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The Rate of Convergence for Chemical Interfacial Reaction Models

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1. Introduction

Recently Yamada and Yotsutani have studied interfacial chemical reaction models from mathematical points of view [3, 4]. Models are described by parabolic systems with non-linear conditions.

We consider the following interfacial reaction model proposed by Kawano and Nakashio [1], which has already been considered by Shinomiya in [2]. Two chemical species U and V are contained between two parallel plates and are separated by an interface parallel to plates (see Figure 1). They stream in the region toward the tangential direction to the interface along, say, a laminar stratified flow, which velocity is given by $a(x)$ and $b(x)$, and diffuse toward the normal direction to the interface. The reaction between U and V occurs on the interface. If m molecules of U and n molecules of V produce one molecules of the product

$$mU + nV \longrightarrow U_m V_n, \tag{1.1}$$

then the gradient on the interface of concentration u and v of U and V is proportional to the probability $u^m v^n$ of collision of molecules. Moreover we

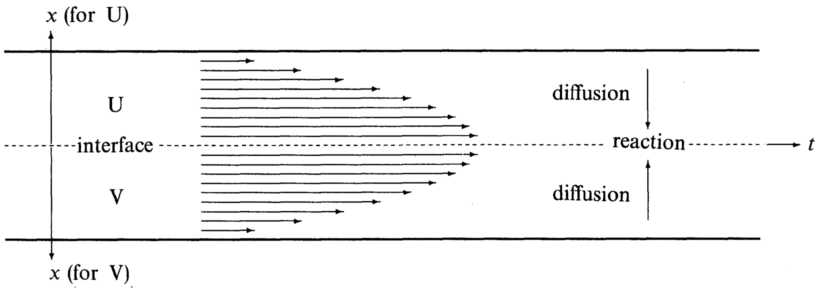

Figure 1: Chemical interfacial reaction

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impose no flux condition on the plates. Thus we obtain our problem

$$
(1.2)\qquad \begin{cases} a(x)u_t = u_{xx} & \text{for } (x, t)\in(0, 1)\times(0, \infty), \\ b(x)v_t = v_{xx} & \text{for } (x, t)\in(0, 1)\times(0, \infty), \\ u_x(1, t) = v_x(1, t) = 0 & \text{for } t\in(0, \infty), \\ u_x(0, t) = k_1 u^m(0, t)v^n(0, t) & \text{for } t\in(0, \infty), \\ v_x(0, t) = k_2 u^m(0, t)v^n(0, t) & \text{for } t\in(0, \infty), \\ u(x, 0) = u_0(x) \geq 0 & \text{for } x\in(0, 1), \\ v(x, 0) = v_0(x) \geq 0 & \text{for } x\in(0, 1), \end{cases}
$$

where k_i are positive constants and $m, n \geq 1$. The functions $a(x)$ and $b(x)$ are non-negative C^∞-functions on $[0, 1]$ satisfying

$$
a(x) > 0 \text{ on } [0, 1), \quad b(x) > 0 \text{ on } [0, 1).
$$

When the flow is a laminar statified flow, $a(x) = \text{const.}(1 - x^2)$ and $b(x) = \text{const.}(1 - x^2)$.

Yamada and Yotsutani [3, 4] treated more complicated system, and established the unique global existence theorem. Their technique for the unique global existence of the solutions is applicable to our system. Shinomiya investigated the asymptotic behavior of the solutions.

Theorem 1.1. [2] *If the initial data* (u_0, v_0) *are non-negative and belong to* $\{L^\infty(0, 1)\}^2$, *then there exists a non-negative unique global solution to* (1.2) *in the class*

$$
(u, v)\in\{C([0, \infty); L^2(0, 1))\cap C((0, \infty); H^2(0, 1))\cap H^1_{loc}(0, \infty; H^1(0, 1)) \cap C^\infty([0, 1]\times(0, \infty))\cap L^\infty((0, 1)\times(0, \infty))\}^2,
$$

and it converges to a steady state:

$$
(1.3)\qquad \|u - u_\infty\|_{L^\infty(0,1)} + \|v - v_\infty\|_{L^\infty(0,1)} + \|u_x\|_{L^\infty(0,1)} + \|v_x\|_{L^\infty(0,1)} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty,
$$

where

$$
u_\infty = k_1 \max\{A, 0\} \Big/ \int_0^1 a\,dx,
$$

$$
v_\infty = k_2 \max\{-A, 0\} \Big/ \int_0^1 b\,dx,
$$

$$
A = \frac{1}{k_1}\int_0^1 a u_0\,dx - \frac{1}{k_2}\int_0^1 b v_0\,dx.
$$

We note that our system has an invariance

$$\frac{1}{k_1}\int_0^1 au\,dx - \frac{1}{k_2}\int_0^1 bv\,dx \equiv A. \tag{1.4}$$

The purpose of this article is to clarify the rate of convergence in the weighted L^p-topology. We choose $a(x)$ and $b(x)$ as weight functions for u and v respectively, because the invariance (1.4) is related to the L^1-norms with such weights

The larger numbers of molecules m and n in the reaction (1.1) are, the harder m plus n molecules collide at the same time. Hence it seems that the rate of convergence goes down as m and n increase. We shall establish this observation as follows:

Theorem 1.2. *It holds that*

$$\int_0^1 a|u-u_\infty|\,dx \le C\rho_1(t), \quad \int_0^1 b|v-v_\infty|\,dx \le C\rho_2(t),$$

where

$$\rho_j(t) = \begin{cases} e^{-\lambda t/(3-j)}\ (\lambda>0) & \text{if } A>0 \text{ and } n=1, \\ (t+1)^{-1/\{(3-j)(n-1)\}} & \text{if } A>0 \text{ and } n>1, \\ (t+1)^{-1/(m+n-1)} & \text{if } A=0, \\ (t+1)^{-1/\{j(m-1)\}} & \text{if } A<0 \text{ and } m>1, \\ e^{-\lambda t/j}\ (\lambda>0) & \text{if } A<0 \text{ and } m=1. \end{cases}$$

Indeed we can show the following theorems. Theorem 1.2 is a consequence of them via Hölder's inequality.

Theorem 1.3. *Let A be nonzero. Then there exist $\lambda>0$ and $C>0$ such that*

1. *For $A>0$ and $n=1$,*

$$\int_0^1 a(u-u_\infty)^2\,dx \le Ce^{-\lambda t}, \quad \int_0^1 bv^2\,dx \le Ce^{-2\lambda t},$$

2. *For $A>0$ and $n>1$,*

$$\int_0^1 a(u-u_\infty)^2\,dx \le C(t+1)^{-1/(n-1)}, \quad \int_0^1 bv^p\,dx \le C(p)(t+1)^{-p/(n-1)}$$

$$(1 \le p < \infty),$$

3. *For* $A < 0$ *and* $m > 1$,

$$\int_0^1 au^p\,dx \le C(p)(t+1)^{-p/(m-1)} \qquad (1 \le p < \infty),$$

$$\int_0^1 b(v - v_\infty)^2\,dx \le C(t+1)^{-1/(m-1)},$$

4. *For* $A < 0$ *and* $m = 1$,

$$\int_0^1 au^2\,dx \le Ce^{-2\lambda t}, \quad \int_0^1 b(v - v_\infty)^2\,dx \le Ce^{-\lambda t}.$$

Theorem 1.4. *Let* $A = 0$ *and* $m = n = 1$. *Then there exists* $C > 0$ *such that*

$$\int_0^1 (au^2 + bv^2)\,dx \le C(t+1)^{-2}.$$

Theorem 1.5. *Let* $A = 0$ *and* $mn > 1$. *Then there exists* $C > 0$ *such that*

$$\int_0^1 (au + bv)\,dx \le C(t+1)^{-1/(m+n-1)}.$$

We shall prove Theorems 1.3–1.5 in §§2–4 respectively. The estimates of Theorem 1.2 are partially the best possible. In §5 we shall mention their sharpness. In the last section §6 we shall discuss the decay rate of derivatives u_x and v_x in some weighted L^2-space. The proofs shall be proceeded under the assumption that (u, v) is a solution constructed by Theorem 1.1, but we shall not use (1.3) taking account of the self-contained argument.

2. The case $A \neq 0$

In this section we prove Theorem 1.3. Without loss of generality we may assume $A > 0$. We put

$$\tilde{A} = M - u_\infty,$$

where

$$M = 2\max\{\|u\|_{L^\infty((0,1)\times(0,\infty))}, \|v\|_{L^\infty((0,1)\times(0,\infty))}\} < \infty. \tag{2.1}$$

We multiply both sides of the first equation of (1.2) by $1/(M-u)$, and integrate by use of the boundary conditions. Combining the result and (1.4), we obtain

$$\frac{d}{dt}\mathscr{E}_1(t) + \int_0^1 \frac{u_x^2}{(M-u)^2}\,dx + \frac{k_1 u^m(0,t)v^n(0,t)}{M - u(0,t)} = 0,$$

where

$$\mathscr{E}_1(t) = \int_0^1 \left\{ a\left(\frac{M-u}{\tilde{A}} - \log \frac{M-u}{\tilde{A}} - 1 \right) + \frac{k_1}{\tilde{A}k_2} bv \right\} dx.$$

Remark that there exists $C = C(M) > 1$ such that

$$C^{-1}(u - u_\infty)^2 \le \frac{M-u}{\tilde{A}} - \log \frac{M-u}{\tilde{A}} - 1 \le C(u - u_\infty)^2$$

holds for $0 \le u \le M/2$. The following is an immediate consequence of the above identity.

Lemma 2.1. *It holds that*

$$\int_0^t \int_0^1 u_x^2 \, dx \, d\tau \le C.$$

It is obvious from (1.3) that $1 \le Cu^m(0, t)$ for large t. For our purpose, however, the following estimate is enough, which is easier to prove than (1.3).

Lemma 2.2. *It holds that*

$$1 \le C\left\{ u^m(0, t) + \left(\int_0^1 u_x^2 \, dx \right)^{1/2} \right\}.$$

Proof. By the first mean value theorem for integrals there exists $x(t) \in [0, 1]$ such that

$$u(x(t), t) \int_0^1 a \, dx = \int_0^1 au \, dx. \tag{2.2}$$

The inequality

$$u(x(t), t) \ge u_\infty > 0$$

holds because of (2.2), (1.4), and the non-negativity of v. On the other hand, by virtue of (2.1)

$$u^m(x(t), t) \le u^m(0, t) + m \int_0^1 |u^{m-1} u_x| \, dx \le u^m(0, t) + C\left(\int_0^1 u_x^2 \, dx \right)^{1/2}$$

is valid. Thus we have proved the assertion. □

Let us define a function $\varphi_{n,p}$ $(n \ge 1,\ 1 \le p < \infty)$ on $[0, (M/2)^p]$ by

$$\varphi_{n,p}(v)=\begin{cases} v^2 & \text{for } n=1,\\ \left.\begin{cases}\exp\left\{-\dfrac{c_{n,p}}{v^{(n-1)/p}}\right\} & (v>0)\\ 0 & (v=0)\end{cases}\right\} & \text{for } n>1,\end{cases}$$

where $c_{n,p}$ $(n>1)$ are determined so large that

$$\text{(2.3)}\qquad \begin{aligned} &0\le v^{-2n}\varphi_{n,1}(v)\le C\varphi''_{n,1}(v) &&\text{for } 0\le v\le \frac{M}{2},\\ &0\le \varphi''_{n,p}(v) &&\text{for } 0\le v\le \left(\frac{M}{2}\right)^p \end{aligned}$$

and $c_{n,p}\nearrow$ as $p\nearrow$. We note that

$$\text{(2.4)}\qquad v^n\varphi'_{n,1}(v)=c\varphi_{n,1}(v),$$

where $c=2$ for $n=1$, $c=c_{n,1}(n-1)$ for $n>1$. Here and in what follows we interpret $v^{-k}\varphi_{n,p}(v)|_{v=0}=0$ for $n>1$ and $k>0$. Multiplying the second equation of (1.2) by $\varphi'_{n,1}(v)$, and integrating with respect to x, we obtain

$$\text{(2.5)}\qquad \frac{d}{dt}\int_0^1 b\varphi_{n,1}(v)\,dx+\int_0^1\varphi''_{n,1}(v)v_x^2\,dx+ck_2u^m(0,t)\varphi_{n,1}(v(0,t))=0,$$

where the constant c is that of (2.4). Next we evaluate $\int_0^1 b\varphi_{n,1}(v)\,dx$ itself.

Lemma 2.3. *It holds that*

$$\int_0^1 b\varphi_{n,1}(v)\,dx$$
$$\le C\left(\int_0^1\varphi''_{n,1}v_x^2\,dx+u^m(0,t)\varphi_{n,1}(v(0,t))+\int_0^1 b\varphi_{n,1}(v)\,dx\int_0^1 u_x^2\,dx\right).$$

Proof. First we remark

$$\begin{aligned}\varphi_{n,1}(v)&\le\varphi_{n,1}(v(0,t))+\int_0^1|\varphi'_{n,1}(v)v_x|\,dx\\ &=\varphi_{n,1}(v(0,t))+c\int_0^1|v^{-n}\varphi_{n,1}(v)v_x|\,dx\\ &\le\varphi_{n,1}(v(0,t))+\varepsilon\int_0^1\varphi_{n,1}(v)\,dx+C\int_0^1\varphi''_{n,1}(v)v_x^2\,dx,\end{aligned}$$

where we use (2.4) and (2.3) for $n > 1$. This estimate yields

$$\int_0^1 b\varphi_{n,1}(v)\,dx \le C\int_0^1 \varphi_{n,1}(v)\,dx \le C\left(\varphi_{n,1}(v(0,t)) + \int_0^1 \varphi''_{n,1}(v)\,v_x^2\,dx\right).$$

In consequence of Lemma 2.2 we have

$$\varphi_{n,1}(v(0,t)) \le C\left\{u^m(0,t)\,\varphi_{n,1}(v(0,t)) + \varphi_{n,1}(v(0,t))\left(\int_0^1 u_x^2\,dx\right)^{1/2}\right\}.$$

On the other hand we get

$$\begin{aligned}\varphi_{n,1}(v(0,t)) &\le C\left(\int_0^{1/2} \varphi_{n,1}(v)\,dx + \int_0^{1/2} |\varphi'_{n,1}(v)\,v_x|\,dx\right)\\ &\le C\left\{\int_0^1 b\varphi_{n,1}(v)\,dx + \left(\int_0^1 b\varphi_{n,1}(v)\,dx\right)^{1/2}\left(\int_0^1 \varphi''_{n,1}(v)\,v_x^2\,dx\right)^{1/2}\right\},\end{aligned}$$

where we use (2.4) and (2.3) for $n > 1$ again. This yields

$$\begin{aligned}&\varphi_{n,1}(v(0,t))\left(\int_0^1 u_x^2\,dx\right)^{1/2}\\ &\quad\le \varepsilon\int_0^1 b\varphi_{n,1}(v)\,dx + C\left(\int_0^1 b\varphi_{n,1}(v)\,dx\int_0^1 u_x^2\,dx + \int_0^1 \varphi''_{n,1}(v)\,v_x^2\,dx\right).\end{aligned}$$

□

It follows from this lemma and (2.5) that

$$\frac{d}{dt}\int_0^1 b\varphi_{n,1}(v)\,dx + \lambda_2\int_0^1 b\varphi_{n,1}(v)\,dx \le C\int_0^1 b\varphi_{n,1}(v)\,dx\int_0^1 u_x^2\,dx,$$

which gives

$$\int_0^1 b\varphi_{n,1}(v)\,dx \le Ce^{-\lambda_2 t}$$

with helps of Lemma 2.1 and Gronwall's lemma. For $n = 1$, we have the weighted L^2-estimate of v.

Since $\varphi_{n,1}(v)$ is convex on $[0, M/2]$, Jensen's inequality

$$\varphi_{n,1}\left(\int_0^1 bv\,dx \Big/ \int_0^1 b\,dx\right) \le \int_0^1 b\varphi_{n,1}(v)\,dx \Big/ \int_0^1 b\,dx$$

holds, which implies

$$\int_0^1 bv\,dx \le C\rho_2(t) = \begin{cases} Ce^{-\lambda_2 t/2} & (n=1), \\ C(t+1)^{-1/(n-1)} & (n>1). \end{cases}$$

Let $x(t)$ be the same as in (2.2). By virtue of (1.4) we have

$$u(x(t), t) - u_\infty = k_1 \int_0^1 bv\,dx \Big/ k_2 \int_0^1 a\,dx.$$

Therefore it holds that

$$\int_0^1 a\left(\frac{M-u}{\tilde{A}} - \log\frac{M-u}{\tilde{A}} - 1\right)dx \le C\int_0^1 a(u-u_\infty)^2\,dx$$

$$\le C\int_0^1 a|u(x(t), t) - u_\infty|^2\,dx + C\int_0^1 \left|\int_0^1 |u_x|\,dx\right|^2 dx$$

$$\le C\int_0^1 \left\{bv^2 + \frac{u_x^2}{(M-u)^2}\right\}dx$$

by (2.1). Hence there exists $\lambda_1 > 0$ such that

$$\frac{d}{dt}\mathscr{E}_1(t) + \lambda_1\mathscr{E}_1(t) \le C\int_0^1 b(v+v^2)\,dx \le C\rho_2(t),$$

which gives the decay estimate of $\mathscr{E}_1(t)$.

We consider the case $p > 1$. Since $\varphi_{n,p}$ is convex on $[0, (M/2)^p]$, we can apply Jensen's inequality to $\varphi_{n,p}\left(\int_0^1 bv^p dx \Big/ \int_0^1 b\,dx\right)$. Moreover $\varphi_{n,p}(v^p) \le \varphi_{n,1}(v)$ because of $c_{n,p} \ge c_{n,1}$. Therefore we get

$$\varphi_{n,p}\left(\int_0^1 bv^p\,dx \Big/ \int_0^1 b\,dx\right) \le Ce^{-\lambda_2 t}.$$

It follows

$$\int_0^1 bv^p\,dx \le C(p)(t+1)^{-p/(n-1)}.$$

Now we complete the proof of Theorem 1.3. $\square$

3. The case $A = 0$, $m = n = 1$

Now we devote ourselves to the proof of Theorem 1.4. In a similar manner to obtain (2.5), we have

(3.1) $$\frac{1}{2}\frac{d}{dt}\mathscr{E}_2(t) + \int_0^1 (u_x^2 + v_x^2)\,dx + k_1 u^2(0, 1)v(0, t) + k_2 u(0, 1)v^2(0, t) = 0,$$

where

$$\mathscr{E}_2(t) = \int_0^1 (au^2 + bv^2)\,dx.$$

If $\mathscr{E}_2(t_0) = 0$ for some $t_0 \geq 0$, then $u = v \equiv 0$ for $t \geq t_0$ by the uniqueness of solutions. Thus we may assume $\mathscr{E}_2(t) > 0$ for all $t \geq 0$.

The following is the key lemma to obtain the decay estimate.

Lemma 3.1. *It holds that*

$$(\mathscr{E}_2(t))^{3/2} \leq C\left\{\int_0^1 (u_x^2 + v_x^2)\,dx + u^2(0, t)v(0, t) + u(0, t)v^2(0, t)\right\}.$$

Proof. We prove the estimate for $\int_0^1 au^2\,dx$ instead of $\mathscr{E}_2(t)$. We get the estimate for $\int_0^1 bv^2\,dx$ in a similar manner. It is easy to see that

$$\int_0^1 au^2\,dx \leq C\left\{u^2(0, t) + \left(\int_0^1 u^2\,dx\right)^{1/2}\left(\int_0^1 u_x^2\,dx\right)^{1/2}\right\}$$

and

$$\left(\int_0^1 au^2\,dx\right)^{1/2} \leq C\left\{\int_0^1 au\,dx + \left(\int_0^1 u_x^2\,dx\right)^{1/2}\right\}.$$

Therefore we have

$$\left(\int_0^1 au^2\,dx\right)^{3/2} \leq C(I_1 + I_2 + I_3 + I_4),$$

where

$$\begin{cases} I_1 = u^2(0, t)\displaystyle\int_0^1 au\,dx, \\ I_2 = \displaystyle\int_0^1 au\,dx\left(\int_0^1 u^2\,dx\right)^{1/2}\left(\int_0^1 u_x^2\,dx\right)^{1/2}, \\ I_3 = u^2(0, t)\left(\displaystyle\int_0^1 u_x^2\,dx\right)^{1/2}, \\ I_4 = \left(\displaystyle\int_0^1 u^2\,dx\right)^{1/2}\displaystyle\int_0^1 u_x^2\,dx. \end{cases}$$

First we estimate I_1. It follows from (1.4) with $A = 0$ that

$$\int_0^1 au\,dx = C\int_0^1 bv\,dx \le C\left\{v(0, t) + \left(\int_0^1 v_x^2\,dx\right)^{1/2}\right\}.$$

On the other hand it holds that

$$u^2(0, t) \le C\left\{\int_0^1 au^2\,dx + \left(\int_0^1 au^2\,dx\right)^{1/2}\left(\int_0^1 u_x^2\,dx\right)^{1/2}\right\}. \tag{3.2}$$

Hence we have

$$\begin{aligned} I_1 &\le C\left\{u^2(0, t)v(0, t) + \int_0^1 au^2\,dx\left(\int_0^1 v_x^2\,dx\right)^{1/2}\right. \\ &\quad \left. + \left(\int_0^1 au^2\,dx\right)^{1/2}\left(\int_0^1 u_x^2\,dx\right)^{1/2}\left(\int_0^1 v_x^2\,dx\right)^{1/2}\right\} \\ &\le \varepsilon\left(\int_0^1 au^2\,dx\right)^{3/2} + C\left\{u^2(0, t)v(0, t) + \int_0^1 (u_x^2 + v_x^2)\,dx\right\}, \end{aligned}$$

where we use the L^∞-estimate (2.1).

By use of

$$\left(\int_0^1 u^2\,dx\right)^{1/2} \le C\left\{\left(\int_0^1 au^2\,dx\right)^{1/2} + \left(\int_0^1 u_x^2\,dx\right)^{1/2}\right\},$$

I_2 is evaluated as follows:

$$\begin{aligned} I_2 &\le C\left(\int_0^1 au^2\,dx\right)^{1/2}\left\{\left(\int_0^1 au^2\,dx\right)^{1/2} + \left(\int_0^1 u_x^2\,dx\right)^{1/2}\right\}\left(\int_0^1 u_x^2\,dx\right)^{1/2} \\ &\le \varepsilon\left(\int_0^1 au^2\,dx\right)^{3/2} + C\int_0^1 u_x^2\,dx. \end{aligned}$$

Estimates (2.1) and (3.2) yield

$$I_3 + I_4 \le \varepsilon\left(\int_0^1 au^2\,dx\right)^{3/2} + C\int_0^1 u_x^2\,dx.$$

□

It follows from this lemma and (3.1) that

$$\frac{d}{dt}(\mathscr{E}_2(t))^{-1/2} \ge c$$

for some $c > 0$, which gives Theorem 1.4. □

4. The case $A = 0$, $mn > 1$

We are concerned with the proof of Theorem 1.5. We multiply the first equation of (1.2) by $\varphi'_{m+n,1}(u)$, where $\varphi_{m+n,1}(u)$ is the same as in §2. Integration with respect to x yields

$$\frac{d}{dt}\int_0^1 a\varphi_{m+n,1}(u)\,dx + \int_0^1 \varphi''_{m+n,1}(u)\,u_x^2\,dx$$
$$+ c_{m+n,1}(m+n-1)k_1 u^{-n}(0,t)v^n(0,t)\varphi_{m+n,1}(u(0,t)) = 0.$$

$\int_0^1 a\varphi_{m+n,1}(u(x,t_0))\,dx = 0$ for some $t_0 \geq 0$ implies $u = v \equiv 0$ for $t \geq t_0$ with help of (1.4) with $A = 0$. Therefore we may assume $\int_0^1 a\varphi_{m+n,1}(u(x,t))\,dx > 0$ for all $t \geq 0$. Let us define sets F_i on $(0, \infty)$ by

$$F_1 = \{t > 0;\, u(0,t) \leq v(0,t)\},$$
$$F_2 = \{t > 0;\, u(0,t) \geq v(0,t)\}.$$

In the same manner as in the proof of Lemma 2.3, if $t \in F_1$, then

$$\int_0^1 a\varphi_{m+n,1}(u(x,t))\,dx \leq C\left(\varphi_{m+n,1}(u(0,t)) + \int_0^1 \varphi''_{m+n,1}(u)u_x^2\,dx\right)$$
$$\leq C\left(u^{-n}(0,t)v^n(0,t)\varphi_{m+n,1}(u(0,t)) + \int_0^1 \varphi''_{m+n,1}(u)\,u_x^2\,dx\right)$$

holds. Therefore we get a differential inequality

$$\frac{d}{dt}\log\int_0^1 a\varphi_{m+n,1}(u)\,dx \leq -\lambda_3\chi_{F_1}(t) \tag{4.1}$$

for some $\lambda_3 > 0$. Here χ_{F_1} is the characteristic function of F_1. (4.1) implies

$$\int_0^1 a\varphi_{m+n,1}(u)\,dx \leq C\exp\left\{-\lambda_3\int_0^t \chi_{F_1}(\tau)\,d\tau\right\}$$

for some $C > 0$.

This estimate and Jensen's inequality imply

$$\varphi_{m+n,1}\left(\int_0^1 au\,dx \middle/ \int_0^1 a\,dx\right) \leq C\exp\left\{-\lambda_3\int_0^t \chi_{F_1}(\tau)\,d\tau\right\}.$$

In a similar way we have

$$\varphi_{m+n,1}\left(\int_0^1 bv\,dx\Big/\int_0^1 b\,dx\right)\leq C\exp\left\{-\lambda_3\int_0^t \chi_{F_2}(\tau)\,d\tau\right\}.$$

Since

$$\int_0^t (\chi_{F_1}+\chi_{F_2})\,d\tau \geq t,$$

we have

$$\varphi_{m+n,1}\left(\int_0^1 au\,dx\Big/\int_0^1 a\,dx\right)\varphi_{m+n,1}\left(\int_0^1 bv\,dx\Big/\int_0^1 b\,dx\right)\leq Ce^{-\lambda_3 t}.$$

Recalling (1.4) with $A=0$, we obtain Theorem 1.5. □

5. Sharpness of estimates

We comment on the sharpness of estimates in Theorem 1.2.

Theorem 5.1. *We assume that the initial data* (u_0, v_0) *satisfy*

$$(5.1)\qquad \begin{cases} (u_0, v_0)\in\{H^2(0,1)\}^2,\ a^{-1/2}u_0''\in L^2(0,1),\ b^{-1/2}v_0''\in L^2(0,1),\\ u_0'(0)=k_1u_0^m(0)v_0^n(0),\ v_0'(0)=k_2u_0^m(0)v_0^n(0),\\ u_0'(1)=0,\ v_0'(1)=0,\\ u_0\geq 0,\ v_0\geq 0,\ u_0'\geq 0,\ v_0'\geq 0,\ u_0\not\equiv 0,\ v_0\not\equiv 0. \end{cases}$$

Then there exists a constant $C>1$ *such that the solution constructed by Theorem* 1.1 *has estimates*

$$C^{-1}\tilde{\rho}_1(t)\leq\int_0^1 au\,dx\leq C\rho_1(t)\qquad \text{if } A\leq 0,$$

$$C^{-1}\tilde{\rho}_2(t)\leq\int_0^1 bv\,dx\leq C\rho_2(t)\qquad \text{if } A\geq 0,$$

where functions $\rho_j(t)$ $(j=1,2)$ *are given in Theorem* 1.2, *and*

$$\tilde{\rho}_1(t)=\begin{cases}\rho_1(t) & \text{if either } A<0 \text{ and } m>1, \text{or } A=0,\\ e^{-\tilde{\lambda}t}(\tilde{\lambda}>0) & \text{if}\quad A<0 \text{ and } m=1,\end{cases}$$

$$\tilde{\rho}_2(t)=\begin{cases}\rho_2(t) & \text{if either } A>0 \text{ and } n>1, \text{ or } A=0,\\ e^{-\tilde{\lambda}t}(\tilde{\lambda}>0) & \text{if}\quad A>0 \text{ and } n=1.\end{cases}$$

To prove the theorem, we give a result of non-negativity of u_x and u_x under (5.1).

Lemma 5.1. *We assume* (5.1). *Then it holds that* $u_x(x, t) \geq 0$, $v_x(x, t) \geq 0$, *for all* $t > 0$.

Proof. We multiply both sides of the first equation of (1.2) by $\{(u_x)_-\}^2 u_t$, where $(u_x)_- = -\min\{u_x, 0\}$. Recalling $u_x(0, t) \geq 0$ by virtue of the non-negativity of u and v, we have

$$\frac{1}{12}\frac{d}{dt}\int_0^1 \{(u_x)_-\}^4\, dx + \int_0^1 a\{(u_x)_-\}^2 u_t^2\, dx = 0$$

via the integration with respect to x.

Under (5.1) the solution (u, v) belongs to $\{C([0, T]; H^2(0, 1)) \cap H^1(0, T; H^1(0, 1))\}^2$ for any $T > 0$ ([2]–[4]), in particular $(u_x)_- \in C([0, T]; L^4(0, 1))$. Consequently we have $u_x \geq 0$ by the integration of the above relation with respect to t. The assertion on v_x is also valid for a similar reason. □

Proof of Theorem 5.1. It is enough to show the estimates from below, and we give its proof for $\int_0^1 au\, dx$ only. We integrate the first equation in (1.2) with respect to x. By virtue of Lemma 5.1 and (2.1) if $A < 0$, (1.4) if $A = 0$, we obtain

$$\frac{d}{dr}\int_0^1 au\, dx = -k_1 u^m(0, t) v^m(0, t) \geq -C\left(\int_0^1 au\, dx\right)^p,$$

where

$$p = \begin{cases} m & \text{if } A < 0, \\ m + n & \text{if } A = 0. \end{cases}$$

This implies the assertion. □

6. A decay estimate for derivatives

We assume the existence of a non-negative C^1-function $\sigma(x)$ on $[0, 1]$ satisfying

(6.1) $$\sigma(x) > 0 \text{ on } [0, 1) \text{ and } \sigma'(x) \leq 0 \text{ on } [0, 1],$$

(6.2) $$\lim_{x \uparrow 1} \frac{a(x)\sigma(x)}{\int_x^1 a(\xi)\, d\xi} \text{ and } \lim_{x \uparrow 1} \frac{b(x)\sigma(x)}{\int_x^1 b(\xi)\, d\xi} \text{ exist and are finite,}$$

(6.3) $$\sup_{x \in [0,1)} \frac{\left(\int_x^1 a(\xi)\, d\xi\right)^2}{a(x)\sigma(x)} + \sup_{x \in [0,1)} \frac{\left(\int_x^1 b(\xi)\, d\xi\right)^2}{b(x)\sigma(x)} \leq C.$$

For example, if $a(x)$ and $b(x)$ satisfy

$$C^{-1}(1-x)^{k_3} \leq a(x) \leq C(1-x)^{k_3},$$
$$C^{-1}(1-x)^{k_4} \leq b(x) \leq C(1-x)^{k_4}$$

around $x=1$ for some $k_i \geq 0$, then $\sigma(x)=1-x$ satisfies our assumptions.

Then

$$\alpha(x)=\int_0^x \left(a(\xi)\sigma(\xi) \Big/ \int_\xi^1 a(\eta)\,d\eta\right) d\xi \int_x^1 a(\xi)\,d\xi$$

and

$$\beta(x)=\int_0^x \left(b(\xi)\sigma(\xi) \Big/ \int_\xi^1 b(\eta)\,d\eta\right) d\xi \int_x^1 b(\xi)\,d\xi$$

are non-negative C^2-functions on $[0, 1]$. The aim in this section is to show the following theorem.

Theorem 6.1. *We have*

$$\int_0^1 (\alpha u_x^2+\beta v_x^2)\,dx \leq \begin{cases} C\tilde{\rho}_2^{2n}(t) & \text{if } A>0, \\ C\rho_1^{2m}(t)\rho_2^{2n}(t) & \text{if } A=0, \\ C\tilde{\rho}_1^{2m}(t) & \text{if } A<0, \end{cases}$$

where functions $\rho_j(t)$ *and* $\tilde{\rho}_j(t)$ $(j=1, 2)$ *are given in Theorems* 1.2 *and* 5.1 *respectively.*

Before we shall prove the theorem, we need some estimates for $u(0, t)$ and $v(0, t)$. Let us define $q_i(x)$ $(i=1, 2)$ by

$$q_1(x)=\frac{a(x)\sigma(x)}{\displaystyle\int_x^1 a(\xi)\,d\xi}, \quad q_2(x)=\frac{b(x)\sigma(x)}{\displaystyle\int_x^1 b(\xi)\,d\xi}.$$

Lemma 6.1. *For any* $\varepsilon>0$ *there exist* $C>0$ *and* $T \geq 0$ *such that if* $A<0$ *and* $k \geq 1$, *or* $A=0$ *and* $k>1$, *then*

$$u^{2k}(0, t) \leq C\rho_1^{2k}(t)+\varepsilon\int_0^1 q_1 u_x^2\,dx \qquad \text{for } t \geq T,$$

and if $A>0$ *and* $k \geq 1$, *or* $A=0$ *and* $k>1$, *then*

$$v^{2k}(0, t) \leq C\rho_2^{2k}(t)+\varepsilon\int_0^1 q_2 v_x^2\,dx \qquad \text{for } t \geq T.$$

Proof. We prove the former estimate only. First we see the case $k > 1$:

$$u^{2k}(0, t) \le C\left\{\left(\int_0^{1/2} u\,dx\right)^k + \int_0^{1/2} u^{k-1}|u_x|\,dx\right\}^2$$
$$\le C\left\{\left(\int_0^1 au\,dx\right)^{2k} + \int_0^1 au^{2k-2}dx \int_0^1 q_1 u_x^2\,dx\right\}.$$

Here we use the fact $a(0) > 0$ and $q_1(0) > 0$. Theorem 1.2 and the estimate

$$\int_0^1 au^{2k-2}\,dx \le \begin{cases} C\|u\|_\infty^{2k-3}\,\rho_1(t) \longrightarrow 0 & \text{as } t \longrightarrow \infty \text{ for } k \ge 3/2, \\ C\rho_1^{2k-2}(t) \longrightarrow 0 & \text{as } t \longrightarrow \infty \text{ for } 1 < k \le 3/2 \end{cases}$$

yield the assertion.

When $A < 0$ and $k = 1$,

$$u^2(0, t) \le C\left\{\left(\int_0^{1/2} u\,dx\right)^2 + \int_0^{1/2} u|u_x|\,dx\right\}$$
$$\le C\left\{\left(\int_0^1 au\,dx\right)^2 + \int_0^1 au^2\,dx\right\} + \varepsilon\int_0^1 q_1 u_x^2\,dx$$

holds by Schwarz's inequality. Theorem 1.3 gives the desired estimate. □

Corollary. *For any $\varepsilon > 0$ there exist $C > 0$ and $T > 0$ such that*

$$u^{2m}(0, t)v^{2n}(0, t) \le \begin{cases} C\rho_2^{2n}(t) + \varepsilon\displaystyle\int_0^1 q_2 v_x^2\,dx & \text{if } A > 0, \\ C\rho_1^{2m}(t)\rho_2^{2n}(t) + \varepsilon\displaystyle\int_0^1 (q_1 u_x^2 + q_2 v_x^2)\,dx & \text{if } A = 0, \\ C\rho_1^{2m}(t) + \varepsilon\displaystyle\int_0^1 q_1 u_x^2\,dx & \text{if } A < 0 \end{cases}$$

is valid for $t \ge T$.

Proof. Since $u, v \in L^\infty((0, 1) \times (0, \infty))$, estimates for $A \ne 0$ follows from Lemma 6.1 directly. If $A = 0$, then the assertion is derived from the fact $\rho_1(t) = \rho_2(t)$ and $u^{2m}v^{2n} \le u^{2(m+n)} + v^{2(m+n)}$. □

Proof of the Theorem 6.1. By the first equation of (1.2) and the boundary condition on $u_x(1, t)$ we have

$$u_x^2(x, t) = -2\int_x^1 a(\xi)u_t(\xi, t)u_x(\xi, t)\,d\xi.$$

Then we get

$$\begin{aligned}\int_0^1 q_1 u_x^2\,dx &= -2\int_0^1 q_1(x)\,dx\int_x^1 a(\xi)u_t(\xi,t)u_x(\xi,t)\,d\xi\\ &= -2\int_0^1 a(\xi)u_t(\xi,t)u_x(\xi,t)u_x(\xi,t)\,d\xi\int_0^\xi q_1(x)\,dx\\ &= -2\int_0^1\left\{\alpha(u_{tx}u_x+u_tu_{xx})+u_tu_xq_1\int_x^1 a(\xi)\,d\xi\right\}dx \qquad (6.4)\\ &= -\frac{d}{dt}\int_0^1 \alpha u_x^2\,dx-\int_0^1\{2\alpha a u_t^2+(u_x^2)_x\sigma\}\,dx\\ &\le -\frac{d}{dt}\int_0^1 \alpha u_x^2\,dx+k_1^2u^{2m}(0,t)v^{2n}(0,t)\sigma(0).\end{aligned}$$

The last inequality is derived from the boundary condition on $u_x(0, t)$ and (6.1). In a similar way we have

$$\frac{d}{dt}\int_0^1 \beta v_x^2\,dx+\int_0^1 q_2v_x^2\,dx\le k_2^2u^{2m}(0,t)v^{2n}(0,t)\sigma(0). \tag{6.5}$$

Assumptions (6.2) and (6.3) imply $\alpha(x)\le Cq_1(x)$ and $\beta(x)\le Cq_2(x)$. Consequently (6.4), (6.5) and Corollary yield the differential inequality

$$\frac{d}{dt}\int_0^1(\alpha u_x^2+\beta v_x^2)\,dx+\lambda_4\int_0^1(\alpha u_x^2+\beta v_x^2)\,dx\le \begin{cases} C\rho_2^{2n}(t) & \text{if } A>0,\\ C\rho_1^{2m}(t)\rho_2^{2n}(t) & \text{if } A=0,\\ C\rho_1^{2m}(t) & \text{if } A<0\end{cases}$$

for some $\lambda_4>0$, $C>0$ and $t\ge T$. The assertion of Theorem 6.1 is given by the integration. □

Remark. While this paper was in the hand of the referee, Professor Yotsutani kindly pointed out that the estimates of Theorem 1.2 can be improved. Due to him, we have decay estimates of $\|u(\cdot,t)-u_\infty\|_{L^\infty(0,1)}$ and $\|v(\cdot,t)-v_\infty\|_{L^\infty(0,1)}$ with the dominant function $\rho(t)=C\min\{\rho_1(t),\rho_2(t)\}$, where $\rho_j(t)$ are given in Theorem 1.2. The proof is in an improved procedure of this paper, in particular Lemma 2.3 and Lemma 3.1. It will be published anywhere by names: Iida, M., Y. Yamada and S. Yotsutani. The author is grateful to Professor Yotsutani for his helpful information.

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