Funkcialaj Ekvacioj, **35** (1992) 255–259

A Functional Equation of Pexider Type

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1. Introduction

In [1] Haruki solved the following functional equation of Pexider type:

$$|f(z+w)|^2+|g(z-w)|^2=|h(z+\overline{w})|^2+|k(z-\overline{w})|^2 \tag{1}$$

where f, g, h, k are unknown entire functions and z, w are complex variables. The exponents in (1) play an essential role in Haruki's argument in [1]. The purpose of the present paper is to use a simpler argument and solve the following more general functional equation:

$$|f(z+w)|+|g(z-w)|=|h(z+\overline{w})|+|k(z-\overline{w})| \tag{2}$$

where f, g, h, k are unknown entire functions. We will obtain the following which immediately implies Haruki's solutions of equation (1).

Theorem 1. *The only systems of entire solutions of equation* (2) *are the following*:

$$\text{(i)}\quad \begin{cases} f(z)=(az+b)^2\,, \\ g(z)=(cz+d)^2\,, \\ h(z)=(pz+q)^2\,, \\ k(z)=(rz+s)^2\,, \end{cases}$$

where a, b, c, d, p, q, r, s are arbitrary complex constants satisfying

$$\begin{aligned} &|a|=|c|=|p|=|r|\,, \qquad |b|^2+|d|^2=|q|^2+|s|^2\,, \\ &a\overline{b}+c\overline{d}=p\overline{q}+r\overline{s}\,, \qquad a\overline{b}-c\overline{d}=\overline{p}q-\overline{r}s\,; \end{aligned} \tag{3}$$

$$\text{(ii)}\quad \begin{cases} f(z)=[a\exp(\lambda z)+b\exp(-\lambda z)]^2\,, \\ g(z)=[c\exp(\lambda z)+d\exp(-\lambda z)]^2\,, \\ h(z)=[p\exp(\lambda z)+q\exp(-\lambda z)]^2\,, \\ k(z)=[r\exp(\lambda z)+s\exp(-\lambda z)]^2\,, \end{cases}$$

where λ *is an arbitrary real constant and* a, b, c, d, p, q, r, s *are arbitrary complex constants satisfying*

$$(4)\qquad \begin{array}{c} |a| = |p|, \quad |b| = |q|, \quad |c| = |r|, \quad |d| = |s|, \\ a\bar{b} = r\bar{s}, \quad c\bar{d} = p\bar{q}; \end{array}$$

$$\text{(iii)} \quad \begin{cases} f(z) = [a \exp(i\lambda z) + b \exp(-i\lambda z)]^2, \\ g(z) = [c \exp(i\lambda z) + d \exp(-i\lambda z)]^2, \\ h(z) = [p \exp(i\lambda z) + q \exp(-i\lambda z)]^2, \\ k(z) = [r \exp(i\lambda z) + s \exp(-i\lambda z)]^2, \end{cases}$$

where λ *is an arbitrary real constant and* a, b, c, d, p, q, r, s *are arbitrary complex constants satisfying*

$$(5)\qquad \begin{array}{c} |a| = |r|, \quad |b| = |s|, \quad |c| = |p|, \quad |d| = |q|, \\ a\bar{b} = p\bar{q}, \quad c\bar{d} = r\bar{s}. \end{array}$$

Note that equation (2) immediately yields

$$(6)\qquad |f(z)| + |g(0)| = |h(x)| + |k(iy)| \qquad (z = x + iy).$$

Here, and later, $z = x + iy$ means that $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. It is easily verified from (6) that f is a solution of the following functional equation:

$$(7)\qquad |F(z)| + |F(0)| = |F(x)| + |F(iy)| \qquad (z = x + iy).$$

where F is an unknown entire function. Similarly, (2) implies that g, h, and k are also solutions of equation (7). The simplicity of the present treatment of Theorem 1 is due mainly to this easy observation that all the solutions of equation (2) must satisfy the above common functional equation. Note that equation (7) can be written in the following equivalent form:

$$(8)\qquad |F(z)| = A(x) + B(y) \qquad (z = x + iy)$$

where F is an unknown entire funcion and A, B are some real-valued functions, uniquely determined by F up to an additive constant, on the real line. We will solve equation (8) on a given nonempty (open and connected) domain Ω in the complex plane. So we are now concerned with the following functional equation:

$$(9)\qquad |F(z)| = A(x) + B(y) \qquad (z = x + iy \in \Omega)$$

where F is an unknown analytic function on Ω and A, B are some real valued functions, depending on F and Ω.

Theorem 2. *The only analytic solutions of equation* (9) *are* $F(z) = (az + b)^2$ *and* $F(z) = [a \exp(\lambda z) + b \exp(-\lambda z)]^2$ *where* a, b *are arbitrary complex constants and* λ *is an arbitrary real or purely imaginary constant.*

The following is an immediate consequence of Theorem 2 and the uniqueness of the analytic continuation.

Corollary 3. *The only entire solutions of equation* (7) *are* $F(z) = (az + b)^2$ *and* $F(z) = [a \exp(\lambda z) + b \exp(-\lambda z)]^2$ *where* a, b *are arbitrary complex constants and* λ *is an arbitrary real or purely imaginary constant.*

The special case of Corollary 3 where $F(0) = 0$ has long been known. See Hille [2] and [3].

2. Proof of the theorems

We first prove Theorem 2 and then apply it to derive Theorem 1. To prove Theorem 2, we need a couple of lemmas first.

Lemma 4. *If* (9) *holds for some analytic function* F, *then* A *and* B *are infinitely differentiable on their domains of definition, respectively.*

Proof. To avoid triviality we assume that F is not identically 0 on Ω. Fix an arbitrary point x_0 in the domain of definition of A. There exists y_0 such that $z_0 = x_0 + iy_0 \in \Omega$ and $|F(z_0)| \neq 0$ by the identity theorem (see for example [4]). Note that $A(x) = |F(x + iy_0)| - B(y_0)$ on some open interval containing x_0. Since $|F|$ is infinitely differentiable at z_0 (as a function of two real variables), A is infinitely differentiable at x_0. The same argument applies to B. □

The following is well-known. See for example [5, Theorem 13.11].

Lemma 5. *Suppose that* F *is analytic and zero-free on a simply connected domain* D. *Then there exists on* D *an analytic function* φ *such that* $F = \varphi^2$ *on* D.

We may now give a proof of Theorem 2.

Proof of Theorem 2. Assume that (9) holds for some analytic function F on Ω. To avoid triviality, we may again assume F is not identically 0 on Ω. Then we have an open disc $D \subset \Omega$ such that F is zero-free on D by the identity theorem. Taking into account that D is simply connected, we may apply Lemma 5 to obtain an analytic function φ on D such that $F = \varphi^2$ on D. By (9) we obtain

$$|\varphi(z)|^2 = A(x) + B(y)$$

and therefore

$$\varphi(z)\overline{\varphi(z)} = A(x) + B(y) \tag{10}$$

for all $z = x + iy \in D$. Since φ satisfies the Cauchy-Riemann equations and A, B are continuously differentiable by Lemma 4, an application of the differential operator $\partial^2/\partial x \partial y$ to both sides of (10) yileds

$$i[\varphi''(z)\overline{\varphi(z)} - \varphi(z)\overline{\varphi''(z)}] = 0$$

for all $z \in D$. Note that φ is zero-free on D. Thus, simplifying and rearranging the above, we have

$$\frac{\varphi''(z)}{\varphi(z)} = \overline{\left(\frac{\varphi''(z)}{\varphi(z)}\right)}$$

for all $z \in D$. This shows that the analytic function φ''/φ maps D into the real line. It follows from the open mapping theorem that there exists a real constant t such that

$$\varphi'' = t\varphi$$

on D. Solve this differential equation. It shows that there are complex contants a, b and a real or purely imaginary constant $\lambda = \sqrt{t}$ such that

$$\varphi(z) = \begin{cases} az + b & \text{if } t = 0 \\ a \exp(\lambda z) + b \exp(-\lambda z) & \text{if } t \neq 0 \end{cases}$$

for all $z \in D$. Consequently, we have

$$F(z) = \begin{cases} (az + b)^2 & \text{if } t = 0 \\ [a \exp(\lambda z) + b \exp(-\lambda z)]^2 & \text{if } t \neq 0 \end{cases} \tag{11}$$

for all $z \in D$ and hence for all $z \in \Omega$ by the uniqueness of the analytic continuation. Finally, a direct substitution shows that the functions in (11) satisfy equation (9) for arbitrary complex constants a, b and for an arbitrary real or purely imaginary constant λ. This completes the proof of the theorem. □

Having proved Theorem 2 and hence its consequence Corollary 3, we can now easily derive Theorem 1.

Proof of Theorem 1. Suppose that equation (2) holds for entire functions f, g, h, and k. Each of f, g, h, and k is then a solution of equation (7), as remarked earlier. Thus, by Corollary 3, each of f, g, h, and k takes one of the following three forms on the complex plane:

$$(\text{i})' \quad z \mapsto (\alpha z + \beta)^2\,;$$

$$(\text{ii})' \quad z \mapsto [\alpha \exp(\lambda z) + \beta \exp(-\lambda z)]^2\,;$$

$$(\text{iii})' \quad z \mapsto [\alpha \exp(i\lambda z) + \beta \exp(-i\lambda z)]^2\,;$$

where α, β are arbitrary complex constants and λ is an arbitrary real constant. By (6), there are some constants c_1 and c_2 such that $|f(x)| = |h(x)| + c_1$ and $|f(ix)| = |k(ix)| + c_2$ for all real x. Similarly, $|g(x)| = |k(x)| + c_3$ for all real x and for some constant c_3. It follows that all of f, g, h, and k must be of the same form. The coefficient conditions (3), (4), and (5) are verified by a routine calculation. □

Acknowledgement. The author would like to thank the referee for his/her suggestion that led to the above proof of Theorem 2. It simplified the original proof of Theorem 2.

The author was in part supported by Korea Science and Engineering Foundation.

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(Ricevita la 23-an de aprilo, 1990)
(Reviziita la 1-an de aŭgusto, 1990)