi-continuous (l.s.c.) on $\bar{\Omega}$. We observe that if u is u.s.c. (resp., l.s.c.) at $x \in \bar{\Omega}$ then $u(x) = u^*(x)$ (resp., $= u_*(x)$).

Let u be a real valued function defined on Ω. For each $x \in \Omega$, we set

$$J^{2,+}u(x) = \left\{ (p, X) \in \boldsymbol{R}^N \times \boldsymbol{S}^N \,\middle|\, u(y) \leqq u(x) + \langle p, y - x \rangle + \frac{1}{2}\langle X(y - x), y - x \rangle + o(|y - x|^2) \text{ as } \Omega \ni y \longrightarrow x \right\}$$

and

$$J^{2,-}u(x) = \left\{(p, X) \in \mathbf{R}^N \times \mathbf{S}^N \,\middle|\, u(y) \geqq u(x) + \langle p, y - x\rangle + \frac{1}{2}\langle X(y - x), y - x\rangle + o(|y - x|^2) \text{ as } \Omega \ni y \longrightarrow x\right\},$$

where $\mathbf{S}^N$ denotes the set of all $N \times N$ real symmetric matrices and $\langle \cdot, \cdot \rangle$ is Euclidian inner product in $\mathbf{R}^N$. We denote by $\bar{J}^{2,+}u(x)$ and $\bar{J}^{2,-}u(x)$ the following sets;

$$\bar{J}^{2,+}u(x) = \{(p, X) \in \mathbf{R}^N \times \mathbf{S}^N \mid \exists (x_n, p_n, X_n) \in \Omega \times \mathbf{R}^N \times \mathbf{S}^N \text{ such that } (p_n, X_n) \in J^{2,+}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \longrightarrow (x, u(x), p, X) \text{ as } n \longrightarrow +\infty\},$$

$$\bar{J}^{2,-}u(x) = \{(p, X) \in \mathbf{R}^N \times \mathbf{S}^N \mid \exists (x_n, p_n, X_n) \in \Omega \times \mathbf{R}^N \times \mathbf{S}^N \text{ such that } (p_n, X_n) \in J^{2,-}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \longrightarrow (x, u(x), p, X) \text{ as } n \longrightarrow +\infty\}.$$

We note that if $\varphi \in C^2(\Omega)$ and $u - \varphi$ attains a local maximum (resp., local minimum) at $x_0 \in \Omega$, then $(D\varphi(x_0), D^2\varphi(x_0)) \in \bar{J}^{2,+}u(x_0)$ (resp., $(D\varphi(x_0), D^2\varphi(x_0)) \in \bar{J}^{2,-}u(x_0)$).

Now we give the definition of viscosity solutions of the nonlinear degenerate elliptic PDEs with implicit boundary condition;

$$\text{(2.1)} \qquad \begin{cases} \max\{F(x, u, Du, D^2u), u - Mu\} = 0 & \text{in } \Omega, \\ \max\{u - g, u - Mu\} = 0 & \text{on } \partial\Omega, \end{cases}$$

where F is a continuous function on $\Omega \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$ satisfying the degenerate ellipticity condition;

$$F(x, r, p, X + Y) \leqq F(x, r, p, X) \qquad \text{for all } x \in \Omega,\ r \in \mathbf{R},\ p \in \mathbf{R}^N,\ X, Y \in \mathbf{S}^N \text{ and } Y \geqq O.$$

Definition 2.1. *Let u be a function defined on $\bar{\Omega}$.*

(1) *u is a viscosity subsolution of* (2.1) *if $u^*(x) < \infty$ on $\bar{\Omega}$ and*

$$\max\{F(x, u^*(x), p, X), u^*(x) - Mu^*(x)\} \leqq 0$$

for all $x \in \Omega$, $(p, X) \in \bar{J}^{2,+}u^*(x)$.

(2) *u is a viscosity supersolution of* (2.1) *if $u_*(x) > -\infty$ on $\bar{\Omega}$ and*

$$\max\{F(x, u_*(x), p, X), u_*(x) - Mu_*(x)\} \geqq 0$$

for all $x \in \Omega$, $(p, X) \in \bar{J}^{2,-}u_*(x)$.

(3) *u is a viscosity solution of* (2.1) *if u is a viscosity subsolution and supersolution of* (2.1).

The boundary condition (1.2) differs from the usual Dirichlet condition. So we introduce the notion of *strong* viscosity solutions. (cf. M. G. Crandall-H. Ishii-P. L. Lions [6] Section 7.)

Definition 2.2. *Let u be a function defined on $\bar{\Omega}$.*
(1) *u is a strong viscosity subsolution of* (2.1) *if u is a viscosity subsolution of* (2.1) *and u satisfies*

$$\max\{u^*(x)-g(x),\ u^*(x)-Mu^*(x)\}\leqq 0 \qquad \textit{for all } x\in\partial\Omega.$$

(2) *u is a strong viscosity supersolution of* (2.1) *if u is a viscosity supersolution of* (2.1) *and u satisfies*

$$\max\{u_*(x)-g(x),\ u_*(x)-Mu_*(x)\}\geqq 0 \qquad \textit{for all } x\in\partial\Omega.$$

(3) *u is a strong viscosity solution of* (2.1) *if u is a strong viscosity subsolution and supersolution of* (2.1).

Remark. It is easily seen that if the strong viscosity solution u of (2.1) is continuous on $\bar{\Omega}$, then u satisfies the boundary condition for all $x\in\partial\Omega$.

We recall the properties of the operator M. In what follows we denote by $USC(\bar{\Omega})$ (resp., $LSC(\bar{\Omega})$) the set of all u.s.c. (resp., l.s.c.) functions on $\bar{\Omega}$.

Proposition 2.3. *Suppose* (A.1), (A.2) *and* (A.7) *hold. Let u, v be real valued functions defined on $\bar{\Omega}$. Then the following properties hold.*
(1) (Monotonicity) *If $u\leqq v$ on $\bar{\Omega}$, then $Mu\leqq Mv$ on $\bar{\Omega}$.*
(2) (Concavity) *$M(tu+(1-t)v)\geqq tMu+(1-t)Mv$ for all $t\in[0, 1]$.*
(3) *$M(u+\lambda)=Mu+\lambda$ for all $\lambda\in\boldsymbol{R}$.*
(4) *If $u\in LSC(\bar{\Omega})$, then $Mu\in LSC(\bar{\Omega})$.*
(5) *If $u\in USC(\bar{\Omega})$, then $Mu\in USC(\bar{\Omega})$.*
(6) (Continuity) *$\|Mu-Mv\|_{C(\bar{\Omega})}\to 0$ as $\|u-v\|_{C(\bar{\Omega})}\to 0$.*

Proof. From the definition of M, (1) ~ (3) are trivial.
(4) We take $\{x_n\}_{n\in\boldsymbol{N}}\subset\bar{\Omega}$, $x\in\bar{\Omega}$ so that $x_n\to x$ $(n\to+\infty)$. The condition $u\in LSC(\bar{\Omega})$ implies that, for each x_n, there exists $\xi_n\geqq 0$ such that

$$x_n+\xi_n\in\bar{\Omega},\quad Mu(x_n)=k(\xi_n)+u(x_n+\xi_n).$$

Since $\{\xi_n\}_{n\in\boldsymbol{N}}$ is bounded, by taking a subsequence, if neccesary, we may suppose that $\xi_n\to{}^{\exists}\xi\geqq 0$ $(n\to+\infty)$ such that $x+\xi\in\bar{\Omega}$. Hence we have

$$\liminf_{n\to+\infty} Mu(x_n) = \lim_{n\to+\infty} k(\xi_n) + \liminf_{n\to+\infty} u(x_n + \xi_n)$$
$$\geqq k(\xi) + u(x + \xi)$$
$$\geqq Mu(x),$$

that is, $Mu \in LSC(\bar{\Omega})$.

(5) We take $\{x_n\}_{n\in N} \subset \bar{\Omega}$ and $x \in \bar{\Omega}$ as in the proof of (4) and fix $\xi \geqq 0$ such that $x + \xi \in \bar{\Omega}$. Then we have by (A.2),

$$Mu(x_n) \leqq k(P(x_n, \xi)) + u(x_n + P(x_n, \xi)).$$

Thus we get

$$\limsup_{n\to+\infty} Mu(x_n) = \lim_{n\to+\infty} k(P(x_n, \xi)) + \limsup_{n\to+\infty} u(x_n + P(x_n, \xi))$$
$$\leqq k(\xi) + u(x + \xi).$$

Taking infimum with respect to $\xi \geqq 0$ satisfying $x + \xi \in \bar{\Omega}$, $Mu \in USC(\bar{\Omega})$ is proved.

(6) The definition of the operator M implies that for any $u, v \in C(\bar{\Omega})$,

$$\sup_{x\in\bar{\Omega}} \left| \inf_{\substack{\xi\geqq 0\\ x+\xi\in\bar{\Omega}}} \{k(\xi) + u(x + \xi)\} - \inf_{\substack{\xi\geqq 0\\ x+\xi\in\bar{\Omega}}} \{k(\xi) + v(x + \xi)\} \right| \leqq \sup_{x\in\bar{\Omega}} |u(x) - v(x)|.$$

Therefore we have the continuity of M. ■

§3. Comparison principle of viscosity solutions

In this section we shall prove the comparison principle of strong viscosity solutions of the problem (1.1)–(1.2).

Theorem 3.1. *Assume* (A.1)–(A.8). *Let* u, v *be a strong viscosity subsolution, supersolution, respectively, of* (1.1)–(1.2). *Then* $u^* \leqq v_*$ *on* $\bar{\Omega}$.

Since the problem (1.1)–(1.2) has the implicit obstacle Mu, we need some perturbation of strong viscosity subsolution to prove Theorem 3.1. (cf. H. Ishii-P. L. Lions [11] V.1.) Moreover the existence of certain *derivatives* plays an important role. So we prepare the following lemmas.

Lemma 3.2. *Let* $u \in USC(\bar{\Omega})$ *be a strong viscosity subsolution of* (1.1)–(1.2) *and let* $C = \max\{(\|f\|_{C(\bar{\Omega})}/\lambda_0, \|g\|_{C(\partial\Omega)}\} + 1$. *Then for each* $m \in N$, $u_m = (1 - 1/m)u - C/m$ *is a strong viscosity subsolution of*

$$(3.1)_m \quad \begin{cases} \max\{Lu_m - f,\, u_m - Mu_m\} + \dfrac{\alpha}{m} = 0 & \text{in } \Omega, \\ \max\{u_m - g,\, u_m - Mu_m\} + \dfrac{\alpha}{m} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha = \min\{1, k_0\}$.

Proof. First, we note that $w \equiv -C$ on $\bar{\Omega}$ satisfies

$$\max\{Lw - f,\, w - Mw\} \leqq \max\{-\lambda_0 C - f,\, -k\} \leqq -\alpha \qquad \text{in } \Omega,$$

$$\max\{w - g,\, w - Mw\} \leqq \max\{-C - g,\, -k\} \leqq -\alpha \qquad \text{on } \partial\Omega,$$

For any fixed $\varphi \in C^2(\Omega)$, we suppose $u_m - \varphi$ attains a local maximum at $x_0 \in \Omega$. Then we have

$$\begin{aligned} u_m(x_0) - \varphi(x_0) &= \left(1 - \frac{1}{m}\right)u(x_0) - \frac{C}{m} - \varphi(x_0) \\ &= \left(1 - \frac{1}{m}\right)\left\{u(x_0) - \left(\frac{m}{m-1}\varphi(x_0) + \frac{C}{m-1}\right)\right\} \end{aligned}$$

and obtain that $u - ((m/m-1)\varphi + C/(m-1))$ attains a local maximum at $x_0 \in \Omega$. Hence using the fact that u is a viscosity subsolution of (1.1), we get

$$(3.2) \quad \max\left\{-\frac{m}{m-1}a_{ij}(x_0)\varphi_{x_i x_j}(x_0) + \frac{m}{m-1}b_i(x_0)\varphi_{x_i}(x_0) \right.$$
$$\left. + c(x_0)u(x_0) - f(x_0),\, u(x_0) - Mu(x_0)\right\} \leqq 0.$$

By (3.2) we have

$$(3.3) \quad -\frac{m}{m-1}a_{ij}(x_0)\varphi_{x_i x_j}(x_0) + \frac{m}{m-1}b_i(x_0)\varphi_{x_i}(x_0) + c(x_0)u(x_0) - f(x_0) \leqq 0,$$

$$(3.4) \quad u(x_0) - Mu(x_0) \leqq 0$$

Thus multiplying (3.3) by $1 - 1/m$ and subtracting C/m from (3.3), we obtain

$$-a_{ij}(x_0)\varphi_{x_i x_j}(x_0) + b_i(x_0)\varphi_{x_i}(x_0) + c(x_0)u_m(x_0) - f(x_0) \leqq -\frac{1}{m},$$

and by a similar calculation and using (A.7) and Proposition 2.3 (2), (3), we get

$$u_m(x_0) - Mu_m(x_0) = \left(1 - \frac{1}{m}\right)u(x_0) - \frac{C}{m} - M\left\{\left(1 - \frac{1}{m}\right)u(x_0) - \frac{C}{m}\right\}$$

$$\leqq \left(1-\frac{1}{m}\right)u(x_0) - M\left\{\left(1-\frac{1}{m}\right)u(x_0) - \frac{1}{m}\cdot 0\right\}$$
$$\leqq \left(1-\frac{1}{m}\right)u(x_0) - \left(1-\frac{1}{m}\right)Mu(x_0) - \frac{k_0}{m}$$
$$\leqq -\frac{k_0}{m}.$$

Thus we obtain

$$\max\{-a_{ij}(x_0)\varphi_{x_ix_j}(x_0) + b_i(x_0)\varphi_{x_i}(x_0) + c(x_0)u_m(x_0) - f(x_0),$$
$$u_m(x_0) - Mu_m(x_0)\} \leqq -\frac{1}{m}\min\{\lambda_0 C + f(x_0), k_0\}.$$

It is easily verified that

$$\max\{u_m - g, u_m - Mu_m\} \leqq -\frac{1}{m}\min\{C + g, k_0\} \leqq -\frac{\alpha}{m} \qquad \text{on } \partial\Omega.$$

Hence the proof is completed. ■

Lemma 3.3. *Let $u \in USC(\bar{\Omega})$ and $v \in LSC(\bar{\Omega})$ and let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega \times \Omega}$ be a local maximum point of the function $(x, y) \to u(x) - v(y) - (1/2\varepsilon)|x-y|^2$. Then there exist $X_\varepsilon, Y_\varepsilon \in \boldsymbol{S}^N$ such that*

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X_\varepsilon & O \\ O & -Y_\varepsilon \end{pmatrix} \leqq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and

$$\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), X_\varepsilon\right) \in \bar{J}^{2,+}u(x_\varepsilon), \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), Y_\varepsilon\right) \in \bar{J}^{2,-}v(y_\varepsilon),$$

where I denotes the identity matrix.

This is proved in M. G. Crandall-H. Ishii [5] Example 1, so we omit the proof.

Proof of Theorem 3.1. We may assume that u is u.s.c. and v is l.s.c. on $\bar{\Omega}$, because if otherwise, we replace u, v with u^*, v_*, respectively.

Let C be the same constant as in Lemma 3.2. For each $m \in \boldsymbol{N}$, the function $u_m = (1 - 1/m)u - C/m$ is a strong viscosity subsolution of $(3.1)_m$. To prove the comparison principle, it is sufficient to show $\max_{\bar{\Omega}}(u_m - v) \leqq 0$ for all $m \geqq 1$ because we obtain the desired result by letting $m \to +\infty$. To the contrary, we suppose $\max_{\bar{\Omega}}(u_{m_0} - v) = {}^{\exists}\theta > 0$ for some $m_0 \geqq 1$ and shall get a contradiction. Then there exist $z \in \bar{\Omega}$ such that $\theta = u_{m_0}(z) - v(z)$.

Case 1. $z \in \partial\Omega$.

In this case we have

$$\max\{u_{m_0}(z) - g(z), u_{m_0}(z) - Mu_{m_0}(z)\} \leqq -\frac{\alpha}{m_0},$$

$$\max\{v(z) - g(z), v(z) - Mv(z)\} \geqq 0.$$

When $v(z) - g(z) \geqq 0$, we obtain a contradiction easily. In the case $v(z) - Mv(z) \geqq 0$ we can find $\xi_z \geqq 0$ such that

$$z + \xi_z \in \bar{\Omega} \quad \text{and} \quad Mv(z) = k(\xi_z) + v(z + \xi_z).$$

Hence we get

$$\begin{aligned} \theta &\leqq k(\xi_z) + u_{m_0}(z + \xi_z) - k(\xi_z) - v(z + \xi_z) - \frac{\alpha}{m_0} \\ &\leqq \theta - \frac{\alpha}{m_0}, \end{aligned}$$

which is a contradiction.

Case 2. $z \in \Omega$.

We note that the function $u_{m_0}(x) - |x - z|^4 - v(x)$ takes the maximum θ and z is a unique maximum point of this function. For each $\varepsilon > 0$ we define the function Φ on $\overline{\Omega \times \Omega}$ by

$$\Phi(x, y) = u_{m_0}(x) - |x - z|^4 - v(y) - \frac{1}{2\varepsilon}|x - y|^2,$$

and let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega \times \Omega}$ be a maximum point of Φ. We observe that the inequality $\Phi(z, z) \leqq \Phi(x_\varepsilon, y_\varepsilon)$ implies that

$$u_{m_0}(z) - v(z) \leqq u_{m_0}(z) - v(z) + \frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \leqq u_{m_0}(x_\varepsilon) - |x_\varepsilon - z|^4 - v(y_\varepsilon). \tag{3.5}$$

Since the functions u_{m_0} and $-v$ are bounded above, we have, from (3.5), $|x_\varepsilon - y_\varepsilon| \to 0$ as $\varepsilon \to 0$. By the compactness of $\bar{\Omega}$ we see that $x_{\varepsilon_n}, y_{\varepsilon_n} \to {}^{\exists}\bar{z} \in \bar{\Omega}$ for a suitable sequence $\{\varepsilon_n\}$ tending to 0. Using (3.5) and the semi-continuity of u_{m_0} and v, we get $\theta \leqq u_{m_0}(\bar{z}) - |\bar{z} - z|^4 - v(\bar{z})$. Hence we have $\bar{z} = z$ and $x_\varepsilon, y_\varepsilon \to z$ $(\varepsilon \to 0)$ because z is a unique maximum point of the function $u_{m_0}(x) - |x - z|^4 - v(x)$. Moreover we obtain

$$u_{m_0}(z) - v(z) \leqq \liminf_{\varepsilon\to 0} u_{m_0}(x_\varepsilon) - \limsup_{\varepsilon\to 0} v(y_\varepsilon) \tag{3.6}$$
$$\leqq \limsup_{\varepsilon\to 0} u_{m_0}(x_\varepsilon) - \liminf_{\varepsilon\to 0} v(y_\varepsilon)$$
$$\leqq u_{m_0}(z) - v(z).$$

Thus we have

$$\liminf_{\varepsilon\to 0} u_{m_0}(x_\varepsilon) - \limsup_{\varepsilon\to 0} v(y_\varepsilon) = \limsup_{\varepsilon\to 0} u_{m_0}(x_\varepsilon) - \liminf_{\varepsilon\to 0} v(y_\varepsilon).$$

This equality implies

$$0 \leqq \limsup_{\varepsilon\to 0} u_{m_0}(x_\varepsilon) - \liminf_{\varepsilon\to 0} u_{m_0}(x_\varepsilon) = \liminf_{\varepsilon\to 0} v(y_\varepsilon) - \limsup_{\varepsilon\to 0} v(y_\varepsilon) \leqq 0.$$

Therefore we get

$$\lim_{\varepsilon\to 0} u_{m_0}(x_\varepsilon) = u_{m_0}(z) \quad \text{and} \quad \lim_{\varepsilon\to 0} v(y_\varepsilon) = v(z). \tag{3.7}$$

It is easily seen that by (3.5) and (3.7)

$$\frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

As $x_\varepsilon, y_\varepsilon \to z \in \Omega$ $(\varepsilon \to 0)$, we have $x_\varepsilon, y_\varepsilon \in \Omega$ for small $\varepsilon > 0$. Then by Lemma 3.3 there exist $X_\varepsilon, Y_\varepsilon \in \boldsymbol{S}^N$ satisfying

$$\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), X_\varepsilon\right) \in \bar{J}^{2,+}(u_{m_0}(x_\varepsilon) - |x_\varepsilon - z|^4), \tag{3.8}$$

$$\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), Y_\varepsilon\right) \in \bar{J}^{2,-} v(y_\varepsilon), \tag{3.9}$$

and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X_\varepsilon & O \\ O & -Y_\varepsilon \end{pmatrix} \leqq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.10}$$

Furthermore, (3.8) implies

$$\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + 4|x_\varepsilon - z|^2(x_\varepsilon - z), X_\varepsilon + Z_\varepsilon\right) \in \bar{J}^{2,+}(u_{m_0}(x_\varepsilon)),$$

where $Z_\varepsilon = 4|x_\varepsilon - z|^2 I + 8(x_\varepsilon - z) \otimes (x_\varepsilon - z)$ and $Z_\varepsilon \to O$ as $\varepsilon \to 0$. Hence using the facts that u_{m_0} is a strong viscosity subsolution of $(3.1)_{m_0}$ and that v

is a strong viscosity supersolution of (1.1)–(1.2), we have the following inequalities;

$$(3.11)\qquad \max\Big\{-\operatorname{tr}\{{}^t\sigma(x_\varepsilon)\sigma(x_\varepsilon)(X_\varepsilon+Z_\varepsilon)\} + b_i(x_\varepsilon)\Big\{\frac{1}{\varepsilon}(x_{\varepsilon,i}-y_{\varepsilon,i})+4|x_\varepsilon-z|^2(x_{\varepsilon,i}-z_i)\Big\} + c(x_\varepsilon)u_{m_0}(x_\varepsilon)-f(x_\varepsilon),\ u_{m_0}(x_\varepsilon)-Mu_{m_0}(x_\varepsilon)\Big\} \leqq -\frac{\alpha}{m_0},$$

$$(3.12)\qquad \max\Big\{-\operatorname{tr}\{{}^t\sigma(y_\varepsilon)\sigma(y_\varepsilon)Y_\varepsilon\}+\frac{1}{\varepsilon}b_i(y_\varepsilon)(x_{\varepsilon,i}-y_{\varepsilon,i}) + c(y_\varepsilon)v(y_\varepsilon)-f(y_\varepsilon),\ v(y_\varepsilon)-Mv(y_\varepsilon)\Big\}\geqq 0.$$

We divide our consideration into two cases.

(I) The case $v(y_\varepsilon)-Mv(y_\varepsilon)\geqq 0$ in (3.12).

(3.11) implies $u_{m_0}(x_\varepsilon)-Mu_{m_0}(x_\varepsilon)\leqq -\alpha/m_0$. Thus we get

$$\theta\leqq u_{m_0}(x_\varepsilon)-v(y_\varepsilon)-|x_\varepsilon-z|^4\leqq Mu_{m_0}(x_\varepsilon)-Mv(y_\varepsilon)-\frac{\alpha}{m_0}-|x_\varepsilon-z|^4.$$

From Proposition 2.3 (4), (5), sending $\varepsilon\to 0$, we have

$$\begin{aligned}\theta &\leqq \limsup_{\varepsilon\to 0} Mu_{m_0}(x_\varepsilon)-\liminf_{\varepsilon\to 0} Mv(y_\varepsilon)-\frac{\alpha}{m_0}\\ &\leqq Mu_{m_0}(z)-Mv(z)-\frac{\alpha}{m_0}.\end{aligned}$$

As in *Case* 1, we get a contradiction.

(II) $$-\operatorname{tr}\{{}^t\sigma(y_\varepsilon)\sigma(y_\varepsilon)Y_\varepsilon\}+b_i(y_\varepsilon)(x_{\varepsilon,i}-y_{\varepsilon,i})/\varepsilon+c(y_\varepsilon)v(y_\varepsilon)-f(y_\varepsilon)\geqq 0 \quad \text{in (3.12).}$$

By (3.11) we have

$$-\operatorname{tr}\{{}^t\sigma(x_\varepsilon)\sigma(x_\varepsilon)(X_\varepsilon+Z_\varepsilon)\}+b_i(x_\varepsilon)\Big\{\frac{1}{\varepsilon}(x_{\varepsilon,i}-y_{\varepsilon,i})+4|x_\varepsilon-z|^2(x_{\varepsilon,i}-z_i)\Big\} + c(x_\varepsilon)u_{m_0}(x_\varepsilon)-f(x_\varepsilon)\leqq -\frac{\alpha}{m_0}.$$

Therefore noting from (3.10) that

$$\begin{aligned}&\operatorname{tr}\{{}^t\sigma(x_\varepsilon)\sigma(x_\varepsilon)X_\varepsilon\} - \operatorname{tr}\{{}^t\sigma(y_\varepsilon)\sigma(y_\varepsilon)Y_\varepsilon\}\\ &\qquad \leqq \frac{3}{\varepsilon}\operatorname{tr}\{{}^t(\sigma(x_\varepsilon)-\sigma(y_\varepsilon))(\sigma(x_\varepsilon)-\sigma(y_\varepsilon))\},\end{aligned}$$

and using (A.3), (A.4) and (A.5) we obtain

$$\begin{aligned}c(x_\varepsilon)u_{m_0}(x_\varepsilon) - c(y_\varepsilon)v(y_\varepsilon) &\leqq \frac{3}{\varepsilon}\operatorname{tr}\{{}^t(\sigma(x_\varepsilon)-\sigma(y_\varepsilon))(\sigma(x_\varepsilon)-\sigma(y_\varepsilon))\}\\ &\quad + \operatorname{tr}\{{}^t\sigma(x_\varepsilon)\sigma(x_\varepsilon)Z_\varepsilon\}\\ &\quad + \frac{1}{\varepsilon}(b_i(x_\varepsilon)-b_i(y_\varepsilon))(x_\varepsilon - y_\varepsilon)\\ &\quad - 4|x_\varepsilon - z|^2 b_i(x_\varepsilon)(x_{\varepsilon,i} - z_i)\\ &\quad + f(x_\varepsilon) - f(y_\varepsilon) - \frac{\alpha}{m_0}\\ &\leqq \frac{K}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 + K|x_\varepsilon - z|^2 + K|x_\varepsilon - z|^3\\ &\quad + \omega_f(|x_\varepsilon - y_\varepsilon|) - \frac{\alpha}{m_0}.\end{aligned}$$

Here and hereafter K denotes a positive constant depending only on known constants. Moreover we have

$$\begin{aligned}\lambda_0\theta &\leqq \lambda_0(u_{m_0}(x_\varepsilon) - v(y_\varepsilon) - |x_\varepsilon - z|^4)\\ &\leqq c(x_\varepsilon)(u_{m_0}(x_\varepsilon) - v(y_\varepsilon))\\ &= c(x_\varepsilon)u_{m_0}(x_\varepsilon) - c(y_\varepsilon)v(y_\varepsilon) - v(y_\varepsilon)(c(x_\varepsilon) - c(y_\varepsilon))\\ &\leqq \frac{K}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 + K|x_\varepsilon - z|^2 + K|x_\varepsilon - z|^3 + \omega_f(|x_\varepsilon - y_\varepsilon|)\\ &\quad - \frac{\alpha}{m_0} - v(y_\varepsilon)(c(y_\varepsilon) - c(x_\varepsilon))\\ &\leqq \frac{K}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 + K|x_\varepsilon - z|^2 + K|x_\varepsilon - z|^3 + \omega_f(|x_\varepsilon - y_\varepsilon|)\\ &\quad + K\omega_c(|x_\varepsilon - y_\varepsilon|) - \frac{\alpha}{m_0},\end{aligned}$$

because $-v$ is bounded above. Letting $\varepsilon \to 0$, we obtain a contradiction.

Thus we conclude that $\max_{\bar{\Omega}}(u_m - v) \leqq 0$ for all $m \geqq 1$. Letting $m \to +\infty$, we complete the proof. ■

§4. Existence of viscosity solutions

In this section we shall show that a strong viscosity solution of (1.1) can be constructed by Perron's method.

Theorem 4.1. *Assume* (A.1)-(A.8). *Furthermore, assume* (A.9) *or* (A.10) *holds*;

(A.9) $(a_{ij}(x)) \geqq \mu I$ *on* $\bar{\Omega}$ *for some* $\mu > 0$,

(A.10) $(a_{ij}(x)) \geqq 0$ *on* $\bar{\Omega}$ *and*

$$b_i(x)\nu_i(x) < 0 \quad on \quad \Gamma = \{x \in \partial\Omega \,|\, a_{ij}(x)\nu_i(x)\nu_j(x) = 0\},$$

where I *denotes the identity matrix and* $\nu(x) = (\nu_1(x), \ldots, \nu_N(x))$ *is the outward unit normal to* Ω *at* $x \in \partial\Omega$.
Then there exists a unique strong viscosity solution $u \in C(\bar{\Omega})$ *of the problem* (1.1)–(1.2), *which is a unique viscosity solution of* (1.1) *satisfying* (1.2).

Before proving Theorem 4.1, we show the existence of strong viscosity supersolution of (1.1)–(1.2).

Lemma 4.2. *Assume* (A.1), (A.3), (A.4), (A.5), (A.6) *and* (A.8). *If* (A.9) *or* (A.10) *holds, there exists a viscosity solution* $\bar{u} \in C(\bar{\Omega})$ *of*

$$\begin{cases} Lu - f = 0 & in \ \ \Omega, \\ u = g & on \ \ \partial\Omega. \end{cases} \tag{4.1}$$

Proof. First of all, we remark that the comparison principle of viscosity solutions of (4.1) holds. (See H. Ishii-P. L. Lions [11] Theorem II.2.) In the case where (A.9) holds, there exists a solution $\bar{u} \in W^{2,p}_{loc}(\Omega) \cap C(\bar{\Omega})$ $(n < p < +\infty)$ of (4.1) satisfying $\bar{u} = g$ on $\partial\Omega$ by D. Gilbarg-N. S. Trudinger [8] Corollary 9.18. Hence it is also a unique viscosity solution of (4.1) satisfying $\bar{u} = g$ on $\partial\Omega$. (See P. L. Lions [14] Theorem I.2.)

In the case where (A.10) holds, we apply the barrier construction argument in A. O. Oleinik-E. V. Radkevic [16] Theorem 1.5.2. (cf. H. Ishii-S. Koike [10] Proposition 4.3 and S. M. Lenhart-N. Yamada [12] Theorem 2.2.) Let $\psi \in C^2(\Omega) \cap C(\bar{\Omega})$ be a function such that $\psi = g$ on $\partial\Omega$. We consider the following degenerate linear elliptic PDE;

$$\begin{cases} Lw - \hat{f} = 0 & \text{in} \ \ \Omega, \\ w = 0 & \text{on} \ \ \partial\Omega, \end{cases} \tag{4.2}$$

where $\hat{f} = a_{ij}\psi_{x_i x_j} - b_i\psi_{x_i} - c\psi + f$. Then for each $z \in \partial\Omega$, there exist a neighborhood V_z of z and a local barrier $\zeta_z \in C^2(\Omega \cap V_z) \cap C(\overline{\Omega \cap V_z})$ satisfying

$$\zeta_z(z) = 0,$$

$$\zeta_z \geqq 0 \qquad \text{on} \quad \overline{\Omega \cap V_z},$$

$$\zeta_z \geqq \frac{\hat{C}}{\lambda_0} \qquad \text{on} \quad \Omega \cap \partial V_z,$$

$$L\zeta_z - \hat{f} \geqq 0 \qquad \text{in} \quad \Omega \cap V_z,$$

where $\hat{C}$ is a constant depending on $\|a_{ij}\|_{C(\bar{\Omega})}$, $\|b_i\|_{C(\bar{\Omega})}$, $\|c\|_{C(\bar{\Omega})}$, $\|f\|_{C(\bar{\Omega})}$, $\|\psi\|_{C(\bar{\Omega})}$, $\|D\psi\|_{L^\infty(\Omega)}$, $\|D^2\psi\|_{L^\infty(\Omega)}$. Hence setting

$$\hat{\zeta}_z(x) = \begin{cases} \min\left\{\zeta_z(x), \dfrac{\hat{C}}{\lambda_0}\right\} & \text{for } x \in \bar{V}_z, \\ \dfrac{\hat{C}}{\lambda_0} & \text{otherwise}, \end{cases}$$

$$\zeta(x) = \inf\{\hat{\zeta}_z(x) | z \in \partial\Omega\},$$

we observe that $\hat{\zeta}_z \in C(\bar{\Omega})$ is a viscosity supersolution of (4.2) and thus ζ is u.s.c and is a viscosity supersolution of (4.2) such that $\zeta = 0$ on $\partial\Omega$. In the same way we can construct a viscosity subsolution ζ' of (4.2) satisfying $\zeta' = 0$ on $\partial\Omega$. Hence by Perron's method there exists a viscosity solution $w \in C(\bar{\Omega})$ of (4.2) satisfying $w = 0$ on $\partial\Omega$. Therefore $\bar{u} = w + \psi$ is a viscosity solution of (4.1) satisfying $\bar{u} = g$ on $\partial\Omega$. Indeed, for any $\varphi \in C^2(\Omega)$, suppose $\bar{u} - \varphi$ attains a local maximum at $x_0 \in \Omega$. Since $w - (\varphi - \psi)$ attains a local maximum at $x_0 \in \Omega$, we get

$$- a_{ij}(x_0)\{\varphi_{x_i x_j}(x_0) - \psi_{x_i x_j}(x_0)\} + b_i(x_0)\{\varphi_{x_i}(x_0) - \psi_{x_i}(x_0)\} + c(x_0)w(x_0) - \hat{f}(x_0) \leqq 0.$$

The definition of $\hat{f}$ implies that

$$- a_{ij}(x_0)\varphi_{x_i x_j}(x_0) + b_i(x_0)\varphi_{x_i}(x_0) + c(x_0)\{w(x_0) + \psi(x_0)\} - f(x_0) \leqq 0.$$

Thus $\bar{u}$ is a viscosity subsolution of (4.1). We can show similarly that $\bar{u}$ is a viscosity supersolution of (4.1). Hence we have obtained the result. ■

By Lemma 4.2, it is easily seen that $\bar{u}$ is a strong viscosity supersolution of the problem (1.1)–(1.2). Next we show that Perron's method can be used for (1.1)–(1.2).

Proposition 4.3. *We define the set S and the function u as follows;*

$$S = \{v : \bar{\Omega} \longrightarrow \boldsymbol{R} \,|\, v \text{ is a strong viscosity subsolution of } (1.1)\text{–}(1.2)\},$$

$$u(x) = \sup\{v(x) | v \in S\} \qquad (x \in \bar{\Omega}).$$

Then the following properties hold.
(P.1) $u \in S$.
(P.2) *If* $v \in S$ *is not a strong viscosity supersolution of* (1.1), *then there exists* $w \in S$ *such that* $v(y) < w(y)$ *for some* $y \in \bar{\Omega}$.

Proof. We note that $S \neq \emptyset$ because, for the same constant C as in Lemma 3.2, $\underline{u} \equiv -C \in S$. Moreover the function $\bar{u}$ in Lemma 4.2 is a strong viscosity supersolution of (1.1)–(1.2). Thus by the definition of u and Theorem 3.1 we observe that $\underline{u} \leqq u \leqq \bar{u}$ on $\bar{\Omega}$.

Now, we shall prove (P.1) holds. Suppose that for $\varphi \in C^2(\Omega)$, $u^* - \varphi$ attains a local maximum at $x_0 \in \Omega$. Without loss of generality, we may assume

$$u^*(x_0) - \varphi(x_0) = 0, \quad u^*(x) - \varphi(x) \leqq 0 \qquad \text{in } \Omega,$$

and

$$u^*(x) - \varphi(x) \leqq -|x - x_0|^4 \qquad \text{in } B(x_0, r) \text{ for some } r > 0.$$

Then there exists $\{x_n\}_{n \in \boldsymbol{N}} \subset B(x_0, r)$ such that

$$x_n \longrightarrow x_0 \quad \text{and} \quad u^*(x_n) - \varphi(x_n) \longrightarrow 0.$$

Because of the definition of u, there exists $\{u_n\}_{n \in \boldsymbol{N}} \in S$ satisfying

$$u_n^*(x_n) - \varphi(x_n) > u^*(x_n) - \varphi(x_n) - \frac{1}{n},$$

$$u_n^*(x) - \varphi(x) \leqq -|x - x_0|^4 \qquad \text{on } \overline{B(x_0, r)}.$$

Let $y_n \in \overline{B(x_0, r)}$ be such that

$$u_n^*(y_n) - \varphi(y_n) = \max_{x \in \overline{B(x_0, r) \cap \Omega}} (u_n^*(x) - \varphi(x)).$$

Thus we get

$$u^*(x_n) - \varphi(x_n) - \frac{1}{n} < u_n^*(x_n) - \varphi(x_n) \leqq u_n^*(y_n) - \varphi(y_n)$$
$$\leqq u^*(y_n) - \varphi(y_n) \leqq -|y_n - x_0|^4.$$

Therefore we have $y_n \to x_0$ and

$$\lim_{n \to +\infty} u^*(y_n) = \lim_{n \to +\infty} \varphi(y_n) = \varphi(x_0) = u^*(x_0).$$

Since u_n is a strong viscosity subsolution of (1.1)–(1.2), we obtain

$$\max\{-a_{ij}(y_n)\varphi_{x_ix_j}(y_n)+b_i(y_n)\varphi_{x_i}(y_n)+c(y_n)u_n^*(y_n)$$
$$-f(y_n),\ u_n^*(y_n)-Mu_n^*(y_n)\}\leqq 0$$

Remarking that $u_n \leqq u$ on $\bar{\Omega}$ and Proposition 2.3 (5), we have

$$\limsup_{n\to+\infty} Mu_n^*(y_n)\leqq \limsup_{n\to+\infty} Mu^*(y_n)\leqq Mu^*(x_0)$$

and sending $n\to+\infty$, we get

$$\max\{-a_{ij}(x_0)\varphi_{x_ix_j}(x_0)+b_i(x_0)\varphi_{x_i}(x_0)$$
$$+c(x_0)u^*(x_0)-f(x_0),\ u^*(x_0)-Mu^*(x_0)\}\leqq 0.$$

As to the boundary condition (1.2), $u\leqq \bar{u}$ on $\bar{\Omega}$ and $\bar{u}\in C(\bar{\Omega})$ imply $u^*-g\leqq 0$ on $\partial\Omega$. Moreover, by the definition of u and Proposition 2.1 (1) we get

$$v^*-Mu^*\leqq v^*-Mv^*\leqq 0 \qquad \text{on } \partial\Omega \text{ for } v\in S,$$

since $v^*-Mv^*\leqq 0$ on $\partial\Omega$. Thus we have $\max\{u^*-g,\ u^*-Mu^*\}\leqq 0$ on $\partial\Omega$. Hence $u\in S$.

Next, we suppose $v\in S$ is not a strong viscosity supersolution of (1.1)–(1.2). Then there exists $x_0\in\bar{\Omega}$ and $\beta>0$ such that

$$\max\{-a_{ij}(x_0)X_{ij}+b_i(x_0)p_i+c(x_0)v_*(x_0)-f(x_0),$$
$$v_*(x_0)-Mv_*(x_0)\}\leqq -\beta \tag{4.3}$$
$$\text{for some } (p,X)\in \bar{J}^{2,-}v_*(x_0) \text{ if } x_0\in\Omega,$$

$$\max\{v_*(x_0)-g(x_0),\ v_*(x_0)-Mv_*(x_0)\}\leqq -\beta \qquad \text{if } x_0\in\partial\Omega, \tag{4.4}$$

We consider the following two cases.

(I) *The case* (4.3). We can find $\varphi\in C^2(\Omega)$ satisfying $D\varphi(x_0)=p$ and $D^2\varphi(x_0)=X$ and fix it. Furthermore, we may assume that

$$v_*(x_0)=\varphi(x_0),\quad v_*(x)\geqq \varphi(x)+|x-x_0|^4\quad (x\in B(x_0,r))$$

for some $r>0$.

We claim $v_*(x_0)<\bar{u}(x_0)$. If otherwise, $v_*(x_0)=\bar{u}(x_0)$ and $\bar{u}(x)-\varphi(x)$ attains its minimum at x_0. Therefore we get

$$\max\{-a_{ij}(x_0)\varphi_{x_ix_j}(x_0)+b_i(x_0)\varphi_{x_i}(x_0)+c(x_0)\bar{u}(x_0)$$
$$-f(x_0),\ \bar{u}(x_0)-M\bar{u}(x_0)\}\geqq 0,$$

because $\bar{u}\in C(\bar{\Omega})$ is a strong viscosity supersolution of (1.1)–(1.2). This is a contradiction.

Thus there exists $\delta_1 > 0$ such that

$$u_*(x_0) + \delta_1 < \bar{u}(x) \qquad \text{for } x \in B(x_0, \delta_1).$$

Using (A.3), (A.4), (A.5), (A.6), the continuity of φ and the lower semi-continuity of Mv_*, we have, for $0 < {}^\exists\delta < \min\{r, \delta_1, 1\}/2$,

$$\begin{aligned} \max\{&- a_{ij}(x)\varphi_{x_i x_j}(x) + b_i(x)\varphi_{x_i}(x) + c(x)\{\varphi(x) + \delta^4\} \\ &- f(x), \varphi(x) + \delta^4 - Mv_*(x)\} \leqq 0 \qquad \text{for } x \in B(x_0, 2\delta). \end{aligned}$$

Hence $\varphi(x) + \delta^4$ is a viscosity subsolution of

$$\max\{Lu - f, u - Mv_*\} = 0 \qquad \text{in } B(x_0, 2\delta).$$

We define

$$w(x) = \begin{cases} \max\{\varphi(x) + \delta^4, v(x)\} & \text{for } x \in B(x_0, \delta), \\ v(x) & \text{otherwise.} \end{cases}$$

We note that if $\delta \leqq |x - x_0| \leqq 2\delta$, then $v_*(x) \geqq \varphi(x) + \delta^4$ and by $Mv(x) \leqq Mw(x)$ on $\bar{\Omega}$, we get

$$\max\{w^* - g, w^* - Mw^*\} \leqq 0 \qquad \text{on } \partial\Omega.$$

Therefore by the similar argument to the proof of (P.1) we can observe that $w \in S$. Since $0 = v_*(x_0) - \varphi(x_0) = \lim_{s \to 0} \inf\{v(x) - \varphi(x) \mid |x - x_0| < s, \ x \in \bar{\Omega}\}$, there exists $y \in B(x_0, \delta)$ such that $\varphi(y) + \delta^4 > v(y)$.

(II) *The case* (4.4). We may assume that $\inf_{\bar{\Omega}} v > -\infty$. Because, if otherwise, there exists a point $y \in \bar{\Omega}$ such that $v(y) < \underline{u}$. Thus $\underline{u}$ is a desired function.

(4.4) implies

$$v_*(x_0) \leqq \min\{g(x_0), Mu_*(x_0)\} - \beta.$$

So we can find $\delta_0 > 0$ such that

$$v_*(x_0) + \frac{1}{2}\beta < \min\{g(x), Mv_*(x)\} \qquad \text{in } B(x_0, \delta_0).$$

Hence by the barrier argument there exist $0 < \varepsilon_0 < \delta_0$ and $\zeta \in C^2(B(x_0, \varepsilon_0) \cap \Omega) \cap C(\overline{B(x_0, \varepsilon_0)} \cap \bar{\Omega})$ satisfying

$$\begin{aligned} &\zeta(x_0) = v_*(x_0) + \frac{1}{2}\beta, \\ &\zeta \leqq g && \text{in } B(x_0, \varepsilon_0) \cap \partial\Omega, \\ &\zeta \leqq Mv_* && \text{in } \overline{B(x_0, \varepsilon_0)} \cap \bar{\Omega}, \end{aligned}$$

$$L\zeta - f \leqq 0 \qquad \text{in} \quad B(x_0, \varepsilon_0) \cap \Omega,$$
$$\zeta < \inf_{\Omega} v \qquad \text{on} \quad \partial B(x_0, \varepsilon_0) \cap \bar{\Omega}.$$

We define

$$w(x) = \begin{cases} \max\{\zeta(x), v(x)\} & \text{for } x \in B(x_0, \varepsilon_0) \cap \bar{\Omega}, \\ v(x) & \text{otherwise.} \end{cases}$$

Noting $v_*(x) \geqq \zeta(x)$ if $|x - x_0| = \varepsilon_1$ for $0 < {}^{\exists}\varepsilon_1 < \varepsilon_0$, we observe $w \in S$ as in (I).

By the definition of ζ, we obtain

$$v_*(x_0) + \frac{1}{2}\beta = \zeta(x_0) = w_*(x_0) > v_*(x_0).$$

Thus we can find $y \in \bar{\Omega}$ such that $v(y) < w(y)$. ■

Now, we can prove Theorem 4.1.

Proof of Theorem 4.1. Let u be as in Proposition 4.3. Then the assertions (P.1) and (P.2) in Proposition 4.3 imply that u is a viscosity sub-and supersolution in the strong sense. Therefore by Theorem 3.1 we have $u^* \leqq u_*$ on $\bar{\Omega}$. Then combining this inequality with $u_* \leqq u \leqq u^*$, we obtain $u_* = u = u^*$ on $\bar{\Omega}$ and $u \in C(\bar{\Omega})$. Moreover u satisfies the boundary condition (1.2) for each $x \in \partial\Omega$. Using Theorem 3.1 again, we get the uniqueness of strong viscosity solutions. Thus the proof is completed. ■

Remark. Of course, we can extend Theorems 3.1 and 4.1 to Hamilton-Jacobi-Bellman equation with impulse control

$$\begin{cases} \max\{\sup_{v \in V}\{L^v u - f^v\}, u - Mu\} = 0 & \text{in} \quad \Omega, \\ \max\{u - g, u - Mu\} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

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