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Global Bounded Weak Solutions for an Abstract Nonlinear Timoshenko Beam Equation with Four Propagation Speeds

By

A. AROSIO and S. PANIZZI

(Università di Parma, Italy)

§1. Introduction and statement of the main result

Let V and H be separable Hilbert spaces over a field $\mathscr{K}$ (which may be indifferently the field $\boldsymbol{R}$ of real numbers, or the field $\boldsymbol{C}$ of complex ones), such that $V \subseteq H$ densely and continuously. Denote by V' the (anti) dual space of V: then, thanks to the Riesz identification of H with its own (anti) dual space, we have

$$V \subseteq H \subseteq V'.$$

Without loss of generality, we may suppose that $\|\cdot\|_V = \langle A\cdot, \cdot\rangle^{1/2}$. Also, for simplicity, we will denote $\| \ \|_V$ simply by $\| \ \|$. We denote the (anti) duality between V and V' by $\langle \cdot, \cdot \rangle$, the inner product in H by $(\cdot, \cdot)$, and the norm of H by $| \ |$. Let $A: V \to V'$ be a symmetric positive definite isomorphism, and let λ_1 be the best constant such that

$$\langle Av, v\rangle \geq \lambda_1 |v|^2, \quad \lambda_1 > 0 \quad (v \in V).$$

Finally, assume that

(1) *$M: V \to V'$ is a weakly sequentially continuous (nonlinear) operator, which admits a real valued potential F.*

In (1), it is sufficient to intend the notion of potential in the sense of Gateaux: this amounts to say that the functional $F: V \to \boldsymbol{R}$ satisfies for each v in V

$$F(v + \rho h) = F(v) + \rho \operatorname{Re} < M(v), h > + o(\rho) \qquad (\boldsymbol{R} \ni \rho \to 0).$$

To fix ideas, we assume that $F(0) = 0$.

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For any real number γ, let us denote $\Box_\gamma = \partial_{tt} + \gamma A$. For a, b, c real numbers, with

$$0 < a < b,$$

let us consider the 4th order evolution equation

$$\Box_a \Box_b u + M(\Box_c u) = 0 \qquad (t > 0). \tag{2}$$

In the case when $M \equiv mI_V$ (i.e. $F(u) = m|u|^2/2$) for some $m \in \boldsymbol{R}$, Eq. (2) reduces to

$$\Box_a \Box_b u + m\Box_c u = 0 \qquad (t > 0), \tag{3}$$

which is the abstract version of the Timoshenko beam equation [T]:

$$(\partial_{tt} - a\partial_{xx})(\partial_{tt} - b\partial_{xx})u + m(\partial_{tt} - c\partial_{xx})u = 0 \qquad (0 < x < L,\ t > 0). \tag{4}$$

For $m > 0$, Eq. (4) describes the small transverse vibrations of a stretched ($c \geq 0$), or compressed ($c \leq 0$) beam in a more accurate way (see [TC] [APR] [W] [AE] [K] [CHU], and also [NS]) than the classical Euler-Bernoulli and Rayleigh-Love ones. For an historical survey on Eq. (4), we refer the reader to [Kr]. Eq. (4) has received in the last years a renewal of interest from the point of view of Control theory, see e.g. [R] [KR] [Sc] [Ko] [IK] [D]. A corresponding two-dimensional model has been given for the plate [M] [U], cf. [LL].

In [APP] the authors of the present paper, jointly with M. G. Paoli, have studied some *linear* complications of Eq. (3), in which the d'Alembertians depends on different non-commuting operators (inhomogeneous Timoshenko beam equation), even varying with time (temporally inhomogeneous Timoshenko beam equation).

We are interested here in the (*global existence* and) *global boundedness* of solutions to the Cauchy problem for equations of the type of Eq. (2).

Results in this direction were already given in the works of M. G. Paoli [Pa] and S. Panizzi [P]. It is established there that the Cauchy problem for Eq. (2) has a (global) globally bounded solution in each one of the following cases:

i) [Pa]: $M = mI_V$ *for some* $m \geq 0$,
and

$$-\lambda_1 ab < mc < mb + 2^{-2}\lambda_1\{[b - a - m\lambda_1^{-1}]^+\}^2;$$

ii) [P]: $0 < a < c < b$,

$$\min\lim_{\|v\|\to 0} 2F(v)\|v\|^{-2} > -abc^{-1},$$

and the initial data are suitably small.

The above condition on F is clearly satisfied if the behaviour at the origin of F is superquadratic. Note that the interval of admissibility of the characteristic speed c is wider in the (non-negative) quadratic case i) than in the superquadratic one ii). It is easily seen that the result in the (non-negative) quadratic case is sharp. One can ask if the result in the superquadratic one is sharp too. For the moment, we have not succeeded in answering the question. What we do here is simply to give a statement which encompasses both of the ones above (see final Remark 1). Actually we prove that the following holds true:

Theorem 1 (Main result). *Assume that M satisfies condition* (1), *and let us consider the* 4th *order abstract nonlinear evolution equation*

$$\Box_a \Box_b u + m \Box_d u + M(\Box_c u) = 0 \qquad (t > 0), \tag{5}$$

where

$$m \geq 0, \tag{6}$$

$$0 < a < c < b, \tag{7}$$

$$-\lambda_1 ab < md < mc + \lambda_1 (b-c)(c-a). \tag{8}$$

Then ($[\cdot]^- =$ *negative part*)

$$\beta_0 \overset{\mathrm{def}}{=} abc^{-1}(1 - m\lambda_1^{-1}\{[d]^- a^{-1} b^{-1} + [c-d]^-(b-c)^{-1}(c-a)^{-1}\}) > 0 \tag{9}$$

and the Cauchy problem for Eq.(5) *admits at least one* (*global and*) *globally bounded weakly continuous solution in the phase space* $D(A^{3/2}) \times D(A) \times V \times H$, *provided that*

$$\min \lim_{\|v\| \to 0} 2F(v) \|v\|^{-2} > -\beta_0 \tag{10}$$

and the initial data are suitably small in the norm of the phase space.

For additional properties of the solution, see the final Remark 2.

Condition (10) is still satisfied if F is superquadratic at the origin. Thus, the above result says that the two different perturbations treated in [Pa] [P] may coexist in the same equation. As (8) shows, the intervals of admissibility of the characteristic speeds c and d interact only for positive values of d.

As for the technique of proof, it combines in a suitable way the ideas of [Pa] [P]. $\Box_a u'$ and $\Box_b u'$ are the obvious multiplicators for Eq.(5): by setting $\gamma = a, b$ in the key identity of [Pa] ($(\cdot)' = \partial_t$)

(11)
$$2\,\mathrm{Re}\int_0^t (\Box_d u,\ \Box_\gamma u')\,ds = d\gamma^{-1}|\Box_\gamma u|^2 + (1-d\gamma^{-1})|u''|^2 + (\gamma-d)\,\|u'\|_V^2 + \\ - d\gamma^{-1}|\Box_\gamma u(0)|^2 + (1-d\gamma^{-1})|u''(0)|^2 + (\gamma-d)\,\|u'(0)\|_V^2,$$

one gets *two non-autonomous* conserved quantities for Eq. (5). By making a suitable *convex* combination of them, with coeficients say λ and μ, one obtains *one autonomous* conserved quantity. Then, following [P], we change variables to

$$x_1 \overset{\mathrm{def}}{=} \mu^{1/2}\,\Box_b u, \quad x_2 \overset{\mathrm{def}}{=} \lambda^{1/2}\,\Box_a u, \tag{12}$$

Thus, Eq. (5) is reduced to a 2nd order one, which may be handled by an argument of *potential-well* type in the spirit of [S], see Theorem 2 below.

Plan of the paper: in §2 we give the proof of the main result, in §3 we outline an application to PDE's; all the remarks are concentrated in §4.

§2. Proof

Orientation. The line of our proof requires to pass to the new variable $x \overset{\mathrm{def}}{=} (x_1, x_2)$ given by (12), and to exhibit a conserved quantity in terms of these variable. In [P], where it is dealt with the case $m = 0$, this is done by considering the equivalent evolution equation satisfied by x. This is a second order equation in the phase space $\mathscr{V} \times \mathscr{H}$, where $\mathscr{V} \overset{\mathrm{def}}{=} V \times V$, $\mathscr{H} \overset{\mathrm{def}}{=} H \times H$:

$$x'' + \mathscr{A}x + \mathscr{M}(x) = 0 \qquad (t > 0).$$

Here $\mathscr{A}$, $\mathscr{M}$ are given respectively by

$$\mathscr{A} \overset{\mathrm{def}}{=} \begin{vmatrix} aA & 0 \\ 0 & bA \end{vmatrix}, \quad \mathscr{M} \overset{\mathrm{def}}{=} \begin{vmatrix} \mu^{1/2} M(\mu^{1/2}x_1 + \lambda^{1/2}x_2) \\ \lambda^{1/2} M(\mu^{1/2}x_1 + \lambda^{1/2}x_2) \end{vmatrix}.$$

where λ and μ are such that

$$\lambda a + \mu b = c, \qquad \lambda + \mu = 1.$$

A potential for $\mathscr{M}$ is given by the functional

$$\mathscr{F}(x) \overset{\mathrm{def}}{=} F(\mu^{1/2}x_1 + \lambda^{1/2}x_2).$$

Therefore one has immediately that (for regular solutions) the following quantity is a conserved one

$$|x'|^2 + \langle \mathscr{A}x, x\rangle + 2\mathscr{F}(x) = \text{constant}. \qquad (t \geq 0).$$

Then [P] concludes using a potential-well argument in the spirit of [S], see [A].

In the case $m > 0$, the new variable x satisfies a slightly complicated equation, namely

$$x'' + \mathscr{A}x + \mathscr{B}x + \mathscr{M}(x) = 0 \qquad (t > 0), \tag{13}$$

where

$$\mathscr{B} \overset{\text{def}}{=} m \begin{vmatrix} \mu' & \mu^{1/2}\lambda^{-1/2}\lambda' \\ \lambda^{1/2}\mu^{-1/2}\mu' & \lambda' \end{vmatrix},$$

μ', λ' being the coefficients such that

$$\Box_d = \lambda' \Box_a + \mu' \Box_b.$$

Now, except for the trivial case $d = c$, the new term $\mathscr{B}$ is an *asymmetric* operator, so it is not clear whether a conserved quantity does actually exist. As a matter of fact, such a quantity does exist, but it is hidden into Eq. (13): it can be more easily found out by starting directly with Eq. (5). We have found more interesting for the reader, instead of presenting our conserved quantity as a *deus ex machina*, to arrive to it through an euristic approach (i.e. by working in the variable u).

Proof of the main result. Thanks to the assumption (7), there exist two positive constants λ, μ such that

$$\lambda a + \mu b = c, \quad \lambda + \mu = 1. \tag{14}$$

For any regular enough function u, one has [Pa]

$$\begin{aligned} 2\,\mathrm{Re}\int_0^t [d(Au, u''') + \gamma(Au', u'')]\,ds \\ = 2\,\mathrm{Re}\int_0^t \{d[(Au, u''') + (Au', u'')] + (\gamma - d)(Au', u'')\}\,ds \\ = 2d\,\mathrm{Re}(Au, u'') + (\gamma - d)\,\|u'\|^2 - 2d\,\mathrm{Re}(Au(0), u''(0)) + (\gamma - d)\,\|u'(0)\|^2. \end{aligned}$$

This easily yields that the identity (11) holds true. Setting first $\gamma \overset{\text{def}}{=} b$, one obtains the following (non-autonomous) conserved quantity for (regular enough) solutions of Eq. (5):

$$\begin{aligned} E_1 \overset{\text{def}}{=} a\,\|\Box_b u\|^2 + |\Box_b u'|^2 \\ + m\{db^{-1}|\Box_b u|^2 + (1 - db^{-1})|u''|^2 + (b - d)\,\|u'\|^2\} \end{aligned}$$

$$+ 2\,\mathrm{Re}\int_0^t < M(\Box_c u),\ \Box_b u' > ds = \text{constant}.$$

Setting then $\gamma = a$ in the identity (11), one gets an analogous conserved quantity E_2. Clearly, any linear combination of those two conserved quantities is again a conserved quantity. In particular, let us form the *convex* combination of E_1 and E_2 with coefficients μ and λ given by (14). Now suppose that u is an (approximating) solution with finite dimensional range (as the approximating solutions of the Ritz-Galerkin method are). Using the identities

$$\Box_c = \lambda \Box_a + \mu \Box_b,$$

$$\mathrm{Re}\int_0^t \langle M(w), w' \rangle\, ds = F(w(t)) - F(w(0)) \qquad (t > 0)$$

(note that the second identity holds true thanks to the fact that (Gateaux diff.) $\wedge$ (finite dimension) $\Rightarrow$ Frechét diff.), and setting

$$\begin{aligned} E_{quadr.} \overset{\text{def}}{=} {} & \mu a \| \Box_b u \|^2 + \lambda b \| \Box_a u \|^2 \\ & + \mu | \Box_b u' |^2 + \lambda | \Box_a u' |^2 \\ & + md\{\mu b^{-1} | \Box_b u |^2 + \lambda a^{-1} | \Box_a u |^2\} \\ & + m\{1 - d(\mu b^{-1} + \lambda a^{-1})\} |u''|^2 \\ & + m(c - d) \| u' \|^2, \end{aligned}$$

we obtain the following (autonomous) conserved quantity:

$$E_{tot.} \overset{\text{def}}{=} E_{quadr.} + 2F(\Box_c u) = \text{constant}. \tag{15}$$

Now, as in [P], we introduce the new variable $\mathscr{x} = (x_1, x_2)$ given by (12). With respect to these new variables, $E_{quadr.}$ becomes

$$\begin{aligned} E_{quadr.} \overset{\text{def}}{=} {} & a \| x_1 \|^2 + b \| x_2 \|^2 && (1^{st}) \\ & + |x_1'|^2 + |x_2'|^2 && (2^{nd}) \\ & + mda^{-1}b^{-1}\{a|x_1|^2 + b|x_2|^2\} && (3^{rd}) \\ & + m\{1 - d(\mu b^{-1} + \lambda a^{-1})\}(b - a)^{-2} |a\mu^{-1/2} x_1 - b\lambda^{-1/2} x_2|^2 && (4^{th}) \\ & + m(c - d)(b - a)^{-2} \| \mu^{-1/2} x_1' - \lambda^{-1/2} x_2' \|_{V'}^2, && (5^{th}) \end{aligned}$$

while the total energy conservation (15) becomes (see also the final Remark 3)

$$(16) \qquad E_{tot.} \overset{\text{def}}{=} E_{quadr.} + 2F(\mu^{1/2}x_1 + \lambda^{1/2}x_2) = \text{constant}.$$

To prove the global existence of a solution $\mathscr{x}$ in the phase space $\mathscr{V} \times \mathscr{H}$ (where $\mathscr{V} \overset{\text{def}}{=} V \times V$, $\mathscr{H} \overset{\text{def}}{=} H \times H$), we will use an argument of potential well type, see Theorem 2 below. For that, it is enough to prove that (the parameters a, b, c, d are such that):

i) $E_{quadr.}$ above defines a topologically equivalent (squared) norm on the phase space $\mathscr{V} \times \mathscr{H}$;

ii) the non-quadratic term F in (16) does not affect the positive quadratic behaviour of $E_{quadr.}$ in a small neighbourhood of the origin in $\mathscr{V} \times \mathscr{H}$.

Now, since $E_{quadr.}$ is a quadratic form, i) will hold true provided that its 3rd and 4th (resp.: 5th) terms do not perturbate the positive definiteness of the 1st (resp.: 2nd) one. Indeed, we claim that for any choice of the parametres a, b, c which satisfies (7), and for any choice of the parametre d, the following inequality holds true:

$$(17) \qquad E_{quadr.} \geq \{1 - m\lambda_1^{-1}([d]^- a^{-1}b^{-1} + [c-d]^-(b-c)^{-1}(c-a)^{-1})\} \times (a\|x_1\|^2 + b\|x_2\|^2 + |x_1'|^2 + |x_2'|^2).$$

To verify that (17) holds true, we divide the discussion in two cases, acording to the sign of the parameter d.

1st *case*: $d < 0$: the claim follows easily from the consideration that in this case the 4th and 5th terms are nonnegative, and from the inequality

$$(18) \qquad -|\cdot|^2 \geq -\lambda_1^{-1}\|\cdot\|^2;$$

2nd *case*: $d \geq 0$: in this case, we need the following simple

Lemma. *Let* $(Z, \|\cdot\|)$ *be a Hilbert space. Let* $y, z \in Z$ *and* $\alpha, \beta \in \boldsymbol{R}$. *Then*

$$-\|\alpha y - \beta z\|^2 \geq -(\alpha^2 + \beta^2)(\|y\|^2 + \|z\|^2).$$

We claim that, thanks to (14), (17) and (18), we have

$$\begin{aligned} &3^{rd} + 4^{th} \\ &\geq m\{da^{-1}b^{-1} - [1 - d(\lambda a^{-1} + \mu b^{-1})]^-(b-a)^{-2}(a\mu^{-1} + b\lambda^{-1})\}\{a|x_1|^2 + b|x_2|^2\} \\ &\geq -m[c-d]^-(b-c)^{-1}(c-a)^{-1}\{a|x_1|^2 + b|x_2|^2\} \\ &\geq -m\lambda_1^{-1}[c-d]^-(b-c)^{-1}(c-a)^{-1}\{a\|x_1\|^2 + b\|x_2\|^2\}. \end{aligned}$$

Indeed, the second inequality is equivalent to

$$d(b-c)(c-a) - [ab - d(b+c-a)]^- c + [c-d]^- ab \geq 0.$$

This is quite evident in the case $ab - d(b+c-a) \geq 0$. On the other hand, we note that the left-hand term in the inequality to be proven equals

$$- [ab - d(b + c - a)]^{+} c + [c - d]^{+} ab,$$

which also in the complementary case (i.e. when $ab - d(b + c - a) < 0$) is clearly non-negative.

Also

$$\begin{aligned} 5^{\text{th}} &\geq - m[c - d]^{-}(b - a)^{-2}(\mu^{-1} + \lambda^{-1})\{\|x_1'\|_{V'}^2 + \|x_1'\|_{V'}^2\} \\ &\geq - m[c - d]^{-}(b - c)^{-1}(c - a)^{-1}\{\|x_1'\|_{V'}^2 + \|x_1'\|_{V'}^2\} \\ &\geq - m\lambda_1^{-1}[c - d]^{-}(b - c)^{-1}(c - a)^{-1}\{|x_1'|^2 + |x_1'|^2\}. \end{aligned}$$

By summing up, the claim (17) is proved. Thanks to (9), the coefficient in (17) is > 0, and therefore point i) is achieved.

Now we proceed to prove point ii): indeed from assumption (10) we have, thanks to the Lemma above, to (14) and to (17), that for any β such that

$$- \beta_0 < - \beta < \min\lim_{\|v\| \to 0} 2F(v)\|v\|^{-2}$$

for small enough $\mathscr{x}$ one has

$$\begin{aligned} (19) \qquad E_{tot.} &= E_{quadr.} + 2F(\mu^{1/2}x_1 + \lambda^{1/2}x_2) \\ &\geq E_{quadr.} - \beta\|\mu^{1/2}x_1 + \lambda^{1/2}x_2\|^2 \\ &\geq E_{quadr.} - [\beta]^{+}(\mu a^{-1} + \lambda b^{-1})\{a\|x_1\|^2 + b\|x_2\|^2\} \\ &= E_{quadr.} - [\beta]^{+} ca^{-1}b^{-1}\{a\|x_1\|^2 + b\|x_2\|^2\} \\ &\geq \varepsilon E_{quadr.} \end{aligned}$$

where

$$\varepsilon \overset{\text{def}}{=} 1 - [\beta]^{+}/\beta_0 > 0.$$

Therefore also point ii) is proved.

Now we need the following

Theorem 2 [A]. *Let $\mathscr{V}$ and $\mathscr{H}$ be separable Hilbert spaces, $\mathscr{V} \subseteq \mathscr{H}$ densely and continuously. Let $\mathscr{M}^{\sim} : \mathscr{V} \to \mathscr{V}'$ be a (nonlinear) weakly sequentially continuous operator, and consider the second order evolution equation*

$$(20) \qquad \mathscr{x}'' + \mathscr{M}^{\sim}(\mathscr{x}) = 0 \qquad (t > 0).$$

Assume that there exists a continuous functional

$$\mathscr{F}^{\sim} : \mathscr{V} \times \mathscr{H} \longrightarrow \mathbf{R}, \qquad \mathscr{F}^{\sim}(0, 0) = 0,$$

such that

i) *there esists some sequence $(\mathscr{x}_n)$ of (maximal) Ritz-Galerkin approximating* *solutions for Eq.*(20), *such that for each* n *the quantity* $\mathscr{F}^{\sim}(x_n(t), x_n'(t))$ *is non-increasing with respect to time*:

ii) $\min \lim_{\|(x, y)\| \to 0} \mathscr{F}^{\sim}(x, y)\, \|(x, y)\|^{-2} > 0.$

Then the Cauchy problem for Eq.(20) *admits at least one* (*global and*) *globally bounded weakly continuous solution in the phase space* $\mathscr{V} \times \mathscr{H}$, *provided that the initial data are suitably small in the norm of the phase space.*

Let us check the hypotheses of the above Theorem 2 for the choice $\mathscr{M}^{\sim} \overset{\text{def}}{=} \mathscr{A} + \mathscr{B} + \mathscr{M}$, $\|(x, y)\| \overset{\text{def}}{=} E_{quadr.}$ and $\mathscr{F}^{\sim} \overset{\text{def}}{=} E_{tot.}$. Indeed, if we choose the finite-dimensional spaces $\mathscr{V}_n$ for the Ritz-Galerkin method of the type $\mathscr{V}_n \overset{\text{def}}{=} V_n \times V_n$, then the n^{th} Ritz-Galerkin approximating equation reads as

$$x_n'' + \mathscr{P}_n \mathscr{A} x_n + \mathscr{B} x_n + \mathscr{P}_n \mathscr{M}(x_n) = 0 \qquad (t > 0).$$

So, if we take $\mathscr{V}_n$, $\mathscr{P}_n\mathscr{A}$, $\mathscr{P}_n\mathscr{M}$, $\mathscr{F}|_{\mathscr{V}_n}$ in place respectively of $\mathscr{V}$, $\mathscr{A}$, $\mathscr{M}$, $\mathscr{F}$ we are allowed to apply the energy equality (16) to each x_n. This relation yields hypotheses i) of Theorem 2. Hypothesis ii) follows immediately from (19), so Theorem 1 is achieved.

§3. An application

We outline an application of Theorem 1 to a mixed problem for a PDE of hyperbolic type.

Let Ω be a *bounded* regular (C^3) open set of $\mathscr{R}^n (n \geq 3)$, and consider the Cauchy-Dirichlet problem

(21)

$$(\partial_{tt} - a\Delta_x)(\partial_{tt} - b\Delta_x)u + m(\partial_{tt} - d\Delta_x)u + g|(\partial_{tt} - c\Delta_x)u|^{p-1}(\partial_{tt} - c\Delta_x)u = 0,$$

$$u(\cdot, t)|_{\partial\Omega} = \Delta_x u(u \cdot t)|_{\partial\Omega} = 0 \qquad (t \geq 0).$$

Let λ_1 denote the first eigenvalue of the laplacian $(-\Delta_x)$ with respect to the Dirichlet boundary value problem, and assume that the real numbers a, b, c, d, s, g, p are subjected to the following limitations:

$$m \geq 0,$$

$$0 < a < c < b,$$

$$-\lambda_1 ab < md < mc + \lambda_1(b - c)(c - a),$$

$$1 < p \leq (n + 2)(n - 2)^{-1} \quad \textit{if} \quad n \geq 3 \ (1 < p \ \textit{if} \ n \leq 2)$$

(*no limitation on* g *is imposed*).

By an application of Theorem 1 (for the critical value of p, see final Remark 4), the Cauchy-Dirichlet problem (21) results to admit a (global and) globally bounded weakly continuous solution in the phase space

$$(H^3 \cap H_0^1 \cap \{-\Delta_x u \in H_0^1\}) \times (H^2 \cap H_0^1) \times H_0^1 \times L^2,$$

for initial data which are suitably small in the phase space.

If we exclude the critical case $p = (n+2)(n-2)^{-1}$ for $n \geq 3$ (or if we assume that $g \geq 0$), then the functional $E_{tot.}$ defined by (16) satisfies $E_{tot.} \leq E_{tot.}(0)$, and the solution is strongly continuous in the phase space at $t = 0^+$ (see the final Remark 2).

If the exponent p is subjected to the stronger restriction

$$1 < p \leq n(n-2)^{-1},$$

then the solution results to be unique: therefore by the final Remark 3 one has that $E_{tot.} =$ constant, and that the solution is strongly continuous in the phase space at each $t \geq 0$.

§4. Final remarks

Remark 1. The result of [P] is easily recovered from Theorem 1 by setting $m = 0$ (or, alternatively, $d = c$). The result of [Pa] too may be recovered, by setting $M \equiv 0$, since in that case condition (7) and (8) may be read as

$$^{\exists}c: a < c < b \quad and \quad md < mc + \lambda_1(b-c)(c-a),$$

and this is equivalent to

$$md < \sup_{a<c<b}\{mc + \lambda_1(b-c)(c-a)\},$$

i.e.

$$md < mb + 2^{-2}\lambda_1\{[b - a - m\lambda_1^{-1}]^+\}^2.$$

Remark 2. If in Theorem 1 (resp.: Theorem 2) some additional assumptions are made, then the solution enjoys some additional properties.

If in Theorem 2 it is assumed that

$$\mathscr{F}^{\sim}(x, y) \overset{\text{def}}{=} \|(x, y)\|^2 + \mathscr{F}(x, y),$$

where

$\mathscr{F}$ is weakly sequentially lower semicontinuous on $\mathscr{V} \times \mathscr{H}$,

then [A] the solution x of Eq. (20) provided by Theorem 2 satisfies

$$\mathscr{F}^{\sim}(x(t), x'(t)) \leq \mathscr{F}^{\sim}(x(0), x'(0)) \qquad (t \geq 0),$$

(x, x') *is strongly continuous in the phase space* $\mathscr{V} \times \mathscr{H}$ *at* $t = 0^+$.

If the Cauchy problem for Eq. (20) is uniquely solvable for any choice of the initial data, then [A]

$$\mathscr{F}^{\sim}(x(t)x'(t)) = constant \qquad (t \geq 0),$$

(x, x') *is strongly continuous in the phase space at each point* $t \geq 0$.

As a consequence, if in Theorem 1 it is assumed that

F is weakly sequentially lower semicontinuous on V,

then for the solution u of Eq. (5) provided by Theorem 1 one has

$$E_{tot.} \leq E_{tot.}(0) \qquad on \ [0, +\infty[,$$

(u, u', u'', u''') *is strongly continuous in the phase space* $D(A^{3/2}) \times D(A) \times V \times H$ *at* $t = 0^+$.

In the case when the Cauchy problem for Eq. (5) is uniquely solvable for any choice of the initial data, then

$$E_{tot.} = constant \qquad on \ [0, +\infty[,$$

(u, u', u'', u''') *is strongly continuous in the phase space at each point* $t \geq 0$.

Remark 3. Of course the reader can check directly that (16) holds true by differentiation with respect to time, noting that any solution of Eq. (13) satisfies the equality

$$A^{-1}(\mu^{-1/2}x_1'' - \lambda^{-1/2}x_2'') = b\lambda^{-1/2}x_2 - a\mu^{-1/2}x_1.$$

Remark 4. A slightly stronger form of Theorem 1 holds true, in which the assumption (1) on M is replaced by the following weaker one:

$$(22) \qquad \begin{aligned} &(u_k) \longrightarrow u \ in \ L^\infty([0, T]; V) - \text{weak*}, \ and \\ &(u_k') \longrightarrow u' \text{ in } L^\infty([0, T]; H) - \text{weak*} \\ &\Longrightarrow (M(u_k)) \longrightarrow M(u) \ in \ L^1([0, T]; V') - \text{weak}. \end{aligned}$$

Such a stronger version of Theorem 1 leans upon a corresponding version of Theorem 2 (see [A]). It permits to prove the statement in the application of §4 to the critical value of p (to check that $M(u) \overset{\text{def}}{=} u^p$ fulfills (22), one may argue as in the proof of Theorem 1.1 of Ch. 1 in [L]).

Remark 5. In Theorem 1 (resp.: Theorem 2), the operator M (resp.: $\mathscr{M}$) need not be defined on the whole space V (resp.: $\mathscr{V}$), but only on a small open ball centered at the origin.

Remark 6. By the same methods one can treat equations of the type

$$\Box_a \Box_b u + \textstyle\sum_{i=1,\dots,N} m_i A^{\alpha_i} \Box_{d_i} u + M(\Box_c u) = 0 \qquad (t > 0),$$

provided that $0 \leq \alpha_i \leq 1$ $(i = 1,\dots,N)$.

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nuna adreso:
Dipartimento di Matematica
Università di Parma
v. D'Azeglio 85/A
43100 Parma-Italy

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