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Viscosity Solutions of a System of Nonlinear Second-Order Elliptic PDEs Arising in Switching Games

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§ 1. Introduction

In this paper we are concerned with a system of nonlinear second-order elliptic partial differential equations and will obtain comparison and existence results for solutions of the Dirichlet problem for the system.

Let Ω be a bounded open subset of $\boldsymbol{R}^n$. Let m be an integer greater than 1, and we denote by A the set $\{1, \dots, m\}$. Let g^{kj} and h^{kj}, with $k, j \in A$, be given families of real-valued functions on Ω. For $k \in A$ and $\boldsymbol{R}^m$-valued function $u = (u^1, \dots, u^m)$ we denote, respectively, by $M^k u$ and $N^k u$ the functions on Ω defined by

$$M^k u(x) = \min \{u^j(x) + g^{kj}(x) | j \in A, j \neq k\},$$

and

$$N^k u(x) = \max \{u^j(x) + h^{kj}(x) | j \in A, j \neq k\}.$$

We consider the system of PDEs for $\boldsymbol{R}^m$-valued function $u = (u^1, \dots, u^m)$:

$$\min \{\max \{L^k u^k - f^k, u^k - M^k u\}, u^k - N^k u\} = 0 \quad \text{in } \Omega, \tag{1.1}$$

for $k \in A$, together with the Dirichlet conditions

$$u^k = 0 \quad \text{on } \partial\Omega, \quad \text{for } k \in A. \tag{1.2}$$

Here the f^k are given functions on Ω and the L^k are linear second-order elliptic operators of the form

$$L^k v(x) = -\sum_{i,j=1}^{n} a_{ij}^k(x) \frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^k(x) \frac{\partial v(x)}{\partial x_i} + c^k(x) v(x) \tag{1.3}$$

for $v \in C^2(\Omega)$, where the a_{ij}^k, b_i^k and c^k are real-valued functions on Ω, the matrices $(a_{ij}^k(x))_{1 \le i,j \le n}$ are nonnegative definite and the condition that $\inf \{c^k(x) | x \in \Omega, k \in A\} > 0$ is assumed.

System (1.1) arises as the Bellman-Isaacs equation for a switching game with the state governed by stochastic differential equations and with g^{kj} and h^{kj}, k, $j \in A$, as its switching costs. It should be remarked here that (1.1) includes, as a special case, a system equivalent to the Bellman equation of optimal switching for stochastic differential equations:

$$\max\{L^k u^k - f^k, u^k - M^k u\} = 0 \quad \text{in } \Omega, \quad \text{for } k \in A. \tag{1.4}$$

Indeed, it is easily checked that if u is an $\boldsymbol{R}^m$-valued function on Ω and if functions h^{kj}, with k, $j \in A$, are chosen so that $h^{kj}(x) < u^k(x) - u^j(x)$ for all $x \in \Omega$ and $k \in A$, then u is a subsolution (resp., super- and solution) of (1.1) if and only if it is a subsolution (resp., super- and solution) of (1.4). We refer to N. Yamada [18], S. M. Lenhart and S. A. Belbas [13] and I. Capuzzo Dolcetta and L. C. Evans [2] for the derivation of (1.1) and (1.4), and for the differential game and the problem of optimal switching corresponding to (1.1) and (1.4).

The assumptions which we will put on the L^k are rather general and allow the L^k to be degenerate elliptic. This, however, requires us to introduce weak solutions of (1.1), and the notion of weak solution adapted here is that of viscosity solution which has been introduced by M. G. Crandall and P.-L. Lions [4] and P.-L. Lions [15]. The notion of viscosity solutions was defined for single equations in [4] and [15], and afterwards, it was extended in [2] to that for systems of PDEs. The definition of viscosity solutions of (1.1) will be recalled in Section 2.

There are several existence and uniqueness results for a solution of the Dirichlet problem for (1.1) and (1.4). In [13] the existence and uniqueness of a solution, in the a.e. sense, of (1.4) and (1.2) in the space $(W^{2,\infty}(\Omega))^m$ has been established. In [18] the existence and uniqueness of a solution, in the a.e. sense, of (1.1) and (1.2) in $\bigcap_{p>1}(W^{2,p}(\Omega))^m$ has been shown. In these results it is assumed that the L^k are uniformly elliptic and the coefficients of the L^k are smooth. In [2] they have established the existence and uniqueness of a bounded, continuous, viscosity solution of (1.4) in the case when $\Omega = \boldsymbol{R}^n$ and the L^k are first-order differential operators. N. Yamada [19] and H. Ishii–N. Yamada [11] have obtained some results on the existence and uniqueness of viscosity solutions of problem (1.1) and (1.2) with uniformly elliptic L^k under some regularity assumptions on the coefficients of L^k weaker than those in [13] and [18]. In all the results above it is assumed that g^{kj} and h^{kj} are constant functions.

Our primary purpose here is to remove from the existence and uniqueness theory the restriction that the g^{kj} and h^{kj} be constant functions. The secondary purpose is to take advantage of recent developments in the uniqueness theory for viscosity solutions of second-order elliptic PDEs in order to formulate our results so that they apply to a wide variety of the operators L^k. We refer the interested readers to R. Jensen [12], H. Ishii [9], H. Ishii–P.-L. Lions [10] and M. G. Crandall [3] for the developments.

Our plan of this paper is as follows. In Section 2 we recall the definition of viscosity solutions to general systems of PDEs. A comparison theorem for solutions of (1.1) and (1.2) is established in Section 3. This is actually done for a system of PDEs which is more general than (1.1). In Section 4 an existence result is formulated, where the existence of sub- and supersolutions of (1.1) and (1.2) is assumed, and it is proved by Perron's method.

Finally, we refer to A. Bensoussan–J.-L. Lions [1], L. C. Evans–A. Friedman [7] and S. M. Lenhart–N. Yamada [14] for some problems and results related to ours.

§ 2. Definition of viscosity solutions

For any function $u: \bar{\Omega} \to \boldsymbol{R}$ we define the function $u^*: \bar{\Omega} - \bar{\boldsymbol{R}} \equiv \boldsymbol{R} \cup \{-\infty, \infty\}$ by

$$u^*(x) = \limsup_{r \downarrow 0} \{u(y) | y \in B(x, r) \cap \bar{\Omega}\},$$

and the function $u_*: \bar{\Omega} \to \bar{\boldsymbol{R}}$ by

$$u_*(x) = \liminf_{r \downarrow 0} \{u(y) | y \in B(x, r) \cap \bar{\Omega}\},$$

where $B(x, r)$ denotes the ball $\{y \in \boldsymbol{R}^n | |x - y| < r\}$.

It is immediate from the definitions that $u^* \geq u \geq u_*$ in $\bar{\Omega}$, u^* is upper semi-continuous (u.s.c. in short) on $\bar{\Omega}$ and u_* is lower semi-continuous (l.s.c. in short) on $\bar{\Omega}$. It is easily seen that if u is u.s.c. (resp., l.s.c.) at $x \in \Omega$, then $u(x) = u^*(x)$ (resp., $u_*(x)$). For $\boldsymbol{R}^m$-valued function $u = (u^1, \dots, u^m)$ u^* and u_* denote the $\boldsymbol{R}^m$-valued functions $((u^1)^*, \dots, (u^m)^*)$ and $((u^1)_*, \dots, (u^m)_*)$, respectively.

Let u be a real-valued function on Ω and x a point of Ω. Let S^n denote the set of real symmetric matrices of order n. We denote by $D^{2,+}u(x)$ the set of those $(p, X) \in \boldsymbol{R}^n \times S^n$ for which

$$u(y) \leq u(x) + \langle p, y - x \rangle + \tfrac{1}{2}\langle X(y - x), y - x \rangle + o(|x - y|^2)$$

as $y \to x$. Likewise, we denote by $D^{2,-}u(x)$ the set of those $(p, X) \in \boldsymbol{R}^n \times S^n$ for which

$$u(y) \geq u(x) + \langle p, y - x \rangle + \tfrac{1}{2}\langle X(y - x), y - x \rangle + o(|x - y|^2)$$

as $y \to x$. It is now easily seen that if $\varphi \in C^2(\Omega)$ and $x \in \Omega$ is a maximum (resp., minimum) point of $u - \varphi$, then $(D\varphi(x), D^2\varphi(x)) \in D^{2,+}u(x)$ (resp., $(D\varphi(x), D^2\varphi(x)) \in D^{2,-}u(x)$).

Now we consider the system of PDEs for $\boldsymbol{R}^m$-valued function $u = (u^1, \dots, u^m)$:

(2.1) $$G^k(x, u, Du^k, D^2u^k) = 0 \quad \text{in } \Omega, \qquad \text{for } k \in A,$$

where the G^k are real-valued continuous functions on $\Omega \times \boldsymbol{R}^m \times \boldsymbol{R}^n \times S^n$.

Definition. Let $u = (u^1, \dots, u^m)$ be an $\boldsymbol{R}^m$-valued function on $\bar{\Omega}$. We call u a viscosity subsolution of (2.1) if $(u^k)^* < \infty$ in Ω for all $k \in A$, and

(2.2) $$G^k(x, u^*(x), p, X) \leq 0$$

for all $x \in \Omega$, $k \in A$ and $(p, X) \in D^{2,+}(u^k)^*(x)$. Similarly, we call u a viscosity supersolution of (2.1) if $(u^k)_* > -\infty$ in Ω for all $k \in A$, and

(2.3) $$G^k(x, u_*(x), p, X) \geq 0$$

for all $x \in \Omega$, $k \in A$ and $(p, X) \in D^{2,-}(u^k)_*(x)$. Finally, we call u a viscosity solution of (2.1) if it is both a viscosity sub- and supersolution of (2.1).

We note that the continuity condition on G^k of viscosity subsolution (resp., supersolution) can be replaced by the lower (resp., upper) semi-continuity in $\Omega \times \boldsymbol{R}^m \times \boldsymbol{R}^n \times S^n$.

When u is a real-valued function on $\bar{\Omega}$ and $x \in \Omega$, we denote by $\bar{D}^{2,+}u(x)$ (resp., $\bar{D}^{2,-}u(x)$) as the set of those points $(r, p, X) \in \boldsymbol{R} \times \boldsymbol{R}^n \times S^n$ for which there is a sequence of points $(x_j, p_j, X_j) \in \Omega \times \boldsymbol{R}^n \times S^n$ such that $(p_j, X_j) \in D^{2,+}u(x_j)$ (resp., $(p_j, X_j) \in D^{2,-}u(x_j)$) for all j and such that $x_j \to x$, $u(x_j) \to r$, $p_j \to p$ and $X_j \to X$ as $j \to \infty$. In other words, $\bar{D}^{2,+}u(x)$, for instance, is the graph closure of the multi-valued mapping $x \to \{(u(x), p, X) | (p, X) \in D^{2,+}u(x)\} \subset \boldsymbol{R} \times \boldsymbol{R}^n \times S^n$.

§ 3. A comparison result

In this and the next sections we consider the system of PDEs for $\boldsymbol{R}^m$-valued function $u = (u^1, \dots, u^m)$:

(3.1) $$\min \{\max \{F^k(x, u^k, Du^k, D^2u^k), u^k - M^k u\}, u^k - N^k u\} = 0 \quad \text{in } \Omega,$$

for $k \in A$.

We will use the following assumptions:

(A.0) The F^k are real-valued continuous functions on $\Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \times S^n$.

(A.1) There is an $m \in C(\boldsymbol{R}^+)$ with $m(0) = 0$ such that

$$F^k(y, r, p, -Y) - F^k(x, r, p, X) \leq m(|x - y|(|p| + 1) + \alpha|x - y|^2)$$

$$\text{if} \quad -\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

for all $k \in A$, $\alpha > 1$, x, $y \in \Omega$, $r \in \boldsymbol{R}$ and X, $Y \in S^n$.

(A.2) There is a positive constant θ for which

$$F^k(x, r, p, X) - F^k(x, s, p, X) \geq \theta(r - s) \qquad \text{if } r \geq s\,,$$

for all $k \in A$, $x \in \Omega$, r, $s \in \boldsymbol{R}$, $p \in \boldsymbol{R}^n$ and $X \in S^n$.

(A.3) There exists $\omega \in C(\boldsymbol{R}^+)$ with $\omega(0) = 0$ for which

$$|F^k(x, r, p, X) - F^k(x, r, q, X)| \leq \omega(|p - q|)$$

for all $k \in A$, $x \in \Omega$, $r \in \boldsymbol{R}$, p, $q \in \boldsymbol{R}^n$ and $X \in S^n$.

(A.4) The g^{kj} and h^{kj} are continuous on Ω.

(A.5) For any sequence $\{k(i) | i = 1, \dots, p\}$, with $2 \leq p \leq m$, of distinct integers in A, if $\eta^i = g^{k(i)k(i+1)}$ or $\eta^i = h^{k(i)k(i+1)}$ for $i = 1, \dots, p$, where $k(p + 1) = k(1)$, then $\sum_{i=1}^{p} \eta^i \neq 0$ in Ω.

This condition (A.5) is called "no loop condition" which was introduced in [18]. As is shown in [9] and [10], the linear operators L^k defined by (1.3) satisfy (A.1) if the a_{ij}^k belong to $C^{1,1}(\bar{\Omega})$, the b_i^k are Lipschitz continuous on Ω and the c^k are continuous on $\bar{\Omega}$. See Theorem 5.2.3 in [17] for the regularity of the square roots of S^n-valued functions. It is clear that the L^k satisfy assumptions (A.0), (A.2) and (A.3) under the above regularity hypothesis on the a_{ij}^k, b_i^k and c^k. Also, the L^k satisfy (A.0)–(A.3) if the L^k are uniformly elliptic on Ω, the a_{ij}^k and b_i^k are Lipschitz continuous on Ω and the c^k are continuous on $\bar{\Omega}$. For this and other classes of F^k satisfying (A.1) see [10], [9] and [12].

In what follows, when $u = (u^1, \dots, u^m)$ and $v = (v^1, \dots, v^m)$ are $\boldsymbol{R}^m$-valued functions on Ω and Γ is a subset of Ω, the notation $u \leq v$ on Γ means that $u^k \leq v^k$ on Γ for all $k \in A$.

Now we are in a position to state our main result.

Theorem 3.1. *Assume* (A.0)–(A.5). *Let u and v be, respectively, viscosity sub- and supersolution of* (3.1). *Assume that $u^* \leq v_*$ on $\partial\Omega$. Then $u^* \leq v_*$ in Ω.*

We need the following two lemmas to prove the above theorem.

Lemma 3.2. *Let u be u.s.c. and bounded above, and v be l.s.c. and bounded below in Ω. Let $(\hat{x}, \hat{y}) \in \Omega \times \Omega$ be a local maximum point of the function $(x, y) \to u(x) - v(y) - \frac{\lambda}{2}|x - y|^2$. Then, there exist X, $Y \in S^n$ such that*

$$-4\lambda\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 2\lambda\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{3.2}$$

and

(3.3) $\quad (u(\hat{x}), \lambda(\hat{x} - \hat{y}), X) \in \bar{D}^{2,+}u(\hat{x}), \qquad (v(\hat{y}), \lambda(\hat{x} - \hat{y}), -Y) \in \bar{D}^{2,-}v(\hat{y}).$

We refer to [3] for a proof of this lemma.

Lemma 3.3. *Let K be a compact subset of $\boldsymbol{R}^n$. Assume that u is u.s.c. on K. Then, for almost all $p \in \boldsymbol{R}^n$, the function $u(x) + \langle p, x\rangle$ attains its strict maximum on K.*

This is proved in [5] in a general setting. For completeness we give a proof.

Proof of Lemma 3.3. Denote $H: \boldsymbol{R}^n \to \boldsymbol{R}$ by

$$H(p) = \max\{u(x) + \langle p, x\rangle \mid x \in K\}.$$

Since H is convex, $H(p)$ is differentiable at a.a. $p \in \boldsymbol{R}^n$. Fix p a point at which $H(p)$ is differentiable. Let x be a point at which $u(x) + \langle p, x\rangle$ attains its maximum. We note that

$$u(x) + \langle p + h, x\rangle \le H(p + h) \le u(x) + \langle p, x\rangle + \langle DH(p), h\rangle + o(|h|)$$

as $|h| \to 0$, where $DH(p) = \operatorname{grad} H(p)$. Thus, we see

$$\langle x - DH(p), h\rangle \le o(|h|).$$

Taking $-h$ into the above inequality instead of h, we obtain

$$|\langle x - DH(p), h\rangle| \le o(|h|).$$

Therefore, $x = DH(p)$. $\square$

Proof of Theorem 3.1. For simplicity of notation we write u^j and v^j for $(u^j)^*$ and $(v^j)_*$, respectively. We note that the functions u^j and $-v^j$ are bounded above.

We suppose that $\max\{u^j(x) - v^j(x) \mid x \in \bar{\Omega}, j \in A\} > 0$, and will obtain a contradiction. Let Q denote the set of those points $p \in \boldsymbol{R}^n$ for which the function $x \to \max_{j \in A}(u^j(x) - v^j(x)) + \langle p, x\rangle$ on $\bar{\Omega}$ has a strict maximum. Note that Q is dense in $\boldsymbol{R}^n$ according to Lemma 3.3. Fix $q \in Q$, and define $w = (w^1, \dots, w^m): \bar{\Omega} \to \boldsymbol{R}^m$ by $w^j(x) = v^j(x) - \langle q, x\rangle$. Let $z \in \bar{\Omega}$ be the unique maximum point of the function $x \to \max_{j \in A}(u^j(x) - w^j(x))$. We will take the limit as $|q| \to 0$, and therefore, in view of the assumption that $u \le v$ on $\partial\Omega$, we may assume that $z \in \Omega$. Set

$$A_0 = \{j \in A \mid u^j(z) - w^j(z) = \max_{k \in A}(u^k(z) - w^k(z))\}.$$

For $\varepsilon > 0$ and $k \in A_0$ define the function Φ^k on $\bar{\Omega} \times \bar{\Omega}$ by

$$\Phi^k(x, y) = u^k(x) - w^k(y) - \frac{|x - y|^2}{\varepsilon},$$

and let $(x_\varepsilon^k, y_\varepsilon^k)$ be a maximum point of Φ^k.

Now we fix $k \in A_0$ and examine the limiting behavior of x_ε^k, y_ε^k, $u^k(x_\varepsilon^k)$ and $w^k(y_\varepsilon^k)$ as $\varepsilon \downarrow 0$. First of all, observe that the inequality $\Phi^k(z,z) \le \Phi^k(x_\varepsilon^k, y_\varepsilon^k)$ implies that

$$(3.4) \qquad u^k(z) - w^k(z) \le u^k(z) - w^k(z) + \frac{|x_\varepsilon^k - y_\varepsilon^k|^2}{\varepsilon} \le u^k(x_\varepsilon^k) - w^k(y_\varepsilon^k).$$

Since the functions u^k and $-w^k$ are bounded above, we see from (3.4) that $x_\varepsilon^k - y_\varepsilon^k \to 0$ as $\varepsilon \downarrow 0$. Using (3.4) once again and the semi-continuity of u^k and w^k, we see that x_ε^k and y_ε^k converge to the unique maximum point z of the function $u^k - w^k$ along any convergent subsequence as $\varepsilon \downarrow 0$, and hence that

$$(3.5) \qquad x_\varepsilon^k,\ y_\varepsilon^k \to z \qquad \text{as } \varepsilon \downarrow 0.$$

Moreover we have

$$u^k(z) - w^k(z) \le \varliminf_{\varepsilon\downarrow 0} u^k(x_\varepsilon^k) - \varlimsup_{\varepsilon\downarrow 0} w(y_\varepsilon^k) \le \varlimsup_{\varepsilon\downarrow 0} u^k(x_\varepsilon^k) - \varliminf_{\varepsilon\downarrow 0} w^k(y_\varepsilon^k) \le u^k(z) - w^k(z).$$

This series of inequalities implies that

$$(3.6) \qquad \lim_{\varepsilon\downarrow 0} u^k(x_\varepsilon^k) = u^k(z) \qquad \text{and} \qquad \lim_{\varepsilon\downarrow 0} w^k(y_\varepsilon^k) = w^k(z).$$

Inequality (3.4) now shows that

$$\frac{|x_\varepsilon^k - y_\varepsilon^k|^2}{\varepsilon} \to 0 \qquad \text{as} \qquad \varepsilon \downarrow 0.$$

To proceed, we divide our considerations into two cases. We first consider the case when either $u^k(z) \le N^k u(z)$ or $w^k(z) \ge M^k w(z)$ for all $k \in A_0$. By the definition of the N^k, for each $k \in A_0$ there is a $j \in A \backslash \{k\}$ such that $N^k u(z) = u^j(z) + h^{kj}(z)$. Fix $k \in A_0$ and such a $j \in A$. We claim that if $u^k(z) \le N^k u(z)$, then $j \in A_0$ and $u^k(z) - u^j(z) = h^{kj}(z)$. To see this, we assume that $u^k(z) \le N^k u(z)$. Since v is a viscosity supersolution of (3.1), we have

$$v^k(y_\varepsilon^k) \ge N^k v(y_\varepsilon^k) \qquad \text{if } \varepsilon > 0 \text{ is small enough}.$$

Hence, using (3.5), (3.6), the definition of N^k and the lower semi-continuity of w^j, we have

$$w^k(z) \ge w^j(z) + h^{kj}(z).$$

Therefore, $u^k(z) - u^j(z) \le h^{kj}(z) \le w^k(z) - w^j(z)$. Thus, recalling the definition of A_0, we conclude that $j \in A_0$ and that

$$u^k(z) - u^j(z) = h^{kj}(z).$$

Similarly, we see that if $w^k(z) \ge M^k w(z) = w^j(z) + g^{kj}(z)$ for some $k \in A_0$ and

$j \in A$, then $j \in A_0$ and $u^k(z) - u^j(z) = g^{kj}(z)$. From these observations, we can choose a sequence $\{k(i) | i = 1, \dots, p\}$ of distinct integers in A_0 such that either $u^{k(i)}(z) - u^{k(i+1)}(z) = g^{k(i)k(i+1)}$ or $= h^{k(i)k(i+1)}$ for $1 \leq i \leq p$, where $k(p+1) = k(1)$. Summing up these equalities together with assumption (A.5) yields a contradiction.

We next consider the remaining case, i.e., the case when $u^k(z) > N^k u(z)$ and $w^k(z) < M^k w(z)$ for some $k \in A_0$. Fix such a $k \in A_0$. Using (3.5), (3.6) and the semi-continuity of $N^k u$ and $M^k w$, we see that if $\varepsilon > 0$ is sufficiently small, then $u^k(x_\varepsilon^k) > N^k u(x_\varepsilon^k)$ and $w^k(y_\varepsilon^k) < M^k w(y_\varepsilon^k)$. Choose $\varepsilon > 0$ so small that $u^k(x_\varepsilon^k) > N^k u(x_\varepsilon^k)$ and $w^k(y_\varepsilon^k) < M^k w(y_\varepsilon^k)$. Hereafter we simply write x and y for x_ε^k and y_ε^k, respectively. By Lemma 3.2 there are $X, Y \in S^n$ for which

$$-\frac{8}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \frac{4}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

$$\left(u^k(x), \frac{2}{\varepsilon}(x-y), X\right) \in \bar{D}^{2,+} u^k(x),$$

and

$$\left(w^k(y), \frac{2}{\varepsilon}(x-y), -Y\right) \in \bar{D}^{2,-} w^k(y).$$

If we recall the definition of w^k, we find that $v^k(y) < M^k v(y)$ and that $\left(v^k(y), \frac{2}{\varepsilon}(x-y) + q, -Y\right) \in \bar{D}^{2,-} v^k(y)$. Since u and v are viscosity sub- and supersolution of (3.1), respectively, it is thus easily seen that

$$F^k\left(x, u^k(x), \frac{2}{\varepsilon}(x-y), X\right) \leq 0$$

and

$$F^k\left(y, v^k(y), \frac{2}{\varepsilon}(x-y) + q, -Y\right) \geq 0.$$

Therefore, if $u^k(x) \geq v^k(y)$, then, using (A.1), (A.2) and (A.3), we obtain

$$\begin{aligned} 0 \geq\ & \theta(u^k(x) - v^k(y)) - \omega(|q|) \\ & - m\left(|x-y|\left(\frac{2}{\varepsilon}|x-y| + 1\right) + \frac{8}{\varepsilon}|x-y|^2\right), \end{aligned}$$

where θ is a positive constant from (A.2). Passing to the limit as $\varepsilon \downarrow 0$ and then $|q| \downarrow 0$, we conclude that $\theta(u^k(z) - v^k(z)) \leq 0$, which is a contradiction. The proof is now complete. $\square$

§4. Existence of continuous solutions

In this section we give some results concerned with the existence of a continuous viscosity solution of (3.1).

Theorem 4.1. *Assume that* (A.0)–(A.5) *hold. Let* $\varphi = (\varphi^1, \dots, \varphi^m)$ *and* $\psi = (\psi^1, \dots, \psi^m)$ *be, respectively, viscosity sub- and supersolutions of* (3.1) *such that* $\varphi \le \psi$ *on* $\bar{\Omega}$. *Assume that the* φ^k *are l.s.c. and the* ψ^k *are u.s.c. on* $\bar{\Omega}$ *and that* $\varphi = \psi$ *on* $\partial\Omega$. *Then there is a unique viscosity solution* u *of* (3.1) *satisfying*

$$\varphi \le u \le \psi \qquad \text{on } \bar{\Omega}. \tag{4.1}$$

Moreover, $u \in (C(\bar{\Omega}))^m$.

A sufficient condition for the existence of a viscosity subsolution φ and a viscosity supersolution ψ of (3.1) with the properties described above is given in Proposition 4.3 below.

Our proof of this theorem is based on Perron's method (see [8], [14]) and Theorem 3.1. A consequence of Perron's method to (2.1) is formulated in the following.

Proposition 4.2. *For* $k \in A$ *let* G^k *be a real-valued, continuous function on* $\Omega \times \boldsymbol{R}^m \times \boldsymbol{R}^n \times S^n$. *Let* $\varphi = (\varphi^1, \dots, \varphi^m)$ and $\psi = (\psi^1, \dots, \psi^m)$ *be viscosity sub- and supersolutions of* (2.1), *respectively. Assume that* $\varphi \le \psi$ *on* $\bar{\Omega}$ *and that*

$$G^k(x, r, p, X) \le G^k(x, s, p, X) \qquad \text{if } r \ge s \text{ and } r^k = s^k,$$

for all $k \in A$, $r = (r^1, \dots, r^m)$, $s = (s^1, \dots, s^m) \in \boldsymbol{R}^m$ *and* $(x, p, x) \in \Omega \times \boldsymbol{R}^n \times S^n$. *Then there is a viscosity solution* u *of* (2.1) *satisfying* $\varphi \le u \le \psi$ *in* $\bar{\Omega}$.

We postpone proving this proposition, and we first apply it to conclude Theorem 4.1.

Proof of Theorem 4.1. Let φ and ψ be as in Theorem 4.1. It is clear that $\varphi^* \le \psi = \varphi \le \psi_*$ on $\partial\Omega$. Hence Theorem 3.1 yields that $\varphi^* \le \psi_*$ on $\bar{\Omega}$. For $k \in A$ define the function G^k on $\Omega \times \boldsymbol{R}^m \times \boldsymbol{R}^n \times S^n$ by

$$G^k(x, r, p, X) = \min\{\max\{F^k(x, r^k, p, X), r^k - M^k(x, r)\}, r^k - N^k(x, r)\},$$

where F^k is from (3.1),

$$M^k(x, r) = \min\{r^j + g^{kj}(x) \mid j \in A, j \ne k\}$$

and

$$N^k(x, r) = \max\{r^j + h^{kj}(x) \mid j \in A, j \ne k\}.$$

Clearly, the functions G^k are continuous and have the monotonicity property required in Proposition 4.2. Thus, applying the proposition, we find that there is a viscosity solution u of (3.1) satisfying (4.1).

Now we prove the uniqueness and continuity of a viscosity solution of (3.1) satisfying (4.1.). Let u and v be two viscosity solutions of (3.1) satisfying (4.1). Inequality (4.1) guarantees that $u^* = v_*$ on $\partial\Omega$. Therefore by Theorem 3.1 we have $u^* \leq v_*$ on $\bar{\Omega}$. By the same reasoning, we have $v^* \leq u_*$ on $\bar{\Omega}$. On the other hand, we have $u^* \geq u \geq u_*$ and $v^* \geq v \geq v_*$ on $\bar{\Omega}$. Hence we see that $u = v = u^* = u_*$ on $\bar{\Omega}$. Thus we conclude the uniqueness and that $u \in (C(\bar{\Omega}))^m$. □

Proof of Proposition 4.2. We will use the notation: For any $\boldsymbol{R}^m$-valued function u on Ω and $k \in A$, G_u^k denotes the real-valued function on $\Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \times S^n$ defined by $G_u^k(x, r, p, X) = G^k(x, u_r^k(x), p, X)$, where $u_r^k(x)$ denotes the vector of $\boldsymbol{R}^m$ having the same components as $u(x)$ except the k-th component and r as its k-th component. Note that if u and v are $\boldsymbol{R}^m$-valued functions on Ω and $u \leq v$, then $G_u^k(x, r, p, X) \geq G_v^k(x, r, p, X)$ for all $k \in A$ and $(x, r, p, X) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \times S^n$.

Now let φ and ψ be as in Proposition 4.2. Let $\mathscr{S}$ be the set of viscosity subsolutions v of (2.1) satisfying $v \leq \psi$ on $\bar{\Omega}$. Define $u = (u^1, \ldots, u^m)$ by setting

$$u^k(x) = \sup\{v^k(x) | v = (v^1, \ldots, v^m) \in \mathscr{S}\} \qquad \text{for } k \in A \text{ and } x \in \bar{\Omega}\,.$$

It is clear that $\varphi \leq u \leq \psi$ on $\bar{\Omega}$. We will show that u is a viscosity solution of (3.1).

In order to see that u is a viscosity subsolution of (2.1), we observe that for any $v \in \mathscr{S}$ and $k \in A$, v^k is a viscosity subsolution of

$$G_{v^*}^k(x, v^k, Dv^k, D^2 v^k) = 0 \qquad \text{in } \Omega\,,$$

and hence a viscosity subsolution of

$$G_{u^*}^k(x, v^k, Dv^k, D^2 v^k) = 0 \qquad \text{in } \Omega\,. \tag{4.2}$$

Observe also that the functions $G_{u^*}^k$ are l.s.c. on $\Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \times S^n$. Therefore by [9, Prop. 2.2] we see that for each $k \in A$, u^k is a viscosity subsolution of (4.2). Thus, u is a viscosity subsolution of (2.1).

To show that u is a viscosity supersolution of (2.1), we suppose to the contrary that u is not a viscosity supersolution of (2.1), and will get a contradiction. It follows that there is a $k \in A$ such that u^k is not a viscosity supersolution of

$$G_{u_*}^k(x, u^k, Du^k, D^2 u^k) = 0 \qquad \text{in } \Omega\,. \tag{4.3}$$

Fix such a $k \in A$. As in the proof of [8, Theor. 3.1], we find that there is a viscosity subsolution v^k of (4.2) such that $u^k \leq v^k \leq \psi^k$ on $\bar{\Omega}$ and $u^k \neq v^k$.

Setting $v^j = u^j$ for $j \in A \backslash \{k\}$, we define the $\boldsymbol{R}^m$-valued function $v = (v^1, \ldots, v^m)$ on $\bar{\Omega}$. In view of the inequality $u^* \leq v^*$ on $\bar{\Omega}$, it is easily seen that v is a viscosity subsolution of (2.1). Noting that $v \leq \psi$ on $\bar{\Omega}$, we conclude that $v \in \mathscr{S}$. Note also that $v \geq u$ and $v \neq u$ on $\bar{\Omega}$. The existence of such a v contradicts the definition of u. Thus we deduce that u is a viscosity supersolution of (2.1). □

Finally, adapting some ideas from [16], we formulate a sufficient condition for the existence of viscosity sub- and supersolutions required in Theorem 4.1 in the case of (1.1).

Let L^k be the linear operators defined by (1.4). Let us introduce the Fichera functions d^k associated with the L^k defined on $\partial\Omega$ by

$$d^k(x) = \prod_{i,j=1}^{n} \left(b_i^k(x) + \frac{\partial a_{ij}^k(x)}{\partial x_j} \right) \nu_i(x) \,,$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ denotes the outward unit normal vector to Ω at $x \in \partial\Omega$. For each $k \in A$ let Σ^k denote the subset of $\partial\Omega$ consisting of those points x for which

$$\sum_{i,j=1}^{n} a_{ij}^k(x)\nu_i(x)\nu_j(x) = 0 \,.$$

Proposition 4.3. *Assume that $\partial\Omega$ is of class C^2, that the f^k, c^k and b_i^k are continuous on $\bar{\Omega}$ and that $c^k > 0$ on $\bar{\Omega}$ for all $k \in A$. Assume that the g^{kj} and $-h^{kj}$ are continuous and nonnegative on $\bar{\Omega}$. Assume that there is a neighborhood U of $\bar{\Omega}$ for which $a_{ij}^k \in C^1(U)$ for $k \in A$ and $1 \leq i$, $j \leq n$ and for which the matrices $(a_{ij}^k(x))_{1 \leq i,j \leq n}$ are nonnegative definite for $k \in A$ and $x \in U$. Also, assume that $d^k(x) < 0$ for $k \in A$ and $x \in \Sigma^k$, and moreover that for each $k \in A$ the interior, relative to $\partial\Omega$, of Σ^k is dense in Σ^k. Then there is a viscosity subsolution φ and a viscosity supersolution ψ of (1.1) such that $\varphi \leq \psi$ on $\bar{\Omega}$ and $\varphi = \psi = 0$ on $\partial\Omega$, and such that the components φ^k are l.s.c. and the ψ^k are u.s.c. on $\bar{\Omega}$.*

Proof of Proposition 4.3. We will only construct a viscosity supersolution of (1.1) with all the required properties since the construction of a subsolution with the required properties is similar.

It follows from the positivity of the c^k that there is a positive constant C such that if $u(x) = C$ for $x \in \Omega$, then $L^k u - f^k \geq 0$ in Ω for $k \in A$. Fix such a constant C henceforth.

Fix $z \in \partial\Omega$. According to the proof of [16, Theor. 1.5.2], we can choose a neighborhood V_z of z and a function $\zeta \in C(\bar{\Omega} \cap \bar{V}_z) \cap C^2(\Omega \cap V_z)$ such that $\zeta(z) = 0$, $\zeta \geq 0$ on $\bar{\Omega} \cap \bar{V}_z$ and $\zeta(x) \geq 1$ for $x \in \bar{\Omega} \cap \partial V_z$ and such that $L^k\zeta(x) \geq 1$ for $k \in A$ and $x \in \Omega \cap V_z$. Multiplying ζ by a sufficiently large constant, we obtain a function $\psi_z \in C(\bar{\Omega} \cap \bar{V}_z) \cap C^2(\Omega \cap V_z)$ having the properties: $\psi_z(z) = 0$, $\psi_z \geq 0$ on $\bar{\Omega} \cup \bar{V}_z$, $\psi_z(x) > C$ for $x \in \bar{\Omega} \cap \partial V_z$ and $L^k \psi_z - f^k \geq 0$ on $\Omega \cap V_z$ for all $k \in A$. Define $\hat{\psi}_z \in C(\bar{\Omega})$ by setting $\hat{\psi}_z(x) = \min\{C, \psi_z(x)\}$ if $x \in \bar{V}_z$ and $=C$ otherwise. Indeed, it is easily seen that $\hat{\psi}_z$ is continuous on $\bar{\Omega}$. Also, it is clear that $\hat{\psi}_z(z) = 0$ and $\hat{\psi}_z \geq 0$ on $\bar{\Omega}$. Moreover, it is easily checked that $\hat{\psi}_z$ is a viscosity supersolution of $L^k \hat{\psi}_z - f^k = 0$ in Ω for all $k \in A$. Now we define a u.s.c. function $\hat{\psi}$ on $\bar{\Omega}$ by $\hat{\psi}(x) = \inf\{\hat{\psi}_z(x) | z \in \partial\Omega\}$. Then, it is clear that $\hat{\psi} = 0$ on $\partial\Omega$, $\hat{\psi} \geq 0$ on $\bar{\Omega}$ and $\hat{\psi}$ is a viscosity supersolution of $L^k \hat{\psi} - f^k = 0$ in Ω for all $k \in A$. Finally, setting $\psi^k = \hat{\psi}$ for $k \in A$, we define the $\boldsymbol{R}^m$-valued function $\psi = (\psi^1, \dots, \psi^m)$ on $\bar{\Omega}$. Since $(\psi^k)_* \geq N^k \psi_*$ in Ω by the non-positivity of the h^{kj}, it is easy to see that ψ is a viscosity supersolution of (1.1). Thus, ψ has all the desired properties. $\square$

References

[1] Bensoussan, A. and Lions, J.-L., *Contrôle Impulsionnel et Inéquations Quasi-Variationelles*, Dunod, Paris, 1982.

[2] Capuzzo Dolcetta, I. and Evans, L. C., Optimal switching for ordinary differential equations, SIAM J. Cont. Optm., **22** (1984), 143–161.

[3] Crandall, M. G., Semidifferentials, quadratic forms and fully nonlinear elliptic equations of second order, Ann. Inst. H. Poincaré Anal. non Lin., **6** (1989), 419–435.

[4] Crandall, M. G. and Lions, P.-L., Viscosity solutions of Hamilton-Jacobi equations, Trans. Amer. Math. Soc., **277** (1983), 1–42.

[5] Ekeland, I. and Lebourg, G., Generic Frechet-differentiability and perturbed optimization problems in Banach spaces, Trans. Amer. Math. Soc., **224** (1976), 193–216.

[6] Evans, L. C., Classical solutions of the Hamilton-Jacobi-Bellman equation for uniformly elliptic operators, Trans. Amer. Math. Soc., **275** (1983), 245–255.

[7] Evans, L. C. and Friedman, A., Optimal stochastic switching and the Dirichlet problem for the Bellman equation, Trans. Amer. Math. Soc., **253** (1979), 365–389.

[8] Ishii, H., Perron's method for Hamilton-Jacobi equations, Duke Math. J., **55** (1987), 369–384.

[9] Ishii, H., On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDE's, Comm. Pure Appl. Math., **42** (1989), 15–45.

[10] Ishii, H. and Lions, P.-L., Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Diff. Eq., **83** (1990), 26–78.

[11] Ishii, H. and Yamada, N., A remark on a system of inequalities with bilateral obstacles, Nonlinear Anal. Theo. Meth. Appl., **13** (1989), 1295–1301.

[12] Jensen, R., The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations, Arch. Rat. Mech. Anal., **101** (1988), 1–27.

[13] Lenhart, S. M. and Belbas, S. A., A system of nonlinear partial differential equations arising in the optimal control of stochastic systems with switching costs, SIAM J. Appl. Math., **43** (1983), 465–475.

[14] Lenhart, S. M. and Yamada, N., Viscosity solutions associated with switching game for piecewise-deterministic processes, preprint.

[15] Lions, P.-L., Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations Part 1, Comm. in P.D.E., **8** (1983), 1101–1134.
[16] Oleinik, A. O. and Radkevic, E. V., *Second Order Equations with Nonnegative Characteristic Form*, Amer. Math. Soc., Providence, Rhode Island and Plenum Press, New York, 1973.
[17] Stroock, D. W. and Varadhan, S. R. S., *Multidimensional Diffusion Processes*, Springer-Verlag, Berlin-Heidelberg-New York, 1979.
[18] Yamada, N., A system of elliptic variational inequalities associated with a stochastic switching game, Hiroshima Math. J., **13** (1983), 109–132.
[19] Yamada, N., Viscosity solutions for a system of inequalities with bilateral obstacles, Funkcial. Ekvac., **30** (1987), 417–425.

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