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Painlevé's Theorem on Automorphic Functions II

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Dedicated to Professor Noboru Tanaka on his 60th birthday

§0. Introduction

Triangular functions of Schwarz satisfy third order algebraic differential equations. Painlevé states these equations in some sense cannot be reduced to a finite number of algebraic differential equations of order at most 2 (confer p. 721 in [6]). We give here the proof of his statement from the standpoint of differential algebra, which was attempted in the previous paper [5].

Let λ, μ and ν be three rational numbers with

$$p=\lambda^{-1},\ q=\mu^{-1},\ r=\nu^{-1}\in \boldsymbol{N} \quad \text{and} \quad \lambda+\mu+\nu<1.$$

Consider the following algebraic differential equation over $\boldsymbol{C}$ with respect to the differentiation $'=d/dx$

$$D(y)=-\frac{1}{2}y'^2Q(y). \tag{1}$$

Here the left hand side denotes the Schwarzian derivative of y with respect to x:

$$D(y)=(y''/y')'-\frac{1}{2}(y''/y')^2$$

and

$$Q(y)=\frac{1-\lambda^2}{y^2}+\frac{1-\mu^2}{(y-1)^2}+\frac{\lambda^2+\mu^2-\nu^2-1}{y(y-1)}.$$

By the use of the result of [5] we shall prove the following theorem.

Theorem. *For any finite chain of differential field extensions of* $\boldsymbol{C}$: $\boldsymbol{C}=R_0\subset R_1\subset\cdots\subset R_m$ *with* $\operatorname{tr.deg} R_i/R_{i-1}\leqq 2$ $(1\leqq i\leqq m)$, R_m *contains no nonconstant solution of* (1).

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§1. Continuity of differentiation

Let K be a field of characteristic 0 and R be an algebraic function field of one variable over K. Suppose R is moreover an ordinary differential field with a single differentiation $'$. Here we need not assume K is a differential subfield of R.

Lemma. *Suppose that K is algebraically closed. If a valuation ring A of R containing K has the property that K' is included in some fractional ideal J of A, then the differentiation $'$ is continuous with respect to the topology induced by A.*

Proof. Let (t) be the maximal ideal of A, $t \in A$. The completion $\tilde{R}$ of R is represented as the field of formal power series $K((t))$. R can be regarded as a subfield of $\tilde{R}$. Let $\imath$ denote the embedding map of R into $\tilde{R}$. In $\tilde{R}$ we introduce the continuous differentiation $*$ as

$$(\sum a_i t^i)^* = \sum \imath(a_i')t^i + \sum i a_i t^{i-1}\imath(t').$$

This is well-defined because $K' \subset J$. In this sense R turns out to be a differential subfield of $\tilde{R}$. In fact clearly $a^* = \imath(a')$ for $a \in K$ and $t^* = \imath(t')$. We must show $\imath(u)^* = \imath(u')$ for any u in R. If u is an element of $K[t]$, the assertion follows readily. Hence the assertion holds true for the subfield $K(t)$ of R. Let u be an arbitrary element of $R \backslash K(t)$. It then satisfies the irreducible equation over $K(t)$

$$u^n + a_1 u^{n-1} + \cdots + a_n = 0, \quad n > 1, \quad a_i \in K(t).$$

Differentiating this equality, we have

$$\{nu^{n-1} + (n-1)a_1 u^{n-2} + \cdots + a_{n-1}\}u' + a_1' u^{n-1} + \cdots + a_n = 0.$$

On the other hand from the equality

$$\imath(u)^n + \imath(a_1)\imath(u)^{n-1} + \cdots + \imath(a_n) = 0$$

we have

$$\{n\imath(u)^{n-1} + (n-1)\imath(a_1)\imath(u)^{n-2} + \cdots + \imath(a_{n-1})\}\imath(u)^* \\ + \imath(a_1)^*\imath(u)^{n-1} + \cdots + \imath(a_n)^* = 0.$$

The fact that $\imath(a_i') = \imath(a_i)^*$ implies

$$\{n\imath(u)^{n-1} + (n-1)\imath(a_1)\imath(u)^{n-2} + \cdots + \imath(a_{n-1})\} \cdot \\ \cdot \{\imath(u') - \imath(u)^*\} = 0$$

Noting the term in the first braces is equal to the image of

$$nu^{n-1}+(n-1)a_1u^{n-2}+\cdots+a_{n-1}\neq 0,$$

we find $\imath(u')=\imath(u)^*$. This completes the proof.

Proposition. *Let k be a differential subfield of R and suppose R is an algebraic function field of n variables over k. If a valuation ring A of R includes some intermediate subfield K between k and R with* tr. deg $R/k=n-1$, *then the differentiation of R is continuous with respect to the topology induced by A.*

Proof. Let v be the valuation of R associated with A. Let L denote the algebraic closure of K. Note that v is trivial on the algebraic closure of K in R. Then v can be extended to the valuation w of LR which is trivial on L. The field extension LR of k turns out to be a differential field extension of k with a unique extension of the differentiation. To say precisely let x_i $(1\leqq i\leqq n-1)$ be a transcendental base of K over k and define a differentiation by

$$u'=D_0u+\sum(D_iu)x_i'$$

for u in L, where D_0 denotes the derivation of L which coincides with $'$ on k and satisfies $D_0x_i=0$ for every i, D_i $(1\leqq i\leqq n-1)$ are derivations of L over k with $D_ix_j=0$ $(i\neq j)$, 1 $(i=j)$. Clearly L' is included in $L+Lx_1'+\cdots+Lx_{n-1}'$, therefore in some fractional ideal of the ring of w. By the above lemma we complete the proof.

§2. Riccati equation

Let K be a differential field of characteristic 0 with a single differentiation $'$ and p and q be two elements of K. Consider the following linear differential equation over K

$$y''+py'+qy=0. \tag{2}$$

Recall that a differential field extension R of K is called a weakly liouvillian extension of K if there is a finite chain of differential field extensions: $K=R_0\subset R_1\subset\cdots\subset R_m=R$ such that for each i, R_i is an algebraic extension of $R_{i-1}(t_i)$ of finite degree and either t_i' or $t_i'/t_i\in R_{i-1}$. If the further condition that the fields of constants of K and R are the same is satisfied R is called liouvillian over K. Elements of a [weakly] liouvillian extension of K are called [weakly] liouvillian over K. (cf. [7].)

In (2) if we let $u=y'/y+p/2$ and $v'/v=y'/y+p/2$, we have

(3) $$u' + u^2 + s/2 = 0,$$

(4) $$v'' + sv/2 = 0,$$

where $s = -p' + 2q - p^2/2 \in K$. The following theorem is due to Liouville (cf. [3] or p. 97 in [4]).

Lemma. *Let C denote the field of constants of K and suppose C is algebraically closed. If the equation* (2) *admits a nonzero solution which is weakly liouvillian over K, then either the equation* (4) *has a fundamental system consisting of algebraic elements over K or the equation* (3) *admits as a solution an algebraic element over K of degree at most* 2.

Proof. Suppose $y_1 \neq 0$ is weakly liouvillian over K, satisfying (2). If we set $y = zy_1$ in (2), then (2) reads

$$y_1 z'' + (2y_1' + py_1)z' = 0.$$

If z is a nonconstant solution then (2) has the fundamental system y_1, zy_1 which are weakly liouvillian over K. By virtue of a theorem of Kolchin [2] there exists a fundamental system for (2) consisting of solutions which are liouvillian over K. Applying the theorem of Kaplansky [1, §19], we get the desired result. (cf. [4, §12–13].)

Now let us consider the hypergeometric differential equation

(5) $$x(1-x)y'' + \{\gamma - (1+\alpha+\beta)x\}y' - \alpha\beta y = 0$$

with complex numbers α, β and γ. This time the equations (3) and (4) read

(6) $$u' + u^2 + s(x)/2 = 0,$$

(7) $$v'' + s(x)v = 0,$$

where

$$s = \frac{1-\lambda^2}{2x^2} + \frac{1-\mu^2}{2(1-x)^2} + \frac{1-\lambda^2-\mu^2+\nu^2}{2x(1-x)},$$

$$\lambda = 1-\gamma, \quad \mu = \gamma-\alpha-\beta, \quad \nu = \alpha-\beta.$$

The equation (5) is reducible (in the sense of linear operator) if and only if (6) has a rational solution. The following facts are known.

(I) The equation (5) is reducible if and only if one of α, β, $\gamma-\alpha$, $\gamma-\beta$ is a rational integer. (cf. p. 7 in [4].)

(II) Under the irreducibility of the equation (5) and the condition that $0<\lambda<1$, $0<\mu<1$, $0<\nu<1$, whenever (5) has a non-trivial algebraic solution, the numbers λ, μ and ν must be rational numbers with $\lambda+\mu+\nu>1$.

(cf. p. 17 in [4].)

(III) Under the irreducibility of the equation (5), whenever (6) has a non-trivial quadratic irrational solution, two of $\lambda - 1/2$, $\mu - 1/2$, $\nu - 1/2$ must be rational integers. (cf. pp. 96–100 in [4], or [7].)

According to these facts and the above lemma, we shall prove the following.

Proposition. *There exists no algebraic solution of*

$$u' + u^2 + Q(x)/4 = 0, \tag{8}$$

where Q denotes the rational function mentioned in the introduction.

Proof. Suppose there exists an algebraic solution u of (8). If u is a rational function, then the equation (5) is reducible. Hence by (I) one of $\alpha, \beta, \gamma - \alpha, \gamma - \beta$ is a rational integer. This is however impossible because

$$\alpha = \frac{1}{2}(1 - \lambda - \mu + \nu), \qquad \beta = \frac{1}{2}(1 - \lambda - \mu - \nu),$$

$$\gamma - \alpha = \frac{1}{2}(1 - \lambda + \mu - \nu), \qquad \gamma - \beta = \frac{1}{2}(1 - \lambda + \mu + \nu),$$

and $\lambda + \mu + \nu < 1$, all λ, μ, ν being positive rational numbers $\leqq 1/2$. The equation (5) is therefore irreducible. If u is a quadratic irrational function, then by (III) two of $\lambda - 1/2$, $\mu - 1/2$ and $\nu - 1/2$ are rational integers. Since each of λ, μ, ν has an inverse in natural numbers, no two of them coincide with $1/2$, we meet a contradiction. From the lemma it follows that the equation (7), therefore the equation (5), has a fundamental system consisting of algebraic solutions. The numbers λ, μ, ν must satisfy the inequality in (II). But this is absurd.

§ 3. Proof of the theorem

Conversely assume that there exists a finite chain of differential field extensions of $\boldsymbol{C}$: $\boldsymbol{C} = R_0 \subset R_1 \subset \cdots \subset R_m$ such that tr. deg $R_i/R_{i-1} \leqq 2$ $(1 \leqq i \leqq m)$ and R_m contains a nonconstant solution of (1). We may assume without loss of generality m is the minimum and each R_i $(1 \leqq i \leqq m-1)$ is algebraically closed. Then tr. deg $R_m/R_{m-1} = 2$. In fact if it is not the case, by Theorem 1 in [5], R_{m-1} contains a nonconstant solution of (1), which contradicts the minimality of m. Let $k = R_{m-1}$ and y be a nonconstant solution of (1) which is contained in R_m. By our assumption y satisfies a second order algebraic differential equation over k, but none of the first order. Hence y' is transcendental over $k(y)$. Let K be the algebraic closure of $k(y)$. Define the differentiation in the field of Puiseux series $K\{\{1/y'\}\}$ in $1/y'$ over K as

$$(\sum a_i y'^{\lambda_i})' = \sum a_i^* y'^{\lambda_i} + \sum a_{iy} y'^{\lambda_i+1} + \sum \lambda_i a_i y'^{\lambda_i-1} y'',$$

where λ_i $(0 \leqq i)$ are descending rational numbers with a common denominator, $a_0 \neq 0$, $*$ denotes a derivation of K which coincides on k with $'$ and satisfies $y^* = 0$, and a_{iy} denotes the derivative of a_i with respect to y. Then KR_m may be regarded as a differential subfield of $K\{\{1/y'\}\}$ according to the proposition in §1. If we let $z = y''/y'$, the equation (1) reads

$$y'' = y'z, \quad z' - \frac{1}{2}z^2 + \frac{1}{2}y'^2 Q(y) = 0. \tag{9}$$

If we express $z = \sum a_i y'^{\lambda_i}$ $(0 \leqq i)$, $a_0 \neq 0$,

$$z' = \sum a_i^* y'^{\lambda_i} + \sum a_{iy} y'^{\lambda_i+1} + \sum \lambda_i a_i y'^{\lambda_i} \sum a_i y'^{\lambda_i}.$$

It is readily seen that $\lambda_0 = 1$ and

$$a_{0y} + \frac{1}{2}a_0^2 + \frac{1}{2}Q(y) = 0.$$

The element $u = a_0/2$ is algebraic over $k(y)$, so that it is algebraic over $k_0(y)$ for some finitely generated field extension k_0 of the rational number field. Hence u is regarded as an algebraic function in y over $\boldsymbol{C}$. This contradicts however the proposition in §2, which completes the proof.

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