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An Application of the Eilenberg-Montgomery Theorem to Measurable Orientor Fields on Manifolds

By

Stanisław DOMACHOWSKI and Tadeuz PRUSZKO
(Gdańsk University, Poland)

This paper is concerned with the initial boundary value problem for convex-valued orientor fields on a closed C^2-manifold and the result of Pliś [6] is extended. The main tool used in the proof of the existence theorem is the Eilenberg-Montgomery Fixed-Point Theorem for acyclic maps and the main idea in the reduction of the existence problem rely on the indication of the Poincaré operator suitable associated with the orientor field. As a result we obtain existence of the integral curve of a continuous vector field on a closed C^2-manifold or the existence of solutions of boundary value problems for orientor fields, in the Euclidean space $\mathbf{R}^n$, with a "local" nonlinear boundary condition, $L: \mathbf{R}^n \to \mathbf{R}^k$.

Note added in Introduction. We have just learned that the orientor fields on a compact subset of $\mathbf{R}^n$ had been considered by S. Plaskacz (J. Math. Anal. Appl., 148 (1990), 202–212). Nevertheless the paper remains unchanged for two reasons: one for completeness and the other that the methods of the proofs presented here are quite different from his and might be used to partial differential equations too.

1. Definitions and assumptions

In this paper the term manifold will be used to mean of a closed C^2-manifold M^n, modeled on Euclidean space $\mathbf{R}^n$, with a metric $\rho: M^n \times M^n \to \mathbf{R}_+$ coincided with the topology on M^n.

Let $\mathscr{A} = \{(U_1, \varphi_1), \dots, (U_k, \varphi_k)\}$ be an atlas on M^n and let $\{W_i\}_{i=1}^k$ be an open covering of M^n such that $\bar{W}_i \subset U_i$ for $i = 1, \dots, k$. Recall that for each point $p \in M^n$ and a chart $(U_i, \varphi_i) \in \mathscr{A}$ such that $p \in U_i$ there is given a linear isomorphism $\lambda_i^p = \lambda_{\varphi_i}^p: \mathbf{R}^n \to T_pM^n$ such that for every C^1-path at p, $l: (-\varepsilon, \varepsilon) \to M^n$ ($l(0) = p$), the following condition is satisfied

$$\lambda_i^p[(\varphi_i \circ l)'(0)] = [l]$$

where $(\varphi_i \circ l)'(0)$ is the gradient of the C^1-map $(\varphi_i \circ l): (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ in the point

$0 \in (-\varepsilon, \varepsilon)$. If $x\colon (a, b) \to M^n$ is a map, $(U_i, \varphi_i) \in \mathscr{A}$ and $t_0 \in (a, b)$ is a point such that $x(t_0) \in U_i$ and $(\varphi_i \circ x)\colon (a, b) \to \mathbf{R}^n$ is differentiable at t_0, then there is defined a tangent vector in the fibre $T_{x(t_0)}M^n$ by the following formula

$$\dot{x}(t_0) := \lambda_i^{x(t_0)}[(\varphi_i \circ x)'(t_0)] \tag{1.1}$$

By properties of the isomorphisms $\lambda_i^p\colon \mathbf{R}^n \to T_pM^n$, it follows that above definition does not depend on the choice of the chart $(U_i, \varphi_i) \in \mathscr{A}$.

A multi-valued map $F\colon X \to Y$ from a metric space X into another metric space Y, is upper semi continuous (u.s.c.) iff its graph $\Gamma_F = \{(x, y) \in X \times Y\colon y \in F(x)\}$ is a closed subset of $X \times Y$.

A convex-valued map $F\colon [a, b] \times M^n \to TM^n$, where $[a, b] \subset \mathbf{R}^1$ is a compact interval, is an orientor field on M^n iff, for every point $(t, p) \in [a, b] \times M^n$, $F(t, p) \neq \emptyset$ and $F(t, p) \subset T_pM^n$.
In what follows we make the assumptions.

Assumptions 1.2 (Carathéodory conditions).

(i) *for every* $t \in [a, b]$, $F(t, \cdot)$ *is an (u.s.c.) map from* M^n *into* TM^n;

(ii) *for every* $p \in M^n$, $F(\cdot, p)$ *is a measurable mapping from* $[a, b]$ *into* TM^n (*i.e. the counter-image of an open subset* $U \subset TM^n$ $\{t \in [a, b]\colon F(t, p) \cap U \neq \emptyset\}$ *is a measurable subset of* $[a, b]$);

(iii) *there exists an integrable function* $\eta\colon [a, b] \to \mathbf{R}_+$ *such that* $|(\lambda_i^p)^{-1}(F(t, p))| \leq \eta(t)$ *for each* $(t, p) \in [a, b] \times W_i$ $(i = 1, 2, \dots, k)$ ($|A|$ *denotes, the Hausdorff distance of the sets* $A \subset \mathbf{R}^n$ *and* $\{0\} \subset \mathbf{R}^n$)

Definition 1.3. Let $L\colon \mathbf{R}^n \to \mathbf{R}^k (k < n)$ be a C^2-map and let $c \in \mathbf{R}^k$ be a point such that the counter-image $L^{-1}(c) \subset \mathbf{R}^n$ is bounded and for each point $x \in L^{-1}(c)$ $\operatorname{rank} DL(x) = k$. A multi-valued map $F\colon [a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ is (L, c)-invariant iff for each $(t, x) \in [a, b] \times L^{-1}(c)$ $F(t, x) \cap \operatorname{Ker} DL(x) \neq \emptyset$.

2. Main theorems

Recall that M^n denotes a closed C^2-manifold.

Theorem 1. *If a convex-valued orientor field* $F\colon [a, b] \times M^n \to TM^n$ *satisfies Assumptions* 1.2, (i)-(iii), *then for each point* $q \in M^n$ *there exists at least one solution of the intial value porblem*

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \textit{a.e. on} \quad [a, b] \\ x(a) = q. \end{cases} \tag{2.1}$$

By a solution of (2.1) we call a map $x\colon [a, b] \to M^n$ such that the correspondence $[a, b] \ni t \to \dot{x}(t) \in TM^n$(cf. (1.1)) defines a.e. on $[a, b]$ a measurable map and such that equations (2.1) are satisfied. A C^1-curve

$x\colon (a, b) \to M^n$, $a < 0 < b$, is an integral curve of a continuous vector field $f\colon M^n \to TM^n$ at $p \in M^n$ provided $x(0) = p$ and $\dot{x}(t) = f(x(t))$ for every $t \in (a, b)$.

Theorem 2. *If $f\colon M^n \to TM^n$ is a continuous vector field on M^n, then for every point $p \in M^n$ and for every real number $r > 0$ there exists an integral curve $x\colon (-r, r) \to M^n$ of $f(\cdot)$ at the point p.*

Theorem 3. *Suppose that $L\colon \mathbf{R}^n \to \mathbf{R}^k$ $(k < n)$ is a C^2-map and $F\colon [a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ is a convex-valued (L, c)-invariant map satisfying the Carathéodory conditions. Then for every point $p \in L^{-1}(c)$ there exists at least one solution of the following boundary value problem*

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \textit{a.e. on} \quad [a, b] \\ L(x(t)) = c & \textit{for every} \quad t \in [a, b] \\ x(a) = p. & \end{cases}$$

3. Reformulation of problem (2.1)

In this section we reduce the problem of the existence of problem (2.1) to an equivalent problem of the existence of fixed points of a multi-valued map in the space of paths on M^n. To this we will need the following theorem.

Theorem 3.1. *Suppose that $f\colon [a, b] \times M^n \to TM^n$ satisfies the following conditions:*

(i) *for every $t \in [a, b]$ $f(t, \cdot)\colon M^n \to TM^n$ is a C^1-vector field on M^n;*

(ii) *for every $p \in M^n$ $f(\cdot, p)\colon [a, b] \to TM^n$ is an integrable map;*

(iii) *there exists an integrable function $\alpha\colon [a, b] \to \mathbf{R}_+$ such that $\left|\frac{\partial f_i}{\partial x}(t, x)\right| \le \alpha(t)$ for $(t, x) \in [a, b] \times \varphi_i(W_i)$ where $f_i\colon [a, b] \times \varphi_i(U) \to \mathbf{R}^n$ is given by*

$$f_i(t, x) = [\lambda_i^{p(x)}]^{-1} f(t, p(x)), \quad p(x) = \varphi_i^{-1}(x) \qquad (i = 1, 2, \dots, k).$$

Then for every $(t_0, p_0) \in [a, b] \times M^n$ there exists exactly one solution of the problem

$$\begin{cases} \dot{u}(t) = f(t, u(t)) \\ u(t_0) = p_0. \end{cases} \tag{3.1.1}$$

The sketch of the proof of the above theorem will be given in the Appendix.

Let $A = \{(U_1, \varphi_1), \dots, (U_k, \varphi_k)\}$ be an atlas on M^n and let $\{\beta_i\}_{i=1}^k$ be a pariton of the unity subordinate to A and let $\{W_i\}_{i=1}^k$ be an open covering of M^n such that the following condition is satisfied

(3.2)

$$\bar{V}_i \subset W_i \subset \bar{W}_i \subset U_i \qquad \text{for every set} \quad V_i = \{p \in M^n \colon \beta_i(p) \neq 0\} \qquad (i = 1, \dots, k).$$

We introduce C^2-functions $\alpha_i\colon M^n \to [0, 1]$ $(i = 1, \dots, k)$ given by

$$\alpha_i(p) = \frac{\beta_i(p)}{(\sum_{i=1}^k \beta_i^2(p))^{1/2}} \qquad \text{for} \quad p \in M^n.$$

These functions satisfy equality

$$\sum_{i=1}^k \alpha_i^2(p) = 1 \qquad \text{for} \quad p \in M^n. \tag{3.3}$$

Denote by $C[a, b]$ the space of all continuous maps (paths) $x\colon [a, b] \to M^n$ with the supremum metric. For every $x \in C[a, b]$ and for every set $W_i \subset U_i$ $(i = 1, \dots, k)$ we put

$$\Delta_i^x = x^{-1}(W_i) \quad \text{and} \quad \delta_i^x = x^{-1}(\bar{W}_i). \tag{3.4}$$

Definition 3.5. Let $F\colon [a, b] \times M^n \to TM^n$ be an orientor field on M^n and let $x \in C[a, b]$. A map $y\colon [a, b] \to TM^n$ is an integrable selector of the composition $F \circ (id \otimes x)$ iff $y(t) \in F(t, x(t))$ a.e. on $[a, b]$ and maps $y_i\colon \delta_i^x \to \mathbf{R}^n$ $(i = 1, \dots, k)$ given by $y_i(t) = [\lambda_i^{x(t)}]^{-1}(y(t))$, for $t \in \delta_i^x$ are integrable.

In what follows an integrable selector $y\colon [a, b] \to TM^n$ of $F \circ (id \otimes x)$ will be denote by $y \in F \circ (id \otimes x)$.

The above definition dose not depend on the choise of the covering $\{W_i\}_{i=1}^k$ of M^n satisfying condition (3.2) (cf. Corollary 4.4).

Now with every couple (x, y), where $x \in C[a, b]$ and $y \in F \circ (id \otimes x)$, we associate a vector field $f_{(x,y)}\colon [a, b] \times M^n \to TM^n$ given as follows:

$$f_{(x,y)}(t, p) = \sum_{i=1}^k f_{(x,y)}^i(t, p) \tag{3.6}$$

where

$$f_{(x,y)}^i(t, p) = \begin{cases} \lambda_i^p(\lambda_i^{x(t)})^{-1}[\alpha_i(x(t))\alpha_i(p)y(t))] & (t, p) \in \Delta_i^x \times W_i \\ 0 & (t, p) \notin \Delta_i^x \times W_i. \end{cases}$$

By Theorem 3.1, for every (x, y), there exists exactly one solution of the problem

$$(3.7)_{(x,y)} \qquad \begin{cases} \dot{u}(t) = f_{(x,y)}(t, u(t)), \\ u(a) = q. \end{cases}$$

Finally, with the convex-valued orientor field $F\colon [a, b] \times M^n \to TM^n$, we can associate a multi-valued map $\phi\colon C[a, b] \to C[a, b]$ given by the following formula

$$\phi(x) = \{z_y \in C[a, b] : z_y \text{ is the unique solution of } (3.7)_{(x,y)},\ y \in F \circ (id \otimes x)\}. \tag{3.8}$$

Remark. If a convex-valued orientor field $F\colon [a, b] \times M^n \to TM^n$ satisfies the Carathéodory conditions, then for every $x \in C[a, b]$ there exists $y \in F \circ (id \otimes x)$ (cf. Lemma 4.5), and so $\phi(x) \neq \emptyset$.

Proposition 3.9. *A map* $x \in C[a, b]$ *is a solution of* (2.1) *iff* $x \in \phi(x)$.

Proof. $x \in \phi(x) \Leftrightarrow x(\cdot)$ *is a solution of problem* $(3.7)_{(x,y)}$ *for an* $y \in F \circ (id \otimes x) \Leftrightarrow \dot{x}(t) = y(t)$ *a.e. on* $[a, b]$ *and* $x(a) = q \Leftrightarrow x(\cdot)$ *is a solution of* (2.1). ■

Now by (3.9) it follows that Theorem 1 is equivalent to the following theorem.

Theorem 3.10. *If a convex-valued orientor field* $F\colon [a, b] \times M^n \to TM^n$ *satisfies the Carathéodory conditions, then the map* $\phi\colon C[a, b] \to C[a, b]$ *given by* (3.8) *has a fixed point.*

4. Auxiliary results

In this section we accept that convex-valued orientor fields satisfy Assumptions 1.2, (i)-(iii). We will need the following two propositions, which are immediate consequences of the respective definitions.

Proposition 4.1. *If sequence* $\{p_m\} \subset M^n$ *is convergent to a point* $p \in M^n$, *then for every* $t \in [a, b]$ *and for a real number* $\varepsilon > 0$

$$(\lambda_i^{p_m})^{-1}(F(t, p_m)) \subset 0_\varepsilon[(\lambda_i^p)^{-1}(F(t, p))]$$

for sufficiently large m, *where* $0_\varepsilon(A)$ *denotes* ε*-neighbourhood of the set* $A \subset \mathbf{R}^n$.

Proposition 4.2. *Let* $\{u_m\} \subset L^1([a, b]; \mathbf{R}^n)$ *be bounded weakly convergent sequence to an itegrable function* $u_0 \in L^1([a, b], \mathbf{R}^n)$ *and let* $\beta_m\colon [a, b] \to L(\mathbf{R}^n; \mathbf{R}^n)$ $(m = 0, 1, 2, \dots)$ *be a uniformly convergent sequence of continuous maps* β_m *to the map* β_0. *Suppose moreover that maps* $w_m\colon [a, b] \to \mathbf{R}^n$ $(m = 1, 2, \dots)$ *are given by* $w_m(t) = \beta_m(t)u_m(t)$ *for* $t \in [a, b]$. *Then* $w_m \in L^1([a, b]; \mathbf{R}^n)$ *for* $m = 0, 1, 2, \dots$ *and the sequence* $\{w_m\}$ *is weakly convergent to* w_0 *in* $L^1([a, b]; \mathbf{R}^n)$.

In what follows we will make use of the sets $\bar{V}_i \subset W_i \subset \bar{W}_i \subset U_i$ (cf. (3.2)) and $\Delta_i^x = x^{-1}(W_i)$, $\delta_i^x = x^{-1}(\bar{W}_i)$ (cf. (3.4)).

Lemma 4.3. *Let* $\{x_m\} \subset C[a, b]$ *be a sequence convergent to* $x \in C[a, b]$ *and let* $y_m \in F \circ (id \otimes x_m)$ $(m = 1, 2, \dots)$ *be a sequence of integrable selectors. Then, for a point* $t_0 \in [a, b]$, *there are* $y \in F \circ (id \otimes x)$ *and a subsequence*

$\{y_{m_s}\} \subset \{y_m\}$ *such that* $x([t_0 - \delta, t_0 + \delta]) \subset W_i$ *implies*

$$\int_{t_0}^{t} (\lambda_i^{x_m(\tau)})^{-1}(y_{m_s}(\tau))d\tau \to \int_{t_0}^{t} (\lambda_i^{x(\tau)})^{-1}(y(\tau))d\tau \tag{4.3.1}$$

for $t \in [t_0 - \delta, t_0 + \delta]$.

Proof. By the continuoity of $x\colon [a, b] \to M^n$ and by the uniform convergence $x_m \to x$, there are a finite sequence $a = t_1 < t_2 < \cdots < t_r = b$ and a subset $\{k_1, \dots k_r\} \subset \{1, \dots, k\}$ such that $x([t_i, t_{i+1}]) \subset W_{k_i}$ and $x_m([t_i, t_{i+1}]) \subset W_{k_i}$ for sufficientlt large m (say $m > M$) and for $i = 1, \dots, r-1$.
1° First we fix the interval $[t_1, t_2]$ and define an intergrable map $v_m^1\colon [t_1, t_2] \to \mathbf{R}^n$ by the formula

$$v_m^1(t) = (\lambda_{k_1}^{x_m(t)})^{-1}(y_m(t)) \qquad \text{for} \quad m > M.$$

Now making use of the known argumentation (see for instance Proposition (3.5) in [7]) by Dunford-Pettis Theorem, by Mazur Theorem and by Proposition 4.1, we confirm the existence of a weakly convergent subsequence $\{v_{m_s}^1\} \subset \{v_m^1\}$ to an integrable map $v^1 \in L^1([t_1, t_2]; \mathbf{R}^n)$ such that $v^1(t) = (\lambda_{k_1}^{x(t)})^{-1}(y_1(t))$, where $y_1(t) \in F(t, x(t))$ a.e. on $[t_1, t_2]$. Therefore by weakly convergence of $\{v_{m_s}\}$ condition (4.3.1) is satisfied for the subsequence $\{y_{m_s}\}$, the map $y_1\colon [t_1, t_2] \to TM^n$ and for the point $t_0, t \in [t_1, t_2]$.

Repeating the above argumentation succesively for the interval $[t_i, t_{i+1}]$ $(i = 2, \dots, r)$ we obtain subsequence $\{x_{m_s}^i\} \subset \{x_{m_s}^{i-1}\} \subset \{x_m\}$, $\{y_{m_s}^i\} \subset \{y_{m_s}^{i-1}\} \subset \{y_m\}$ and maps $v^i \in L^1([t_1, t_{i+1}]; \mathbf{R}^n)$ given by the equality $v^i(t) = (\lambda_k^{x(t)})^{-1}(y_i(t))$, where $y_i(t) \in F(t, x(t))$ a.e. on $[t_i, t_{i+1}]$ and the subsequences $\{y_{m_s}^i\}$, $\{x_{m_s}^i\}$ together with the map $y\colon [t_i, t_{i+1}] \to TM^n$ satisfy condition (4.3.1). Finally we put $\{y_{m_s}\} = \{y_{m_s}^{r-1}\}$.
2° Let $y\colon [a, b] \to TM^n$ be a map given by $y(t) = y_i(t)$ for $t \in [t_i, t_{i+1}]$ $(i = 1, \dots, r-1)$. We show that the above map is an integrable selector of $F \circ (id \otimes x)$. According to definition (3.5) we show that the maps $h_j\colon \delta_j^x \to \mathbf{R}^n$, $h_j(t) = (\lambda_j^{x(t)})^{-1}(y(t))$ for $t \in \delta_j^x$, are integrable. Since the sets $\delta_j^x = \bigcup_{i=1}^{r-1} \delta_j^x \cap [t_i, t_{i+1}]$ are measurable and since the maps $g_i\colon \delta_j^x \to \mathbf{R}^n$, given by

$$g_i(t) = \begin{cases} D(\varphi_j \circ \varphi_{k_i}^{-1})(\varphi_{k_i}(x(t)))v^i(t) & \text{for} \quad t \in [t_i, t_{i+1}] \cap \delta_j^x \\ 0 & \text{for} \quad t \notin [t_i, t_{i+1}] \cap \delta_j^x \end{cases}$$

where $v^i(t) = (\lambda_{k_i}^{x(t)})^{-1}(y_i(t))$, are integrable too by (4.2), then $h_j = \sum_{i=1}^{r-1} g_i$ is an integrable map. Therefore $y \in F \circ (id \otimes x)$.
3° Finally we show that condition (4.3.1) is satisfied. Let $t_0 \in [t_i, t_{i+1}]$ be a point for an $i \in \{1, \dots, r-1\}$ and $t > t_0$. Then there exists $j \geq i$ such that $t \in [t_j, t_{j+1}]$. Now applying Proposition 4.2 to every term and making use of

overlap diffeomorphisms we obtain condition (4.3.1) from the next equality

$$\lim_s \int_{t_0}^{t} (\lambda_i^{x_{ms}(\tau)})^{-1}(y_{m_s}(\tau)d\tau = \lim_s \Bigg[\int_{t_0}^{t_{i+1}} B_s^i(\tau))d\tau$$

$$+ \int_{t_{i+1}}^{t_{i+2}} B_s^{i+1}(\tau)(w_s^{i+1}(\tau))d\tau + \cdots + \int_{t_j}^{t} B_s^j(\tau)(w_s^j(\tau))d\tau \Bigg]$$

where

$$B_s^l(\tau) = D(\varphi_i \circ \varphi_{k_l}^{-1})(\varphi_{k_l}(x_{m_s}(\tau)), \quad w_s^l(\tau) = (\lambda_{k_l}^{x_{ms}(\tau)})^{-1}(y_{m_s}(\tau))$$

for $l = i, \ldots, j$. The proof is finished. ■

From the method used in part 2° of the above proof we obtain the following corollary.

Corollary 4.4. *If* $a = t_1 < t_2 < \cdots < t_r = b$ *and* $v^i \in L^1([t_i, t_{i+1}]; \mathbf{R}^n)$ $(i = 1, 2, \ldots, r-1)$ *are given by equality* $v^i(t) = (\lambda_{k_i}^{x(t)})^{-1}(y_i(t))$, *where* $y_i(t) \in F(t, x(t))$ *a.e. on* $[t_i, t_{i+1}]$, *then the map* $y: [a, b] \to \mathbf{R}^n$, *given by* $y(t) = y_i(t)$ *for* $t \in [t_i, t_{i+1}]$ $(i = 1, 2, \ldots, r-1)$, *is an integrable selector of* $F \circ (id \otimes x)$.

Lemma 4.5. *For every* $x \in C[a, b]$ *there exists an integrable selector* $y \in F \circ (id \otimes x)$.

Proof. Keeping up notations of the proof of the Lemma 4.3 we put $Z_i = \varphi_{k_i}(W_{k_i}) \subset \mathbf{R}^n$ and define maps $F_i: [t_i, t_{i+1}] \times Z_i \to \mathbf{R}^n$ by the formula $F_i(t, z) = (\lambda_{k_i}^{p(z)})^{-1}[F(t, p(z))]$, where $p(z) = \varphi_{k_i}^{-1}(z)$ $(i = 1, 2, \ldots, r-1)$. Since the above maps are convex-valued in $\mathbf{R}^n$ and satisfy Carathéodory conditions, so for every continuous $z: [t_i, t_{i+1}] \to \mathbf{R}^n$, $z(t) = \varphi_{k_i} \circ x(t)$, there exists an integrable map $v^i: [t_i, t_{i+1}] \to \mathbf{R}^n$ such that $v^i(t) \in F_i(t, z(t)) = (\lambda_{k_i}^{x(t)})^{-1}[F(t, x(t))]$ $(i = 1, 2, \ldots, r-1)$. Now by Corollary 4.4 there exists an integrable selectors $y \in F \circ (id \otimes x)$. ■

Lemma 4.6. *The graph of the map* $\phi: C[a, b] \to C[a, b]$, *given by condition* (3.8) *is closed in* $C[a, b] \times C[a, b]$.

Proof. We have to demonstrate, that for every couple of sequences $\{x_m\}$, $\{z_m\} \subset C[a, b]$ such that $x_m \to x$, $z_m \to z$ and z_m is the unique solution of $(3.7)_{(x_m, y_m)}$ with an integrable selector $y_m \in F \circ (id \otimes x_m)$, there exists $y \in F \circ (id \otimes x)$ such that $z(\cdot)$ is the unique solution of $(3.7)_{(x,y)}$. For a point $t_0 \in [a, b]$ we consider two cases.

1° There exists a subset $\{i_1, \ldots, i_r\} \subset \{1, 2, \ldots, k\}$ such that $x(t_0), z(t_0) \in W_{i_1} \cap \cdots \cap W_{i_r}$ and $x(t_0) \notin W_j$ or $z(t_0) \notin W_j$ for $j \neq i_s$, $s = 1, \ldots, r$. By uniform convergence of maps there exists a closed neighbourhood $[t_0 - \delta, t_0 + \delta] \equiv \Delta$ of the point t_0 such that $\{x_m(t), x(t), z_m(t), z(t)\} \subset \bigcap_{s=1}^{r} W_{i_s}$ for $t \in \Delta$ and for sufficiently large

m (say $m > M$). Since z_m is the solution of $(3.7)_{(x_m, y_m)}$, therefore the maps $w_m = \varphi_{i_1} \circ (z_m|_\varDelta)$ satisfies conditions

$$w_m(t) = w_m(t_0) + \int_{t_0}^{t} (\lambda_{i_1}^{z_m(\tau)})^{-1} f_{(x_m, y_m)}(\tau, z_m(\tau)) d\tau \qquad \text{for} \quad m > M.$$

Let us see that $w_m \to w$ for $w = \varphi_{i_1} \circ (z|_\varDelta)$ and the other hand, by Lemma 4.3. and Proposition 4.2, there exists an integrable selector $y \in F \circ (id \otimes x)$ such that

$$\lim_m w_m(t) = w(t_0) + \int_{t_0}^{t} (\lambda_{i_1}^{z(\tau)})^{-1} f_{(x,y)}(\tau, z(\tau)) d\tau \qquad \text{for} \quad t \in \varDelta.$$

For that $w'(t) = (\lambda_{i_1}^{z(t)})^{-1} f_{(x,y)}(t, z(t))$ a.e. on $\varDelta$ and so $\dot{z}(t) = f_{(x,y)}(t, z(t))$ a.e. on $[t_0 - \delta, t_0 + \delta]$.

2° Now we consider the secend case, where for every $j = \{1, \dots, k\}$ $z(t_0) \notin W_j$ or $x(t_0) \notin W_j$. Then by analogy there exists a closed neighbourhood $\varDelta = [t_0 - \delta, t_0 + \delta]$ of t_0 such that; $\{z(t), z_m(t)\} \cap V_j = \emptyset$ or $\{x(t), x_m(t)\} \cap V_j = \emptyset$ for $t \in \varDelta$ and $m > M$. Now by definition (3.6),

$$f_{(x_m, y_m)}(t, z_m(t)) = 0 = f_{(x,y)}(t, z(t)) \qquad \text{a.e. on} \quad [t_0 - \delta, t_0 + \delta].$$

Therefore $z_m(\cdot)$ satisfies the equality $z_m(t) = z_m(t_0)$ a.e. on $\varDelta$ and so $z(t) = z(t_0)$ a.e on $\varDelta$ because $z_m \to z$. From the above equalities we have

$$\dot{z}(t) = 0 = f_{(x,y)}(t, z(t)) \qquad \text{a.e. on} \quad [t_0 - \delta, t_0 + \delta]. \quad \blacksquare$$

Lemma 4.7. *Let* $\{z_{y_m}\} \subset \phi(x)$ *be a sequence convergent to* $z_{y_0} \in \phi(x)$ *in* $C[a, b]$, *let* $\{s_m\} \subset [a, b]$ *be sequence convergent to* $s \in [a, b]$ *and let* $\bar{y} \in F \circ (id \otimes x)$ *be an integrable selecotor. Then the sequence* $\{z_{\bar{y}_m}\} \subset \phi(x)$, *where*

$$\bar{y}_m(t) = \begin{cases} y_m(t) & \text{for} \quad t \in [a, s_m] \\ \bar{y}(t) & \text{for} \quad t \in (s_m, b] \end{cases} \qquad (m = 1, 2, \dots)$$

is convergent to $z_{\bar{y}_0} \in \phi(x)$.

Proof. By analogy we consider two cases as in the proof of Lemma 4.6, make use of definitions of solutions z_{y_m}, $z_{\bar{y}_m}$, z_{y_0}, $z_{\bar{y}}$ $(m = 1, 2, \dots)$ and pass to the limits in suitable integrable formulas, which locally represents the above solutions. $\blacksquare$

5. The space of the paths associated with the measurable convex-valued orientor field

Let $\eta \colon [a, b] \to \mathbf{R}_+$ be an integrable function satisfying condition (1.2) (iii) and let $i \colon M^n \to \mathbf{R}^{2n+1}$ be the Whitney imbedding of class C^2. For a point

$q \in M^n$ and for a real number $K > 0$ $C_{q,k}[a, b] \subset C[a, b]$ denote the subspace of all paths $x:[a, b] \to M^n$ such that $x(a) = q$ and

$$(5.1) \qquad |i \circ x(t) - i \circ x(\bar{t})| \leq K \int_t^{\bar{t}} \eta(s)ds \qquad \text{for all} \quad t, \bar{t} \in [a, b].$$

Recall that a metric space X is an absolute retract ($X \in AR$) iff for each metric space Y and for each homeomorphism $h\colon X \to h(X) \subset Y$ onto a closed subset the image $h(X)$ is a retract of Y. The metric space X is and absolute neighbourhood retract ($X \in ANR$) iff for each metric space Y and for each homeomorphism $h\colon X \to h(X) \subset Y$ onto a closed subset there exists an open subset $U \subset Y$ such that $h(X)$ is a retract of U.

Lemma 5.2. *For all $q \in M^n$ and $K > 0$ the space $C_{q,K}[a, b]$ is a compact ANR and has the homotopy type of the point.*

Proof. 1° Define X as a subset of all paths $X\colon [a, b] \to \operatorname{Im}(i) \subset \mathbf{R}^{2n+1}$ with the common module of continuity which is given by condition (5.1) and $x(a) = i(q)$. By Arzela-Ascoli Theorem, X is a compact subset of the Banach space $C([a, b]; \mathbf{R}^{2n+1})$ with the supremum norm. On the othere hand, the homeomorphism $i\colon M^n \to \operatorname{Im}(i) \subset \mathbf{R}^{2n+1}$ induces homeomorphism $i_{\#}\colon C_{q,K}[a, b] \to X$, given by formula

$$[i_{\#}(x)](t) = i \circ (x(t)) \qquad \text{for each} \quad t \in [a, b].$$

Therefore $C_{q,K}[a, b]$ is a compact subset of $C[a, b]$.
2° We shall demonstrate that $C_{q,K} \in ANR$. Denote by $Q = i(M^n)$. Since $M^n \in ANR$ (cf. [2]), there exists an open subset $U \subset \mathbf{R}^{2n+1}$ and a retraction $r\colon U \to Q$. For a point $v \in \mathbf{R}^{2n+1}$ and for a set $V \subset \mathbf{R}^{2n+1}$ by $C_{v,K}([a, b]; V)$ we will denote the space of all paths $z\colon [a, b] \to V$ satisfying condition (5.1) and such that $z(a) = v$. Since the convex subset $C_{0,K}([a, b]; \mathbf{R}^{2n+1}) \subset C([a, b]; \mathbf{R}^{2n+1})$ is an AR so its homeomorphic image $C_{v,K}([a, b]; \mathbf{R}^{2n+1}) = v + C_{0,k}([a, b]; \mathbf{R}^{2n+1})$ is an AR too. Therefore the open subset $C_{v,K}([a, b]; U)$ of $C_{v,K}([a, b]; \mathbf{R}^{2n+1})$ is an ANR. Now we shall construct a retraction of $C_{v,K}([a, b]; U)$ on $C_{r(v),K}([a, b]; Q)$. We put for every $y \in C_{v,K}(a, b]; U)$ a map $L(y)(t) = (r \circ y(t), \prod_{y(t)}(y'(t)))$ a.e. on $[a, b]$ where $\prod_{y(t)}\colon \{r(y(t))\} \times \mathbf{R}^{2n+1} \to T_{r \circ y(t)}Q$ is the canonical projection. Let us see that $r_1\colon C_{v,K}([a, b]; U) \to C_{r(v),K}([a, b]; Q)$, such that for all $y \in C_{v,K}([a, b]; U)$ $r_1(y)$ is the unique solution of $(3.7)_{L(y)}$ is a retraction. In fact, the continuoity of $r_1(\cdot)$ follows from: Proposition 4.2, Definition 3.6 of the field $f_{L(y)}$ and the weakly compactness of the set $\{y'(\cdot)\colon y \in C_{v,K}([a, b]; U)\} \subset L_1([a, b]; U)$. However by definition of $f_{L(y)}$ the map $r_1(\cdot)$ is the identity on $C_{r(v),K}([a, b]; Q)$. Therefore $C_{r(v),K}([a, b]; Q) \in ANR$ and so the homeomorfphic image $i_{\#}^{-1}[C_{i(q),K}([a, b]; Q)]$

is an *ANR* too.

3° Finally the map $H\colon [0,1] \times C_{q,K}[a,b] \to C_{q,K}[a,b]$ given by $H(s,x)(t) = x(a + s(t-a))$ for $(s,t) \in [0,1] \times [a,b]$, is a homotopy joining the constant map $H(0,\cdot)$ with the identity map $H(1,\cdot)$. ■

Corollary 5.3. *$C_{q,K}[a,b]$ is an acyclic space with respect to Čech homology theory.*

Lemma 5.4. *There exists a real number $K > 0$ such that the subspace $C_{q,K}[a,b] \subset C[a,b]$ is an invariant space for the map $\phi\colon C[a,b] \to C[a,b]$ (comp.* (3.8)).

Proof. Let $\{W_i\}_{i=1}^{k}$ be an open convering of M^n of such that $\overline{W}_j \subset U_j$ for $j = 1, \dots k$, and let $K > 0$ be a real number such that $|D(i \circ \varphi_j^{-1})(x)| \le K$ for $x \in W_j$ and $j = 1, 2, \dots k$. For every $u \in C_{q,K}[a,b]$ and $y \in F \circ (id \otimes u)$ the unique solution $z\colon [a,b] \to M^n$ of $(3.7)_{(u,y)}$ satisfies condition

$$(\varphi_j \circ z)'(t)\,[\lambda_j^{z(t)}]^{-1} f_{(u,y)}(t, z(t)) \qquad \text{a.e. on} \quad \varDelta_j^z = z^{-1}(W_j) \qquad (j = 1, \dots k).$$

Therefore by Assumption 1.2 (iii) we have

$$\begin{aligned} |(i \circ z)'(t)| &= |[(i \circ \varphi_j^{-1}) \circ (\varphi_j \circ z)]'(t)| \le |D(i \circ \varphi_j^{-1})_{(\varphi_j \circ z(t))}((\varphi_j \circ z)'(t))| \\ &\le K\eta(t) \qquad \text{a.e. on} \quad \varDelta_j^z. \end{aligned}$$

From the above the solution $z(\cdot)$ satisfies condition (5.1) and so for every $u \in C_{q,K}[a,b]$ we have $\phi(u) \subset C_{q,K}[a,b]$. ■

6. Proofs of theorems

The main tool used in thus section is the Eilenberg-Montgomery fixed point theorem for acyclic maps [4].

Theorem A. *Let M be an acyclic compact ANR and let $T\colon M \to M$ be an (u.s.c) multi-valued map such that for every $x \in M$ the set $T(x)$ is acyclic. Then T has fixed point.*

6.1 Proof of Theorem 3.10. By Lemma 5.4 there exists a constant $K > 0$ such that the map $\phi\colon C_{q,K}[a,b] \to C_{q,K}[a,b]$ given by formula (3.8) is correctly defined. In virtue of (5.2), (5.3) and (4.6), the space $C_{q,K}[a,b]$ is an acyclic compact *ANR* and $\phi\colon C_{q,K}[a,b] \to C_{q,K}[a,b]$ is an (u.s.c.) map. Now Theorem 3.10 will be an immediate concequence of Theorem A, provided we show that $\phi\colon C_{q,K}[a,b] \to C_{q,K}[a,b]$ is an acyclic map.

Let $\bar{y} \in F \circ (id \otimes x)$ be an integrable selector and let $c\colon \phi(x) \to \phi(x)$ be a constant map such that $c(z) = z_{\bar{y}}$ for every $z \in \phi(x)$ where $z_{\bar{y}}$ is the unique solution of $(3.7)_{(x,\bar{y})}$. By Lemma (4.7) we have a continuous map $H\colon [0, 1] \times \phi(x) \to \phi(x)$ given by formula $H(s, z) = z_{y_s}$ for $z \in \phi(x)$ and $s \in [0, 1]$, where

$$y_s = \begin{cases} y(t) & a \leq t \leq a + s(b - a) \\ \bar{y}(t) & a + s(b - a) < t \leq b \end{cases}$$

and z_{y_s} is the unique solution $(3.7)_{(x,y_s)}$. The above map is the homotopy joining the constant map $H(0,\cdot) = c(\cdot)$ with the identity $H(1,\cdot)$ on $\phi(x)$ is an acyclic space. ■

6.2 Proof of Theorem 1. In virtue of Theorem 3.10 and Proposition 3.9, we obtain Theorem 1. ■

6.3 Proof of Theorem 2. It is an immediate consequence of Theorem 1 provided the convex-valued orientor field is a single-valued field on M^n. ■

6.4 Proof of Theorem 3. By assumptions of this theorem $M^{n-k} = L^{-1}(c) = \{x \in \mathbf{R}^n \colon L(x) = c\}$ is a closed C^2-manifold and the map $G\colon [a, b] \times M^{n-k} \to TM^{n-k}$ given by formula $G(t, x) = F(t, x) \cap \operatorname{Ker}(D(L(x)))$ for $(t, x) \in [a, b] \times M^{n-k}$ is a convex-valued orientor field on M^{n-k} satisfying Assumptions 1.2, (i)-(iii). Therefore, making use of Theorem 1 for $G\colon [a, b] \times M^{n-k} \to TM^{n-k}$, we obtaine the thesis of Theorem 3.

Appendix

In this section we shall sketch main idea of the proof of Theorem 3.1. To this we will make use of Lasota-Opial fixed point theorem [5].

Theorem B. *Let U be a neighbourhood of zero in a Banach space E and let* $H\colon U \to n(E)$ *be a completely continuous map such that* $x \in H(x) \Rightarrow x = 0$. *Then, for any continuous map* $h\colon E \to E$, *the condition* $h(x) - h(y) \in H(x - y)$ *for* $x - y \in U$ *implies that the equation* $x = h(x)$ *has exactly one solution.*

7.1 Lemma. *If a singlue-valued map* $f\colon [a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ *satisfies the Carathéodory conditions and there exists an intergrable function* $\alpha\colon [a, b] \to \mathbf{R}_+$ *such that* $|f(t, x) - f(t, y)| \leq \alpha(t)|x - y|$ *for* $t \in [a, b]$ $x, y \in \mathbf{R}^n$, *then, for every* $x_0 \in \mathbf{R}^n$ *there exists exactly one solution of the problem*

$$\begin{cases} x'(t) = f(t, x(t)) & \text{a.e. on} \quad [a, b] \\ x(a) = x_0. \end{cases} \tag{7.1.1}$$

Proof. The maps $h, H\colon C([a, b], \mathbf{R}^n) \to C([a, b], \mathbf{R}^n)$ given respectively by formulas:

$$h(x)(t) = x_0 + \int_a^t f(s, x(s))ds,$$

$$H(x)(t) = \left\{\left(\int_a^t y(s)ds\right) : y \in L^1([a, b]; \mathbf{R}^n), |y(t)| \leq \alpha(t)|x(t)|\right\}$$

are completely continuous (cf. for instance [7]). By definitions of the above maps we have $h(x) - h(y) \in H(x - y)$ for $x, y \in C([a, b]: \mathbf{R}^n)$. Moreover in virtue of Gronwall inequality we obtain the condition $x \in H(x) \Rightarrow x = 0$. Now the thesis of this lemma follows by Theorem B. ■

Lemma 7.2. *Let U, $W \subset \mathbf{R}^n$ be open sets in $\mathbf{R}^n$ such that $\bar{W} \subset U$ and $\bar{W}$ is compact. Suppose f: $[a, b] \times \mathbf{R}^n$ satisfies the following conditions*:

(i) *for every $x \in U$, $f(\cdot, x)$: $[a, b] \to \mathbf{R}^n$ is an integrable maps*;

(ii) *for every $t \in [a, b]$, $f(t, \cdot)$: $U \to \mathbf{R}^n$ is C^1-map*;

(iii) *there exists an inegrable function α: $[a, b] \to \mathbf{R}_+$ such that*

$$\left|\frac{\partial f}{\partial x}(t, x)\right| \leq \alpha(t) \qquad \text{for} \quad (t, x) \in [a, b] \times U,$$

then there exists a real number $\delta > 0$ such that for every $(t_0, x_0) \in [a, b] \times \bar{W}$ the probelm

$$\begin{cases} x'(t) = f(t, x(t)) & \text{a.e. on} \quad [t_0, t_0 + \delta] \\ x(t_0) = x_0 \end{cases} \tag{7.2.1}$$

has exactly one solution.

Proof. Let $0 < r < \operatorname{dist}(\bar{W}, \mathbf{R}^n - U)$ and let $\overline{K(x_0, r)} \subset U$ be closed ball with the center x_0 and of the radius r. Decompose the retraction of $\overline{K(x_0, r)}$ with the map $f(t, \cdot)$ for every $t \in [a, b]$, we obtain an extention $\tilde{f}$: $[a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ of $f|_{[a,b] \times \overline{K(x_0, r)}}$. Since the map $\tilde{f}$ satisfies all assumptions of lemma (7.1), we obtain the unique solution $x = x(\cdot, t_0, r)$: $[a, b] \to \mathbf{R}^n$ of problem (7.1.1) with the map $\tilde{f}$. By the assumption (iii) there exists $\delta > 0$ such that for every $t \in [t_0, t_0 + \delta]$ we have $x(t, t_0, r) \in K(x_0, \delta)$ for all $(t_0, x_0) \in [a, b] \times \bar{W}$. Therefore $x|_{[t_0, t_0 + \delta]}$ is the unique solution of (7.2.1). ■

Proof of Theorem 3.1. Let $A = \{(U_1, \varphi_1), \ldots, (U_k, \varphi_k)\}$ be an atlas on a closed C^2-manifold M^n and let f_i: $[a, b] \times \varphi_i(U_i) \to \mathbf{R}^n$ $(i = 1, \ldots, k)$ be the map given in assumption (3.1) (iii). By (7.2) there exists $\delta > 0$ such that for every $(t_0, x_0) \in [a, b] \times \varphi_i(W_i)$ and for all $i = 1, \ldots, k$ the problem

$$\begin{cases} x'(t) = f_i(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

has exactly one solution.

Now making use of the extension method of solution of the equation $\dot{x}(t)=f(t, x(t))$ on the maniforld M^n we obtain the thesis of Theorem 3.1. ■

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nuna adreso:
Department of Mathematics
Gdańsk University
ul. Wita Stwosza 57
80–952 Gdańsk
Poland

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