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Kneser Type Theorems for Functional Differential Equations in a Banach Space

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§1. Introduction

Let E be an infinite dimensional Banach space with a norm $|\cdot|_E$ and let $\mathscr{B}$ be the phase space satisfying the fundamental axioms introduced by Hale and Kato [3] (cf. [5, 8]). If $x:(-\infty, \sigma+a)\to E$, $0<a\le\infty$, then for any $t\in(-\infty, \sigma+a)$ we define $x_t:(-\infty, 0]\to E$ by $x_t(\theta)=x(t+\theta)$, $-\infty<\theta\le 0$.

The purpose of this paper is to show Kneser type theorems for the functional differential equation (FDE) with infinite delay in a Banach space (for brevity, CP (1.1) or CP (f, σ, φ))

$$\frac{dx}{dt}=f(t, x_t), \qquad x_\sigma=\varphi\in\mathscr{B}, \tag{1.1}$$

under the condition that $f:[\sigma, \sigma+a]\times\mathscr{B}(\varphi, r)\to E$ is uniformly continuous (cf. [2]), where $\mathscr{B}(\varphi, r):=\{\psi\in\mathscr{B}\mid |\varphi-\psi|_{\mathscr{B}}\le r\}$.

Kneser type theorems for ordinary differential equations (ODEs) in Banach spaces have been shown by Szufla [15] and Kubiaczyk [7], etc.. For FDEs with infinite delay in finite dimensional space, Kaminogo [4] has proved Kneser's theorem. Recently, the basic theorems on the existence of solutions and the uniqueness of solutions for CP (1.1) in Banach spaces have been developed by Shin [9, 10, 11, 12, 13].

In this paper we shall prove Kneser type theorems for CP (1.1) in a Banach space by using a property (Theorem 2.2) on Kuratowski's measure of noncompactness. In particular, the results given in [10, Theorem 3.1 and 12, Theorem 2.4] are, in some sense, contained in our existence theorem (Theorem 4.3).

§2. A property of the α-measure of noncompactness

In this section, we shall consider a property on Kuratowski's measure of noncompactness (for brevity, α-measure). Let Y be a Banach space with a norm $|\cdot|_Y$. For a bounded set Ω of Y the α-measure of Ω is defined as follows:

$$\alpha(\Omega) = \inf\{d > 0 \mid \Omega \text{ has a finite cover of diameter } < d\}.$$

For properties of the α-measure refer to [1, 6]. Henceforth, we will use the same notation $\alpha(\cdot)$ for the α-measure of noncompactness in any Banach space. We denote by N the set of all positive integers. To prove the main theorem in this section, we need the following result which is well known (cf. [6, Theorem 1.4.1]).

Lemma 2.1. *If $\{A_n\}$ is a family of nonempty bounded subsets of Banach space Y such that $A_{n+1} \subset A_n$ for every $n \in N$ and $\alpha(A_n) \to 0$ as $n \to \infty$, then $\bigcap_{n=1}^{\infty} \mathrm{Cl}\, A_n$ is nonempty and compact, where $\mathrm{Cl}\, A$ stands for the closure of A.*

Using the above result, we can obtain the main result in this section, which plays an essential role in showing Kneser type theorems for CP (1.1).

Theorem 2.2. *Let $\{S_n\}$ be a family of nonempty bounded subsets of Banach space Y such that $S_{n+1} \subset S_n$ for every $n \in N$. If the set S_n is connected for every $n \in N$ and if $\alpha(S_n) \to 0$ as $n \to \infty$, then the set $\bigcap_{n=1}^{\infty} \mathrm{Cl}\, S_n$ is nonempty, compact and connected.*

Proof. Set $S = \bigcap_{n=1}^{\infty} \mathrm{Cl}\, S_n$. Then it follows from Lemma 2.1 that the set S is nonempty and compact. Thus it remains to show that S is connected. Suppose that S is not connected in Banach space Y. Then there exist nonempty closed sets F^1 and F^2 such that $S = F^1 \cup F^2$ and $F^1 \cap F^2 = \phi$. Hence there exist two disjoint open sets U^1 and U^2 such that $F^1 \subset U^1$ and $F^2 \subset U^2$. Put $U = U^1 \cup U^2$. Clearly, we have $S \subset U$. Now, we shall prove that there is an $m \in N$ such that $\mathrm{Cl}\, S_m \subset U$. Suppose, for a contradiction, that $\mathrm{Cl}\, S_n \setminus U \neq \phi$ for every $n \in N$. Since $\alpha(\mathrm{Cl}\, S_n \setminus U) \to 0$ as $n \to \infty$ and $\mathrm{Cl}\, S_{n+1} \subset \mathrm{Cl}\, S_n$, we have $\phi = S \setminus U = \bigcap_{n=1}^{\infty} (\mathrm{Cl}\, S_n \setminus U) \neq \phi$ by Lemma 2.1. This is a contradiction. Hence we have $S \subset \mathrm{Cl}\, S_m \subset U$. It is obvious that $U^1 \cap \mathrm{Cl}\, S_m \neq \phi$ and $U^2 \cap \mathrm{Cl}\, S_m \neq \phi$. Set $P_m = (U^1)^c \cap \mathrm{Cl}\, S_m$ and $Q_m = (U^2)^c \cap \mathrm{Cl}\, S_m$, where $(U^i)^c = Y \setminus U^i$, $i = 1, 2$. Then $\mathrm{Cl}\, S_m = P_m \cup Q_m$, $P_m \neq \phi \neq Q_m$, and $P_m \cap Q_m = \phi$. Thus $\mathrm{Cl}\, S_m$ is disconnected, which yields a contradiction with the connectedness of S_m. Therefore the proof of the theorem is completed.

§3. The connectedness of the set of approximate solutions

Let $R^- = (-\infty, 0]$, $R^+ = [0, \infty)$ and $R = (-\infty, \infty)$. We denote by $\mathscr{B}^X = \mathscr{B}(R^-, X)$ a linear space of functions from R^- into X with semi-norm

$|\cdot|_{\mathscr{B}^X}$, where $X = E$ or R. Let $\mathscr{X}_\sigma[\delta]$ (resp. $\mathscr{X}_\sigma(\delta)$) be the set of all functions x from $(-\infty, \sigma+\delta]$ (resp. $(-\infty, \sigma+\delta)$), $\delta > 0$, to X such that $x_\sigma \in \mathscr{B}^X$ and $x(t)$ is continuous on $[\sigma, \sigma+\delta]$ (resp. $[\sigma, \sigma+\delta)$). In particular, for a given $\varphi \in \mathscr{B}^X$ we denote by $\mathscr{X}_\sigma^\varphi[\delta]$ the set of all $x \in \mathscr{X}_\sigma[\delta]$ satisfying $x_\sigma = \varphi$. Also, we will use the notation $\mathscr{E}_\sigma[\delta]$ for $\mathscr{X}_\sigma[\delta]$ if $X = E$, etc.. Throughout this paper we assume that the following axioms on the phase space $\mathscr{B}^X$ are always satisfied:

(B_1) If $x \in \mathscr{X}_\sigma(a)$, $0 < a \le \infty$, then x_t lies in $\mathscr{B}^X$ for all $t \in [\sigma, \sigma+a)$ and x_t is continuous in t on $[\sigma, \sigma+a)$.

(B_2) There exist a constant $L > 0$ and functions $K(t) > 0$ and $M(t) \ge 0$ with the following properties:

(i) $K(t)$ is continuous for t in R^+,

(ii) $M(t)$ is locally bounded on R^+.

(iii) For every function $x \in \mathscr{X}_\sigma(a)$, it holds that

$$\frac{1}{L}|x(t)|_X \le |x_t|_{\mathscr{B}^X} \le K(t-\sigma)\sup\{|x(s)|_X \mid \sigma \le s \le t\} + M(t-\sigma)|x_\sigma|_{\mathscr{B}^X}$$

for $t \in [\sigma, \sigma+a)$.

(B_3) The quotient space $\tilde{\mathscr{B}}^X := \mathscr{B}^X/|\cdot|_{\mathscr{B}^X}$ is a Banach space.

If E is understood from the context, we shall write $\mathscr{B}$ for $\mathscr{B}^E$ and $|\cdot|$ for $|\cdot|_E$. Set $K_a = \sup\{K(t) | 0 \le t \le a\}$ and $M_a = \sup\{M(t) | 0 \le t \le a\}$ in Axiom (B_2).

Let f: $[\sigma, \sigma+a] \times \mathscr{B}(\varphi, r) \to E$ in CP (1.1) be a continuous function such that $|f| \le M-1$, $M > 1$. Then for each $n \in N$ a function u: $(-\infty, \sigma+\gamma] \to E$, $0 < \gamma \le a$, is said to be an $\frac{1}{n}$ – approximate solution for CP (1.1) if it satisfies the following conditions:

(1) u belongs to $\mathscr{E}_\sigma^\varphi[\gamma]$;

(2) u has the right hand derivative D_+u such that $|(D_+u)(t)| \le M$ on $[\sigma, \sigma+\gamma)$, and satisfies

$$u(t) = \varphi(0) + \int_\sigma^t (D_+u)(s)\,ds \qquad \text{for } t \in [\sigma, \sigma+\gamma]\text{; and}$$

(3) $|(D_+u)(t) - f(t, u_t)| \le \frac{1}{n}$ for all $t \in [\sigma, \sigma+\gamma)$.

Let $C([a, b], X)$ be the set of all continuous functions x: $[a, b] \to X$ with supremum norm. For a subset $\mathscr{E}$ of $\mathscr{E}_\sigma[\gamma]$ the following notations will be used in the present paper;

$$\mathscr{E}(t)=\{x(t)\in E\,|\,x\in\mathscr{E}\},\ \mathscr{E}_t=\{x_t\in\mathscr{E}\,|\,x\in\mathscr{E}\}\qquad\text{for } t\in[\sigma,\sigma+\gamma],$$
$$\mathscr{E}\,|\,[c,d]=\{x\,|\,[c,d]\in C([c,d],E)\,|\,x\in\mathscr{E}\}$$

and

$$\mathscr{E}\,|_{[c,d]}=\{\zeta\,|\,[c,d]\in C([c,d],\mathscr{B})\,|\,\zeta(t):=x_t\quad\text{for } t\in[\sigma,\sigma+\gamma]\text{ and } x\in\mathscr{E}\},$$

where c, $d\in[\sigma,\sigma+\gamma]$ and $x\,|\,[c,d]$ is the restriction of x to $[c,d]$.

Define a linear operator $S_X(t)\colon \mathscr{B}^X\to\mathscr{B}^X$, $t\geq 0$, by

$$[S_X(t)\varphi](\theta)=\begin{cases}\varphi(t+\theta) & \text{if } t+\theta<0\\ \varphi(0) & \text{if } t+\theta\geq 0.\end{cases}$$

We shall now prove the connectedness of the set of all $\frac{1}{n}$-approximate solutions for CP (1.1), which corresponds to Proposition 2 in [7].

Lemma 3.1. *Suppose the phase space* $\mathscr{B}^E$ *satisfies Axioms* (B_1) *and* (B_2). *Assume that* $f\colon[\sigma,\sigma+a]\times\mathscr{B}(\varphi,r)\to E$ *is uniformly continuous and* $|f|\leq M-1$, $M>1$, *on* $[\sigma,\sigma+a]\times\mathscr{B}(\varphi,r)$. *Let* δ_0 *be the largest number such that* $|S_E(t)\varphi-\varphi|_{\mathscr{B}}\leq r/2$ *for all* $t\in[0,\delta_0]$. *Then for each* $n\in N$ *the set* Q^n *of all* $\frac{1}{n}$*-approximate solutions, defined on* $(-\infty,\sigma+\gamma]$, *for* CP (1.1) *is nonempty, and* $Q^n\,|\,[\sigma,\sigma+\gamma]$ *and* $Q^n|_{[\sigma,\sigma+\gamma]}$ *are connected in* $C([\sigma,\sigma+\gamma],E)$ *and in* $C([\sigma,\sigma+\gamma],\mathscr{B})$, *respectively, where* $\gamma=\min\{a,r/2K_aM,\delta_0\}$.

Proof. The proof will be divided into three parts as follows.

Step 1. We shall show that for any $n\in N$ the set Q^n is nonempty. By the method of Euler-Cauchy polygons [13, Lemma 2.3], we define a function x^ε for each ε, $\gamma>\varepsilon>0$, as follows:

$$x^\varepsilon(t)=\begin{cases}\varphi(t-\sigma) & t\leq\sigma\\ x^\varepsilon(\varepsilon(i))+(t-\varepsilon(i))f(\varepsilon(i),x^\varepsilon_{\varepsilon(i)}) & t\in[\varepsilon(i),\varepsilon(i+1)),\end{cases}$$

where $\varepsilon(i)=\sigma+i\varepsilon$, $i=0,1,\ldots,n(\varepsilon)$, $n(\varepsilon)=\left[\dfrac{\gamma}{\varepsilon}\right]$, $\varepsilon(n(\varepsilon)+1)=\sigma+\gamma$ (if $n(\varepsilon)\varepsilon<\gamma$).

Define a mapping $\tau_\varepsilon\colon[\sigma,\sigma+\gamma]\to[\sigma,\sigma+\gamma]$ by $\tau_\varepsilon(t)=\varepsilon(i)$ for $t\in[\varepsilon(i),\varepsilon(i+1))$, where $i=0,1,\ldots,n(\varepsilon)$. Then the above function x^ε can be expressed in the form

$$x^\varepsilon(t) = \begin{cases} \varphi(t-\sigma) & t \le \sigma \\ \varphi(0) + \displaystyle\int_\sigma^t f(\tau_\varepsilon(s), x^\varepsilon_{\tau_\varepsilon(s)})\,ds & t \in [\sigma, \sigma+\gamma]. \end{cases}$$

It is easy to see that $x_t^\varepsilon \in \mathscr{B}(\varphi, r)$ for all $t \in J := [\sigma, \sigma+\gamma]$. Since f is uniformly continuous, there is a $\delta(n) > 0$ such that $|f(t, \varphi_1) - f(s, \varphi_2)| \le \frac{1}{n}$ if $|t-s| \le \delta(n)$ and $|\varphi_1 - \varphi_2|_{\mathscr{B}} \le \delta(n)$ for $(t, \varphi_1), (s, \varphi_2) \in J \times \mathscr{B}(\varphi, r)$. Since $\{S_E(t)\}_{t \ge 0}$ is a strongly continuous semigroup in $\mathscr{B}$, we can choose a $\delta_0(n) > 0$ such that $|S_E(\tau_\varepsilon(t) - \sigma)\varphi - S_E(t-\sigma)\varphi|_{\mathscr{B}} < \delta(n)/2$ for all $\varepsilon \in (0, \delta_0(n))$. Put $\beta(n) = \min\{\delta(n)/2(K_a+1)M, \delta_0(n)\}$. By Lemma 2.1 in [12], we have

$$\begin{aligned} |x^\varepsilon_{\tau_\varepsilon(t)} - x^\varepsilon_t|_{\mathscr{B}} &\le K_a M|\tau_\varepsilon(t) - t| + |S_E(\tau_\varepsilon(t)-\sigma)\varphi - S_E(t-\sigma)\varphi| \\ &\le K_a M\beta(n) + \delta(n)/2 \le \delta(n) \qquad \text{if } 0 < \varepsilon < \beta(n), \end{aligned}$$

which imlies that $x^\varepsilon(\cdot) \in Q^n$ for all $\varepsilon \in (0, \beta(n))$. Put

$$V_\beta^n = \{x^\varepsilon(\cdot) \in Q^n | 0 < \varepsilon < \beta(n)\}.$$

Step 2. We shall prove that $V_\beta^n | J$ is connected in $C(J, E)$. To do this, it is sufficient to prove the continuity of the mapping $\varepsilon \to x^\varepsilon | J$, $x^\varepsilon \in V_\beta^n$; that is, for any fixed ε, $0 < \varepsilon < \beta(n)$, we must prove that $x^\delta(t) \to x^\varepsilon(t)$ uniformly on J as $\delta \to \varepsilon$. Without loss of generality, we may assume that $\left[\frac{\gamma}{\delta}\right] = \left[\frac{\gamma}{\varepsilon}\right]$ and $\varepsilon(n(\varepsilon)+1) = \sigma + \gamma$. Set $\delta(i) = \sigma + i\delta$, $i = 0, 1, \ldots, n(\varepsilon)$, $\delta(n(\varepsilon)+1) = \sigma + \gamma$. Clearly, $\zeta(\delta, \varepsilon) := \max\{|\varepsilon(i) - \delta(i)| \mid i = 1, 2, \ldots, n(\varepsilon)+1\} \to 0$ as $\delta \to \varepsilon$.

We shall at first prove that for each j, $1 \le j \le n(\varepsilon)+1$,

$$|x^\delta_{\varepsilon(j)} - x^\varepsilon_{\varepsilon(j)}|_{\mathscr{B}} \longrightarrow 0 \quad \text{and} \quad |x^\delta_{\delta(j)} - x^\varepsilon_{\varepsilon(j)}|_{\mathscr{B}} \longrightarrow 0 \text{ as } \delta \longrightarrow \varepsilon. \tag{3.1}$$

The proof is by induction on j. If $j = 1$, then we have, by Axiom (B_2),

$$\begin{aligned} &|x^\delta_{\varepsilon(1)} - x^\varepsilon_{\varepsilon(1)}|_{\mathscr{B}} \\ &\le K_a \sup_{\sigma \le t \le \varepsilon(1)} \left| \int_\sigma^t [f(\tau_\delta(s), x^\delta_{\tau_\delta(s)}) - f(\tau_\varepsilon(s), x^\varepsilon_{\tau_\varepsilon(s)})]\,ds \right| \\ &\le K_a \int_\sigma^{\varepsilon(1)} |f(\tau_\delta(s), x^\delta_{\tau_\delta(s)}) - f(\tau_\varepsilon(s), x^\varepsilon_{\tau_\varepsilon(s)})|\,ds \\ &\le K_a |\delta(1) - \varepsilon(1)|\, |f(\delta(1), x^\delta_{\delta(1)}) - f(\sigma, \varphi)| \\ &\le 2K_a M\zeta(\delta, \varepsilon) \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow \varepsilon. \end{aligned}$$

Since $|x^\delta(t) - x^\delta(s)| \le M|t-s|$ for all $t, s \in J$ and for all $\delta \in (0, \beta(n))$ and $\{x_t^\delta\}_{0<\delta<\beta(n)}$ is uniformly equicontinuous on J, we see that $x^\delta(\delta(1)) \to x^\varepsilon(\varepsilon(1))$

and $x^\delta_{\delta(1)} \to x^\varepsilon_{\varepsilon(1)}$ as $\delta \to \varepsilon$. Assume that (3.1), $j = 1,\ldots, i$, hold. Then, we have the estimate

$$\begin{aligned}
&\sup\{|x^\delta(s) - x^\varepsilon(s)| \mid \varepsilon(i) \le s \le \varepsilon(i+1)\} \\
&\le \sup\left\{\left|\int_{\varepsilon(i)}^{s} [f(\tau_\delta(s), x^\delta_{\tau_\delta(s)}) - f(\tau_\varepsilon(s), x^\varepsilon_{\tau_\varepsilon(s)})]\,ds\right| \mid \varepsilon(i) \le s \le \varepsilon(i+1)\right\} \\
&\qquad + |x^\delta(\varepsilon(i)) - x^\varepsilon(\varepsilon(i))| \\
&\le |I| + |x^\delta(\varepsilon(i)) - x^\varepsilon(\varepsilon(i))|
\end{aligned}$$

where

$$I = \int_{\varepsilon(i)}^{\tilde{t}} [f(\tau_\delta(s), x^\delta_{\tau_\delta(s)}) - f(\tau_\varepsilon(s), x^\varepsilon_{\tau_\varepsilon(s)})]\,ds \qquad \text{for some } \tilde{t} \in (\varepsilon(i), \varepsilon(i+1)].$$

(1) Case $\varepsilon(i) \le \delta(i) \le \tilde{t}$. Then

$$\begin{aligned}
|I| &\le M|\delta(i) - \varepsilon(i)| + |(\tilde{t} - \delta(i))f(\delta(i), x^\delta_{\delta(i)}) - (\tilde{t} - \varepsilon(i))f(\varepsilon(i), x^\varepsilon_{\varepsilon(i)})| \\
&\le 2M|\delta(i) - \varepsilon(i)| + |\tilde{t} - \varepsilon(i)|\,|f(\delta(i), x^\delta_{\delta(i)}) - f(\varepsilon(i), x^\varepsilon_{\varepsilon(i)})|.
\end{aligned}$$

(2) Case $\tilde{t} < \delta(i) < \varepsilon(i+1)$. Then

$$\begin{aligned}
|I| &\le |\tilde{t} - \varepsilon(i)|\,|f(\delta(i-1), x^\delta_{\delta(i-1)}) - f(\varepsilon(i), x^\varepsilon_{\varepsilon(i)})| \\
&\le 2M|\delta(i) - \varepsilon(i)|.
\end{aligned}$$

(3) Case $\varepsilon(i) < \delta(i+1) \le \tilde{t}$. Then

$$\begin{aligned}
|I| &\le M|\delta(i+1) - \varepsilon(i+1)| \\
&\qquad + |(\delta(i+1) - \varepsilon(i))f(\delta(i), x^\delta_{\delta(i)}) - (\tilde{t} - \varepsilon(i))f(\varepsilon(i), x^\varepsilon_{\varepsilon(i)})| \\
&\le 2M|\delta(i+1) - \varepsilon(i+1)| \\
&\qquad + |\tilde{t} - \varepsilon(i)|\,|f(\delta(i), x^\delta_{\delta(i)}) - f(\varepsilon(i), x^\varepsilon_{\varepsilon(i)})|.
\end{aligned}$$

(4) Case $\tilde{t} < \delta(i+1) < \varepsilon(i+1)$. Then

$$|I| \le |\tilde{t} - \varepsilon(i)|\,|f(\delta(i), x^\delta_{\delta(i)}) - f(\varepsilon(i), x^\varepsilon_{\varepsilon(i)})|.$$

Summarizing the above cases, we have

$$\begin{aligned}
|I| &\le 2M\zeta(\delta, \varepsilon) + \varepsilon|f(\delta(i), x^\delta_{\delta(i)}) - f(\varepsilon(i), x^\varepsilon_{\varepsilon(i)})| \\
&\longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow \varepsilon,
\end{aligned}$$

which implies that

$$\sup\{|x^\delta(s) - x^\varepsilon(s)| \mid \varepsilon(i) \le s \le \varepsilon(i+1)\} \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow \varepsilon. \tag{3.2}$$

Thus we get, together with Axiom (B_2),

$$\frac{1}{L}|x^\delta(\varepsilon(i+1)) - x^\varepsilon(\varepsilon(i+1))| \leq |x^\delta_{\varepsilon(i+1)} - x^\varepsilon_{\varepsilon(i+1)}|_{\mathscr{B}}$$

$$\leq K_a \sup\{|x^\delta(s) - x^\varepsilon(s)| \,|\, \varepsilon(i) \leq s \leq \varepsilon(i+1)\} + M_a|x^\delta_{\varepsilon(i)} - x^\varepsilon_{\varepsilon(i)}|_{\mathscr{B}}$$

$$\longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow \varepsilon.$$

Consequently, we see that $\max\{|x^\delta_{\delta(j)} - x^\varepsilon_{\varepsilon(j)}|_{\mathscr{B}} \,|\, j = 1, 2,\dots, n(\varepsilon)+1\} \to 0$ and $\max\{|x^\delta(\delta(j)) - x^\varepsilon(\varepsilon(j))| \,|\, j = 1, 2,\dots, n(\varepsilon)+1\} \to 0$ as $\delta \to \varepsilon$.

Next, we shall show that for each $t \in J$, $|x^\delta_t - x^\varepsilon_t|_{\mathscr{B}} \to 0$ as $\delta \to \varepsilon$. For each $t \in (\sigma, \sigma+\gamma]$, there exists an $i \in N$ such that $\varepsilon(i) < t \leq \varepsilon(i+1)$. Using Axiom ($B_2$), (3.1) and (3.2), we can obtain

$$|x^\delta_t - x^\varepsilon_t|_{\mathscr{B}}$$

$$\leq K_a \sup\{|x^\delta(s) - x^\varepsilon(s)| \,|\, \varepsilon(i) \leq s \leq \varepsilon(i+1)\} + M_a|x^\delta_{\varepsilon(i)} - x^\varepsilon_{\varepsilon(i)}|_{\mathscr{B}}$$

$$\longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow \varepsilon,$$

and hence

$$|x^\delta(t) - x^\varepsilon(t)| \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow \varepsilon.$$

Step 3. We shall prove that the set $Q^n|J$ is connected in $C(J, E)$. For any fixed ε_0, $0 < \varepsilon_0 < \beta(n)$, and for each $z(\cdot) \in Q^n$ we define a set $T^n_z := \{z^p(\cdot) \in Q^n \,|\, \sigma \leq p \leq \sigma+\gamma\}$ consisting of the function

$$z^p(t) = \begin{cases} z(t) & t \leq p \\ z(p) + \displaystyle\int_p^t f(\tau(p, s), z^p_{\tau(p,s)})\,ds & t \in (p, \sigma+\gamma], \end{cases}$$

where $\tau(p, t) = \tau_{\varepsilon_0}(t)$, if $t \in [p, \sigma+\gamma] \setminus [p, \varepsilon_0(j+1))$, $p \in [\varepsilon_0(j), \varepsilon_0(j+1))$, while $\tau(p, t) = p$ if $t \in [p, \varepsilon_0(j+1))$. It is not difficult to see that $z^\sigma(\cdot) = x^{\varepsilon_0}(\cdot) \in V^n_\beta$ and $z^p(\cdot) \in Q^n$. Now we will prove that $z^q(t) \to z^p(t)$ as $q \to p$ for every $t \in J$. Without loss of generality, we may assume that $p < q < t$, $p, q \in [\varepsilon_0(j), \varepsilon_0(j+1))$, $t \in [\varepsilon_0(k), \varepsilon_0(k+1))$, where $k \geq j$. Clearly, $|z^q(q) - z^p(p)| \to 0$ and $|z^q_q - z^p_p|_{\mathscr{B}} \to 0$ as $q \to p$. Furthermore, we have

$$|z^q_{\varepsilon_0(j+1)} - z^p_{\varepsilon_0(j+1)}|_{\mathscr{B}} \leq K_a \sup_{p \leq s \leq \varepsilon_0(j+1)} |z^q(s) - z^p(s)|$$

$$\leq K_a \max\{\sup_{p \leq s \leq q} |z^q(s) - z^p(s)|, \sup_{q \leq s \leq \varepsilon_0(j+1)} |z^q(s) - z^p(s)|\}$$

$$\leq K_a \max\Big\{2M|p-q|, |z(p) - z(q)| + |p-q|\,|f(p, z^p_p)|$$

$$+ \sup_{q \le s \le \varepsilon_0(j+1)} \int_q^s |f(q, z_q^q) - f(p, z_p^p)| ds \Big\}$$
$$\le K_a \{2M|p - q| + \varepsilon_0 |f(p, z_p^p) - f(q, z_q^q)|\}$$
$$\longrightarrow 0 \quad \text{as} \quad q \longrightarrow p.$$

Assume that $|z^q_{\varepsilon_0(i)} - z^p_{\varepsilon_0(i)}|_{\mathscr{B}} \to 0$ as $q \to p$, where $i = j+1, \dots, k-1$.

Then we have

$$\sup_{\varepsilon_0(k-1) \le s \le \varepsilon_0(k)} |z^q(s) - z^p(s)|$$
$$\le \int_{\varepsilon_0(k-1)}^{\varepsilon_0(k)} |f(\varepsilon_0(k-1), z^q_{\varepsilon_0(k-1)}) - f(\varepsilon_0(k-1), z^p_{\varepsilon_0(k-1)})| ds$$
$$+ |z^q(\varepsilon_0(k-1)) - z^p(\varepsilon_0(k-1))|$$
$$\le \varepsilon_0 |f(\varepsilon_0(k-1), z^q_{\varepsilon_0(k-1)}) - f(\varepsilon_0(k-1), z^p_{\varepsilon_0(k-1)})|$$
$$+ |z^q(\varepsilon_0(k-1)) - z^p(\varepsilon_0(k-1))|$$
$$\longrightarrow 0 \quad \text{as} \quad q \longrightarrow p,$$

and hence

$$|z^q_{\varepsilon_0(k)} - z^p_{\varepsilon_0(k)}|_{\mathscr{B}}$$
$$\le K_a \sup_{\varepsilon_0(k-1) \le s \le \varepsilon_0(k)} |z^q(s) - z^p(s)| + M_a |z^q_{\varepsilon_0(k-1)} - z^p_{\varepsilon_0(k-1)}|_{\mathscr{B}}$$
$$\longrightarrow 0 \quad \text{as} \quad q \longrightarrow p.$$

Thus we get

$$|z_t^q - z_t^p|_{\mathscr{B}} \le K_a \sup_{\varepsilon_0(k) \le s \le t} |z^q(s) - z^p(s)| + M_a |z^q_{\varepsilon_0(k)} - z^p_{\varepsilon_0(k)}|_{\mathscr{B}}$$
$$\le K_a |t - \varepsilon_0(k)| \, |f(\varepsilon_0(k), z^q_{\varepsilon_0(k)}) - f(\varepsilon_0(k), z^p_{\varepsilon_0(k)})|$$
$$+ M_a |z^q_{\varepsilon_0(k)} - z^p_{\varepsilon_0(k)}|_{\mathscr{B}} + K_a |z^q(\varepsilon_0(k)) - z^p(\varepsilon_0(k))|$$
$$\longrightarrow 0 \quad \text{as} \quad q \longrightarrow p,$$

and hence, by Axiom (B_2),

$$|z^q(t) - z^p(t)| \longrightarrow 0 \quad \text{as} \quad q \longrightarrow p. \tag{3.3}$$

Since $T_z^n \subset Q^n$ and $Q^n|J$ is uniformly equicontinuous, (3.3) holds uniformly on J. Thus $T_z^n|J$ is connected in $C(J, E)$. Since $z^\sigma(\cdot) = x^{\varepsilon_0}(\cdot) \in V_\beta^n \cap T_z^n$, the set $V_\beta^n|J \cup T_z^n|J$ is connected in $C(J, E)$. It is obvious that $Q^n|J = \bigcup_{z \in Q^n} V_\beta^n|J \cup T_z^n|J$.

Thus the set $Q^n|J$ is also connected in $C(J, E)$. It is not difficult to prove that $Q^n|_J$ is connected in $C(J, \mathscr{B})$. Hence the proof of the lemma is completed.

§ 4. Main result

In this section, we shall prove Kneser type theorems for CP (1.1).

A function $\eta: (\sigma, \sigma + a] \times [0, 2r] \to R$ is said to be a Kamke-type function if the following conditions hold:

(η_1) $\eta = \eta(t, s)$ is a real-valued function, defined on $(\sigma, \sigma + a] \times [0, 2r]$, which is Lebesgue measurable in t for each fixed $s \in [0, 2r]$ and is continuous in s for *a.a.* $t \in [\sigma, \sigma + a]$.

(η_2) There exists a function α, defined on $(\sigma, \sigma + a]$ and locally integrable there, such that $|\eta(t, s)| \leq \alpha(t)$ for *a.a.* $t \in (\sigma, \sigma + a]$ and all $s \in [0, 2r]$.

To state Kneser type theorem, we need the following.

Hypothesis A; For each subset $\mathscr{E}$ of $\mathscr{E}_\sigma[\gamma]$ such that $\alpha(\mathscr{E}_\sigma) = 0$ and $\mathscr{E}|[\sigma, \sigma + \gamma]$ is H-Lipschitzian for some $H > 0$, the inequality $\alpha(\mathscr{E}_t) \leq |\hat{\alpha}_t|_{\mathscr{B}^R}$ holds for $t \in [\sigma, \sigma + \gamma]$, where $\hat{\alpha}(t) = \alpha(\mathscr{E}(t))$ for $t \in [\sigma, \sigma + \gamma]$ and $\hat{\alpha}(t) = 0$ for $t \in (-\infty, \sigma]$.

Here we note that the α-measure of noncompactness on $\mathscr{B}$, which is not a Banach space, is well defined and has usual properties (cf. [12]).

Example 4.1. Let $\gamma \in R$. The space $\mathscr{C}_\gamma$ is the space of continuous functions $\varphi: R^- \to E$ having the limit $\lim_{s \to -\infty} e^{\gamma s}\varphi(s)$ with the norm $|\varphi|_{\mathscr{C}_\gamma} = \sup_{s \in R^-} e^{\gamma s}|\varphi(s)|$. Then, by Proposition 1.7 in [12], we have

$$\begin{aligned}\alpha(\mathscr{E}_t) &= e^{-\gamma t} \sup_{\sigma \leq s \leq t} e^{\gamma s}\alpha(\mathscr{E}(s)) \\ &= |\hat{\alpha}_t|_{\mathscr{C}_\gamma^R}.\end{aligned}$$

Example 4.2. Let

$$\mathscr{L} = \{\varphi: R^- \longrightarrow E | \text{measurable on } (-\infty, -r], \text{ continuous on } [-r, 0] \text{ and } |\varphi|_{\mathscr{L}} < \infty\},$$

where $0 \leq r < \infty$ and

$$|\varphi|_{\mathscr{L}} = \sup_{-r \leq s \leq 0} |\varphi(s)| + \int_{-\infty}^0 e^s|\varphi(s)|\, ds.$$

Then we have, by Lemma 2.2 in [10],

$$\alpha(\mathscr{E}_t) \le \sup_{-r \le s \le 0} \alpha(\mathscr{E}(t+s)) + \int_{\sigma - t}^{0} e^s \alpha(\mathscr{E}(t+s))\, ds$$
$$= |\hat{\alpha}_t|_{\mathscr{S}^R}.$$

We are now in a position to prove the main theorem in this paper.

Theorem 4.3. *Suppose that* $\mathscr{B}^E$ *and* $\mathscr{B}^R$ *satisfy Axioms* (B_1)-(B_3) *and Hypothesis A is satisfied. Assume that* $f\colon [\sigma, \sigma + a] \times \mathscr{B}(\varphi, r) \to E$ *is uniformly continuous and* $|f| \le M - 1$, $M > 1$, *on* $[\sigma, \sigma + a] \times \mathscr{B}(\varphi, r)$ *and that there exists a Kamke-type function* $\omega\colon (\sigma, \sigma + a] \times [0, 2r] \to R^+$ *such that*

1) $\alpha(f(t, B)) \le \omega(t, \alpha(B))$ *for a.a.* $t \in (\sigma, \sigma + a]$ *and*

for all $B \subset \mathscr{B}(\varphi, r)$;

2) $\omega(t, s)$ *is nondecreasing in* s; *and*

3) $z \equiv 0$ *is the unique absolutely continuous function from* $(-\infty, \sigma + a]$ *into* R^+, *which satisfies the initial condition* $(D^+ z)(\sigma) := \lim_{t \to \sigma +} \dfrac{z(t)}{t - \sigma} = 0$ *and* $z(t) = 0$ *for* $t \le 0$ *and the scalar differential inequality*

$$\left|\frac{dz}{dt}\right| \le \omega(t, |z_t|_{\mathscr{B}^R}) \quad \text{for a.a. } t \in (\sigma, \sigma + a]. \tag{4.1}$$

Then the set S *of all solutions, defined on* $(-\infty, \sigma + \gamma]$, *for* CP (1.1) *is nonempty and the set* $S|J$, $J = [\sigma, \sigma + \gamma]$, *is compact and connected in* $C(J, E)$, *where* γ *is as in Lemma* 3.1.

Proof. Let Q^n be the same set as in Lemma 3.1. We note that $x\colon (-\infty, \sigma + \gamma] \to E$ belongs to S if and only if x satisfies the conditions $x|J \in \bigcap_{n=1}^{\infty} \mathrm{Cl}(Q^n|J)$ and $x_\sigma = \varphi$. From Theorem 2.2 and Lemma 3.1 it is sufficient to see that $\alpha(Q^n|J) \to 0$ as $n \to \infty$. Since $Q^n|J$ is an equicontinuous subset of $C(J, E)$, we have $\alpha(Q^n|J) = \sup\{\alpha(Q^n(t)) | t \in J\}$ by Lemma 1.4.1 in [6]. Thus we must prove that $\alpha(Q^n(t)) \to 0$ uniformly on J as $n \to \infty$. Put $z^n(t) = \alpha(Q^n(t))$ for $t \in J$, while $z^n(t) = 0$ for $t \in (-\infty, \sigma]$. Then for $t, s \in J$ and any $n \in N$ we have

$$z^{n+1}(t) \le z^n(t) \quad \text{and} \quad |z^n(t) - z^n(s)| \le M|t - s|,$$

which implies that the sequence $\{z^n(t)\}$ converges to a function $z^0(t)$ uniformly on $(-\infty, \sigma + \gamma]$. Clearly, we have $|z^0(t) - z^0(s)| \le M|t - s|$ for $t, s \in J$.

Next, we shall show that $z^0(t) \equiv 0$ on J. From the definition of the family Q^n, properties (cf. [6. Theorem 1.4.1]) of the α-measure and Theorem 1.3.1 in [6] it follows that for $h > 0$,

$$\frac{1}{h}|\alpha(Q^n(t+h)) - \alpha(Q^n(t))|$$

$$\le \frac{1}{h}\alpha(\{u(t+h) - u(t) | u \in Q^n\})$$

$$\le \alpha(\{D_+u)(s) | s \in [t, t+h), u \in Q^n\})$$

$$\le \alpha(\{(D_+u)(s) - f(s, u_s) + f(s, u_s) | s \in [t, t+h), u \in Q^n\})$$

$$\le \alpha(\{(D_+u)(s) - f(s, u_s) | s \in [t, t+h), u \in Q^n\})$$

$$+ \alpha(\{f(s, u_s) | s \in [t, t+h), u \in Q^n\})$$

$$\le \alpha(\{f(s, u_s) | s \in [t, t+h), u \in Q^n\}) + \frac{2}{n}.$$

Since $Q^n|_J$ is an equicontinuous subset of $C(J, \mathscr{B})$ and f is uniformly continuous, for any $\varepsilon > 0$ we have

$$0 \le \alpha(\{f(s, u_s) | s \in [t, t+h), u \in Q^n\}) - \alpha(f(t, Q_t^n)) < \varepsilon,$$

provided h is sufficiently small and n is sufficiently large. Hence we have

$$\left|\frac{dz^n(t)}{dt}\right| \le \alpha(f(t, Q_t^n)) + \frac{2}{n} + \varepsilon \qquad \text{for } a.a.\ t \in J.$$

By Hypothesis A and the assumptions 1) and 2), we get

$$\left|\frac{dz^n(t)}{dt}\right| \le \omega(t, \alpha(Q_t^n)) + \frac{2}{n} + \varepsilon \qquad \text{for } a.a.\ t \in J$$

$$\le \omega(t, |z_t^n|_{\mathscr{B}^R}) + \frac{2}{n} + \varepsilon \qquad \text{for } a.a.\ t \in J.$$

Letting $n \to \infty$, we have

$$\left|\frac{dz^0(t)}{dt}\right| \le \omega(t, |z_t^0|_{\mathscr{B}^R}) + \varepsilon \qquad \text{for } a.a.\ t \in J,$$

because of Lemma 3.1 in [13]. Since ε is arbitrary, we obtain finally

$$\left|\frac{dz^0(t)}{dt}\right| \le \omega(t, |z_t^0|_{\mathscr{B}^R}) \qquad \text{for } a.a.\ t \in J.$$

Moreover, it is easy to see that

$$z^0(\sigma + h) \le z^n(\sigma + h) \le \frac{2h}{n} + \varepsilon h,$$

which implies that $(D^+z^0)(\sigma)=z^0(\sigma)=0$. Thus we have $z^0(t)\equiv 0$ by the assumption 3). This completes the proof of the theorem.

Corollary 4.4. *The differential inequality* (4.1) *in Theorem* 4.3 *can be replaced by the differential equation*

$$\frac{dz}{dt}=K(0)\omega(t, z(t))+\beta_v^R z(t) \qquad \text{for } a.a.\ t\in(\sigma, \sigma+a] \tag{4.2}$$

provided $\beta_v^R\in R$, *or*

$$\frac{dz}{dt}=\omega(t, K(t-\sigma)z(t)) \qquad \text{for } a.a.\ t\in(\sigma, \sigma+a], \tag{4.3}$$

where $\beta_v^R=\limsup_{t\to 0+}\{\|S_R(t)\|-1\}/t,\ -\infty\le\beta_v^R<\infty$, *and* $\|S_R(t)\|$ *stands for the operator norm of* $S_R(t)$.

For the proof refer to [13, Proposition 4.3 and 11, Theorem 3.1].

Remark 4.5. From Corollary 4.4 we can see that Theorem 4.3 generalizes, in some sense, Theorem 2.4 in [12] and Theorem 3.1 in [10]. For the case of phase space $\mathscr{B}=\mathscr{C}_\gamma$, refer to [14].

§5. Related results

We make the following hypothesis on a mapping $f\colon \Omega\to E,\ \Omega\subset R\times\mathscr{B}$.

Hypothesis B. For every point $(\tau, \psi)\in\Omega$, there exist positive numbers a, r and a Kamke-type function $\omega(t, s)\colon(\tau, \tau+a]\times[0, 2r]\to R^+$ satisfying the following conditions:

(1) $[\tau-a, \tau+a]\times\mathscr{B}(\psi, r)\subset\Omega$.

(2) f is uniformly continuous and bounded on $[\tau-a, \tau+a]\times\mathscr{B}(\psi, r)$.

(3) $\alpha(f(t, B))\le\omega(t, \alpha(B))$ for $a.a.\ t\in(\tau, \tau+a]$ and every subset $B\subset\mathscr{B}(\psi, r)$.

(4) $u(t)\equiv 0$ is the unique absolutely continuous function from $(-\infty, \tau+a]$ into R^+, which satisfies the initial condition $(D^+u)(\tau)=0$ and $u(t)=0$ for $t\le\sigma$ and the scalar differential inequality (4.1)

The following result is a modification of Theorem 3.2 in [12]. Since the proof is essentially based on the argument used in the proof of Theorem 3.2 in [12], it is omitted.

Lemma 5.1. *Suppose that* $\mathscr{B}^E$ *and* $\mathscr{B}^R$ *satisfy Axioms* (B_1)-(B_3) *and*

Hypothesis A is satisfied. Let $f^n: \Omega \to E$, $n \in N$, be a continuous function, where Ω is an open subset of $R \times \mathscr{B}$, and let $x^n: (-\infty, \tau_n) \to E$ be a noncontinuable solution of CP$(f^n, \sigma_n, \varphi^n)$, $\sigma_n \in (-\infty, \tau_n)$. *Assume that*

1) *for every point $(\tau, \psi) \in \Omega$,* CP (f^n, τ, ψ) *has a solution;*
2) *f satisfies Hypothesis B;*
3) *if $\{f^n\}_{n\in N}$ is uniformly bounded on a closed bounded set F of Ω, then*

$$\alpha(\{f^n(t, \psi) - f(t, \psi) | (t, \psi) \in F, \text{n} \geq k\}) \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty;$$

4) *$\{f^n\}_{n\in N}$ converges to the function f uniformly on every compact subset of Ω; and*
5) *$\sigma_n \to \sigma$ and $\varphi^n \to \varphi$ as $n \to \infty$ and $(\sigma, \varphi) \in \Omega$.*

Then there exist a subsequence $\{x^{n(i)}\}$ of $\{x^n\}$ and a noncontinuable solution x, defined on $(-\infty, \tau)$, of CP (f, σ, φ) *such that the following conditions hold:*

(i) $\tau \leq \limsup_{i\to\infty} \tau_{n(i)}$.

(ii) $\sup\{|x^{n(i)}(t) - x(t)| \,|\, s(n(i)) \leq t \leq d\} \to 0$ *as $i \to \infty$ for every $d \in (\sigma, \tau)$, where* $s(n) = \max\{\sigma, \sigma_n\}$.

Set

$$S_E(\varphi) = \{x: I \longrightarrow E \,|\, x \text{ is a solution of CP } (f, \sigma, \varphi)\},$$
$$S_{\mathscr{B}}(\varphi) = \{\zeta: I \longrightarrow \mathscr{B} \,|\, \zeta(t) = x_t \text{ for } t \in I \text{ and } x \in S_E(\varphi)\},$$

where I is a subinterval of $[\sigma, \infty)$, and

$$S_Z(Q) = \bigcup_{\varphi\in Q} S_Z(\varphi),$$

where $Q \subset \mathscr{B}$ and $Z = E$ or $\mathscr{B}$. Clearly, we have $S_Z(Q) \subset C(I, Z)$.

Proposition 5.2. *Let f in* CP (1.1) *be a mapping defined on $I \times \mathscr{B}$, $I := [\sigma, \sigma + a]$. Suppose that all the conditions in Theorem* 4.3 *are satisfied. Then for any compact and connected subset Q in $\mathscr{B}$ the set $S_Z(Q)$ is a compact and connected subset in $C(I, Z)$.*

Proof. We shall prove that the set $S_E(Q)$ is compact in $C(I, E)$. Let $\{u^n\}_{n\in N}$ be any sequence in $S_E(Q)$. Then for each u^n there exists a solution $x^n: (-\infty, \sigma + a] \to E$ of CP (f, σ, φ^n) such that $x^n | I = u^n$ and $x^n_\sigma = \varphi^n \in Q$. By Lemma 5.1 we can choose a subsequence $\{x^{n_i}\}$ of $\{x^n\}$ and a solution x^0 of CP (f, σ, φ^0) such that $x^{n_i}_\sigma \to \varphi^0$ in $\mathscr{B}$ and $x^{n_i}(t) \to x^0(t)$ uniformly on I as $i \to \infty$. Thus the set $S_E(Q)$ is compact in $C(I, E)$. By Theorem 2.1 in [10] we have $\alpha(S_{\mathscr{B}}(Q)) \leq K_a \alpha(S_E(Q)) + M_a \alpha(Q)$ and hence $\alpha(S_{\mathscr{B}}(Q)) = 0$. Clearly, the set $S_{\mathscr{B}}(Q)$ is compact in $C(I, \mathscr{B})$. The proofs of the connectedness for $S_E(Q)$ and $S_{\mathscr{B}}(Q)$ follow from Kaminogo [4]. Hence the proof is complete.

Corollary 5.3. *If all the assumptions in Proposition* 5.2 *are satisfied, then*

$$\sum(Q) = \{(t, x_t) | x \in S_E(Q) \quad \text{and} \quad t \in I\}$$

is compact and connected in $R \times \mathscr{B}$.

Proof. The mapping on $I \times S_{\mathscr{B}}(Q)$ defined by $(t, \zeta) \to (t, \zeta(t))$ is continuous onto $\sum(Q)$. Since $I \times S_{\mathscr{B}}(Q)$ is compact and connected, the set $\sum(Q)$ is also compact and connected in $R \times \mathscr{B}$.

The following result extends Theorem 1 in [7]. Since the proof is similar to that of Theorem 4.3 (or [14, Theorem 3.4]), it is omitted.

Theorem 5.4. *Suppose that* $\mathscr{B}E$ *and* $\mathscr{B}R$ *satisfy Axioms* (B_1)-(B_3) *and Hypothesis A is satisfied. Assume that* $f: [\sigma, \sigma + a] \times \mathscr{B}(\varphi, r) \to E$ *is a uniformly continuous function such that* $|f| \leq M - 1$, $M > 1$, *on* $[\sigma, \sigma + a] \times \mathscr{B}(\varphi, r)$, *and that there exists a Kamke-type function* $\omega(t, s): (\sigma, \sigma + a] \times [0, 2r] \to R^+$ *such that*

1) $$\liminf_{h \to 0-} \frac{1}{h} [\alpha(A(t)) - \alpha(\{x(t) - hf(t, x_t): x \in A\})] \leq \omega(t, \alpha(A_t))$$

for a.a. $t \in (\sigma, \sigma + a]$ *and for any subset* $A \subset \mathscr{E}_\sigma^\varphi[a]$ *such that* $A|[\sigma, \sigma + a]$ *is equicontinuous and* $A_t \subset \mathscr{B}(\varphi, r)$ *for all* $t \in [\sigma, \sigma + a]$;

2) $\omega(t, s)$ *is nondecreasing in s for a.a.* $t \in [\sigma, \sigma + a]$; *and*

3) *the condition* 3) *of Theorem* 4.3 *is satisfied, provided* (4.1) *is replaced by the scalar differential inequality*

$$\frac{dz}{dt} \leq \omega(t, |z_t|_{\mathscr{B}R}) \qquad \text{for a.a. } t \in (\sigma, \sigma + a]. \tag{5.1}$$

Then the conclusion of Theorem 4.3 *remains valid.*

Remark 5.5 [13, *Proposition* 4.3]. The differential inequality (5.1) in Theorem 5.4 can be replaced by the differential equation (4.2) or (4.3) under an additional condition:

4) $$\omega(t, |z_t|_{\mathscr{B}R}) \longrightarrow 0 \text{ as } t \longrightarrow \sigma + 0 \quad \text{and} \quad \int_\sigma^t \omega(s, |z_s|_{\mathscr{B}R})\, ds < \infty,$$

whenever $z: (-\infty, \sigma + a] \to [0, 2r]$ is an absolutely continuous function satisfying the conditions $(D^+ z)(\sigma) = 0$ and $z(t) = 0$ for $t \leq \sigma$.

Remark 5.6. (1) The condition 1) in Theorem 5.4 is weaker than the condition 1) in Theorem 4.3 (cf. [6, p. 48]).

(2) Theorem 5.4 is a generalization of Theorem 4.3 if ω in Theorem 5.4 is the Nagumo-type function $\omega(t, s) = s/(t - \sigma)$. Indeed, the function ω satisfies the condition 4) in Remark 5.5 (cf. [9. Proposition 6.1]), which means that the condition 3) in Theorem 5.4 and the condition 3) in Theorem 4.3 are equivalent to each other.

(3) The condition 3) in Theorem 5.4 is weaker than the condition (iii) in [7, Definition 3] even for ODEs.

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