

Deleting Homeomorphisms and the Failure of Peano's Existence Theorem in Infinite-Dimensional Banach Spaces

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0. Given an infinite-dimensional Banach space, we construct an ordinary differential equation $D_t x = F(t, x)$ with F continuous such that 1) the initial value problem $x(0) = 0_x$ has no local solution and 2) for all $(t_0, x_0) \in \mathbf{R} \times X \setminus \{(0, 0_x)\}$, the initial value problem $x(t_0) = x_0$ has a solution which is unique and depends continuously on (t_0, x_0) and it is defined for all $t \in \mathbf{R}$. Such examples are shown to be closely related to the existence of deleting homeomorphisms/diffeomorphisms. In particular, for nonreflexive spaces, the existence of deleting homeomorphisms is proved via differential equations.

1. Introduction and background results

Let $(X, \|\cdot\|)$ be a Banach space. The origin of X is denoted by 0_x . For $r > 0$, let $B(r) = \{x \in X \mid \|x\| \leq r\}$. Recall that, for a given set Γ , $c_0(\Gamma)$ is the linear space consisting of all $x = (x_\gamma) \in \mathbf{R}^\Gamma$ with $\{\gamma \in \Gamma \mid |x_\gamma| > \varepsilon\}$ finite for any $\varepsilon > 0$; $c_0(\Gamma)$ is regarded as a Banach space under the norm $\|x\| = \sup\{|x_\gamma| \mid \gamma \in \Gamma\}$.

Let X and Y be Banach spaces. If $U \subset X$ is open and $V \subset Y$, then $C^p(U, V)$ denotes the set of all mappings $f: U \rightarrow V$ (with domain U) having continuous p -th Fréchet derivative, $p = 0, 1, 2, \dots$ ($C^0(U, V)$ is simply the set of all continuous mappings.). We also let $C^\infty(U, V) = \bigcap \{C^p(U, V) \mid p \in \mathbf{N}\}$. The derivative of $f \in C^p(U, V)$ at $u \in U$ is denoted by $D_x f(u)$.

Recall that a norm $\|\cdot\|$ on X is said to be of class C^p if the function $x \mapsto \|x\|$, $x \neq 0_x$, is in $C^p(X \setminus \{0_x\}, \mathbf{R})$. The space X is said to be C^p smooth if it admits a non-trivial function $\varphi \in C^p(X, [0, 1])$ whose support $\text{supp } \varphi = \{x \in X \mid \varphi(x) > 0\}$ is bounded. It is obvious that if X admits an equivalent C^p norm, then X is C^p smooth, $p = 1, 2, 3, \dots, \infty$.

Definition. For brevity, we say that a mapping $h: X \rightarrow X \setminus \{0_x\}$ is a deleting homeomorphism if it is a homeomorphism of X onto $X \setminus \{0_x\}$ with the property that $h(x) = x$ whenever $\|x\| \geq 1$. If in addition, $h \in C^p(X, X \setminus \{0_x\})$ and $h^{-1} \in C^p(X \setminus \{0_x\}, X)$, then h is called a deleting C^p diffeomorphism, $p = 1, 2, 3, \dots, \infty$.

The geometry of deleting homeomorphisms/diffeomorphisms is well understood [2], [5], [6]. The study of deleting homeomorphisms/diffeomorphisms is one of the major topics of negligibility theory, a part of infinite-dimensional topology [2].

Lemma 1 [5, Corollary 1]. *Let X be an infinite-dimensional Banach space. Then X admits a deleting homeomorphism. Moreover, the deleting homeomorphisms $h: X \rightarrow X \setminus \{0_x\}$ can be chosen so that there exists an invertible isotopy i.e. a homeomorphism $H: [0, 1] \times X \rightarrow [0, 1] \times X \setminus \{(1, 0_x)\}$ preserving the first coordinate (i.e. if $(t, y) = H(s, x)$ then $t = s$) with the properties that $H(0, x) = x$ and $H(1, x) = h(x)$ for all $x \in X$ and $H(t, x) = (t, x)$ whenever $t \in [0, 1]$, $\|x\| \geq 1$.*

Lemma 2 [6, p. 138]. *Let X be an infinite-dimensional Banach space with the properties that*

- (1) *there exists a continuous, linear injection $T: X \rightarrow c_0(\Gamma)$ for some set Γ ; and*
- (2) *X is C^p smooth, $p = 1, 2, 3, \dots, \infty$.*

Then X admits a deleting C^p diffeomorphism.

Note that none of the Lemmas remains true if X is allowed to be finite-dimensional. The problem of characterizing all Banach spaces admitting deleting diffeomorphisms seems to be open. In particular, Lemma 2 is stated (in form of a remark, with the proof outlined) as a partial answer to Question 6 in [6].

Remark. It is well-known that every reflexive Banach space admits an equivalent norm of class C^1 [13], [14, proof of Corollary 6]. In particular, all reflexive spaces satisfy (2) with $p = 1$. On the other hand, all reflexive spaces satisfy (1) [12]. For a list of spaces satisfying (1) and/or (2), we refer the reader to [6], [13], [14].

2. The main result

It is well-known [9] that Peano's existence theorem fails in all infinite-dimensional Banach spaces. For a survey on counterexamples, see [11]. The overwhelming majority of these counterexamples (including [4], [9]) has a local character: it is pointed out that a particular initial value problem has no solution but no attention is paid to neighbouring initial data. On the failure of Peano's existence theorem in some locally convex spaces, see [1].

The aim of the present paper is to show that the existence of deleting homeomorphisms/diffeomorphisms is closely related to the failure of Peano's existence theorem in infinite-dimensional Banach spaces. We hope that our approach provides a better understanding of these phenomena. Besides, for

nonreflexive spaces, we give a simple proof of Lemma 1 via differential equations (Section 3 and the fourth paragraph of Section 2).

For $F \in C^0(\mathbf{R} \times X, X)$, consider the ordinary differential equation (ODE)

$$(3) \quad D_t x = F(t, x).$$

Let $W = \mathbf{R} \times X \setminus \{(0, 0_x)\}$. As usual, $F|W$ denotes the restriction of F on W . Let $\mathcal{F} = \mathcal{F}(X)$ denote the class of functions $F \in C^0(\mathbf{R} \times X, X)$ satisfying

- (4) $\|F(t, x)\| \leq 8|t|$ for all $(t, x) \in \mathbf{R} \times X$;
- (5) $F(t, x) = 0_x$ whenever $\|x\| \geq 2t^2$;
- (6) for every $(t_0, x_0) \in W$, the ODE (3) has a unique (global) solution $x \in C^1(\mathbf{R}, X)$ with $x(t_0) = x_0$ and, in addition, this solution depends continuously on the initial data;
- (7) the ODE (3) has no (local) solution through $(0, 0_x)$.

Given an $F \in \mathcal{F}$, observe that the mapping

$$h: X \rightarrow X \setminus \{0_x\}, \quad x_0 \mapsto x(0, 2^{-1/2}, x_0)$$

is a deleting homeomorphism satisfying the requirements of Lemma 1: the isotopy H can be chosen as

$$H(t, x_0) = (t, x(2^{-1/2}(1-t), 2^{-1/2}, x_0)), \quad (t, x_0) \in [0, 1] \times X.$$

Here of course, $x(\cdot, t_0, x_0)$ is the solution of (3) through $(t_0, x_0) \in W$. Note that h is a deleting C^p diffeomorphism (and H is a C^p diffeomorphism) provided that $F|W \in C^p(W, X)$, $p = 1, 2, 3, \dots, \infty$.

For $f \in C^0(X, X)$, consider the ODE

$$(8) \quad D_t y = f(y).$$

Let $\Phi = \Phi(X)$ denote the class of functions $f \in C^0(X, X)$ satisfying

- (9) $\|f(y) - y\| \leq 4$ for all $y \in X$;
- (10) $f(y) = y$ whenever $\|y\| \geq 2$;
- (11) the ODE (8) induces a (global) dynamical system on X ;
- (12) the ODE (8) has no bounded trajectories.

Theorem 1. *Let X be an infinite-dimensional Banach space. Then $\Phi(X) \neq \emptyset$.*

The proof of Theorem 1 is postponed to Section 3 and 4. We distinguish two cases according as X is reflexive or not. The proof of the nonreflexive case is based on James' characterization of reflexivity [10]. The proof of the reflexive case is based on Lemma 2.

Theorem 2. *Let X be an infinite-dimensional Banach space. Then $\mathcal{F}(X) \neq \emptyset$.*

Proof. We derive Theorem 2 from Theorem 1 via a simple duality argument. Consider the transformation $J: \mathbf{R} \times X \rightarrow \mathbf{R} \times X$ defined by

$$(t, x) = J(\tau, y) := (\exp(-\tau/2), y \exp(-\tau)), (\tau, y) \in \mathbf{R} \times X.$$

Clearly J is one-to-one and maps $\mathbf{R} \times X$ onto $(0, \infty) \times X$. Given an $f \in \Phi$, the ODE (8) is transformed into

$$D_t x = g(t, x) := 2t(t^{-2}x - f(t^{-2}x)), \quad t > 0, \quad x \in X.$$

In fact, $D_t x = D_\tau t(D_\tau y - y) \exp(-\tau) = -2t^{-1}(f(y) - y)t^2 = g(t, x)$. By (9) and (10), we have $\|g(t, x)\| \leq 8t$ for all $(t, x) \in (0, \infty) \times X$ and $g(t, x) = 0_x$ whenever $\|x\| \geq 2t^2$, $t > 0$. It follows immediately that

$$F(t, x) = \begin{cases} g(t, x) & \text{if } t > 0, x \in X \\ 0_x & \text{if } t = 0, x \in X \\ -g(-t, x) & \text{if } t < 0, x \in X \end{cases}$$

defines a continuous extension of g . Hence, $F \in C^0(\mathbf{R} \times X, X)$. (It is worth to mention here that $F|_W \in C^p(W, X)$ provided that $f \in C^p(X, X)$, $p = 1, 2, 3, \dots, \infty$.) We prove that $F \in \mathcal{F}$. Conditions (4) and (5) are trivially satisfied. Using (5), it is easily seen that condition (6) is implied by (11). Finally, condition (7) is an immediate consequence of (12). In fact, if φ is a solution of (3) with $\varphi(0) = 0_x$, then, in virtue of (9), $\|\varphi(t)\| \leq 4t^2$ for all $t > 0$. Since J maps $\mathbf{R} \times B(r)$, $r > 0$ onto $\{(t, x) \in \mathbf{R} \times X \mid t > 0, \|x\| \leq rt^2\}$, we conclude that φ corresponds to a bounded solution of (8), a contradiction.

3. Proof of Theorem 1. The nonreflexive case

Throughout this section, let X be a nonreflexive Banach space.

By James' characterization of reflexivity [10], there exists a $\varphi \in X^*$, $\|\varphi\| = 1$ such that, for every $x \in B(1)$, $\varphi(x) < 1$. For $c > 0$, define

$$Q_c = \{x \in X \mid \|x\| \leq 1 + c\} \cap \{x \in X \mid \varphi(x) \geq 1 - 3c\}.$$

Note that

$$(13) \quad Q_d \supset Q_c \neq \emptyset \text{ whenever } d > c > 0 \text{ and } \bigcap \{Q_c \mid c > 0\} = \emptyset;$$

$$(14) \quad Q_c, c > 0 \text{ is a bounded convex closed subset of } X;$$

$$(15) \quad Q_c = B(c + 1) \text{ if } c \geq 1.$$

For $x \in X$, define

$$v(x) = \inf\{c > 0 \mid x \in Q_c\}.$$

Observe that $v(x) = \alpha$ if and only if $\|x\| = 1 + \alpha$, $\varphi(x) \geq 1 - 3\alpha$ or $\|x\| \leq 1 + \alpha$, $\varphi(x) = 1 - 3\alpha$. By (13),

$$(16) \quad v(x) > 0 \quad \text{for all } x \in X.$$

Claim 1. $|v(x) - v(y)| \leq \|x - y\|$ for all $x, y \in X$.

Proof. For brevity, we write $v(x) = \alpha$, $v(y) = \beta$. If $\|x\| = 1 + \alpha$, $\|y\| = 1 + \beta$, then $|\alpha - \beta| = \left| \|x\| - \|y\| \right| \leq \|x - y\|$. If $\varphi(x) = 1 - 3\alpha$, $\varphi(y) = 1 - 3\beta$, then $|\alpha - \beta| = 3^{-1}|\varphi(x) - \varphi(y)| = 3^{-1}|\varphi(x - y)| \leq 3^{-1}\|x - y\|$. Finally, if $\|x\| = 1 + \alpha$, $\varphi(y) = 1 - 3\beta$, $\|y\| \leq 1 + \beta$, $\varphi(x) \geq 1 - 3\alpha$, then $\alpha - \beta = \|x\| - (1 + \beta) \leq \|x\| - \|y\| \leq \|x - y\|$ and $\beta - \alpha = 3^{-1}(1 - 3\alpha - \varphi(y)) \leq 3^{-1}(\varphi(x) - \varphi(y)) = 3^{-1}\varphi(x - y) \leq 3^{-1}\|x - y\|$. Thus, in all cases, $|\alpha - \beta| \leq \|x - y\|$.

For $n \in \mathbf{N}$, choose an $x_n \in \{x \in B(1) \mid \varphi(x) \geq 1 - 2^{-n+1}\}$. We may assume that $x_0 = 0_x$ and $x_n \neq x_m$ if $n \neq m$. By letting

$$\Gamma(c) = (2 - 2^{n+1}c)x_{n+1} + (2^{n+1}c - 1)x_n \quad \text{if } 2^{-n-1} < c \leq 2^{-n}, \quad n \in \mathbf{N},$$

we define a continuous function $\Gamma: (0, 1] \rightarrow B(1)$. Geometrically, since $\Gamma(2^{-n}) = x_n$, $n \in \mathbf{N}$, Γ is the polygonal arc $x_0x_1x_2\dots$ i.e. the union of straight line segments connecting x_n to x_{n+1} , $n \in \mathbf{N}$. For $c > 1$, extend Γ by letting

$$\Gamma(c) = 0_x \quad \text{if } c > 1.$$

Observe that

$$(17) \quad \varphi(\Gamma(c)) \geq 1 - 2c \quad \text{for all } c > 0.$$

In fact, $\varphi(\Gamma(c)) = (2 - 2^{n+1}c)\varphi(x_{n+1}) + (2^{n+1}c - 1)\varphi(x_n) \geq (2 - 2^{n+1}c)(1 - 2^{-n}) + (2^{n+1}c - 1)(1 - 2^{-n+1}) = 1 - 2c$ whenever $2^{-n-1} < c \leq 2^{-n}$, $n \in \mathbf{N}$ and $\varphi(\Gamma(c)) = 0 > 1 - 2c$ whenever $c > 1$.

It is easily seen that for each $k > 0$ there exists a constant $L(k) > 0$ such that

$$(18) \quad |\Gamma(c_1) - \Gamma(c_2)| \leq L(k)|c_1 - c_2| \quad \text{whenever } c_1, c_2 > k.$$

Claim 2. $\|x - \Gamma(v(x))\| \geq v(x)$ for all $x \in X$.

Proof. For brevity, we write $v(x) = \alpha$. Observe that $\Gamma(\alpha) \in B(1)$ and, in virtue of (17), $\varphi(\Gamma(\alpha)) \geq 1 - 2\alpha$. We distinguish two cases according as $\|x\| = 1 + \alpha$ or $\varphi(x) = 1 - 3\alpha$. If $\|x\| = 1 + \alpha$, then $\|x - \Gamma(\alpha)\| \geq \|x\| - \|\Gamma(\alpha)\| = 1 + \alpha - \|\Gamma(\alpha)\| \geq \alpha$. If $\varphi(x) = 1 - 3\alpha$, then

$$\begin{aligned} \|x - \Gamma(\alpha)\| &\geq |\varphi(x - \Gamma(\alpha))| = |\varphi(x) - \varphi(\Gamma(\alpha))| \\ &\geq \varphi(\Gamma(\alpha)) - \varphi(x) = \varphi(\Gamma(\alpha)) - (1 - 3\alpha) \geq (1 - 2\alpha) - (1 - 3\alpha) = \alpha. \end{aligned}$$

Hence, in virtue of (16), by letting

$$e(x) = \|x - \Gamma(v(x))\|^{-1}(x - \Gamma(v(x))) \quad \text{for } x \in X,$$

we define a function $e: X \rightarrow B(1)$.

Claim 3. $v(x + \lambda e(x)) > v(x)(1 + \lambda/9)$ whenever $\|x\| < 2$, $\lambda > 0$.

Proof. For brevity, we write $v(x) = \alpha$. Note that $\alpha > 0$. Since $\Gamma(\alpha) \in B(1)$, it follows that $\|x - \Gamma(\alpha)\| < 3$. We distinguish two cases according as $\|x\| = 1 + \alpha$, $\varphi(x) \geq 1 - 3\alpha$ or $\|x\| \leq 1 + \alpha$, $\varphi(x) = 1 - 3\alpha$. First suppose that $\|x\| = 1 + \alpha$. Then

$$\begin{aligned} \|x + \lambda e(x)\| &= \|x + \lambda\|x - \Gamma(\alpha)\|^{-1}x - \lambda\|x - \Gamma(\alpha)\|^{-1}\Gamma(\alpha)\| \\ &\geq (1 + \lambda\|x - \Gamma(\alpha)\|^{-1})\|x\| - \lambda\|x - \Gamma(\alpha)\|^{-1}\|\Gamma(\alpha)\| \\ &\geq (1 + \lambda\|x - \Gamma(\alpha)\|^{-1})(1 + \alpha) - \lambda\|x - \Gamma(\alpha)\|^{-1} \\ &= 1 + \alpha(1 + \lambda\|x - \Gamma(\alpha)\|^{-1}) > 1 + \alpha(1 + \lambda 3^{-1}) \end{aligned}$$

and consequently,

$$x + \lambda e(x) \notin Q_{\alpha(1+\lambda/3)}, \quad v(x + \lambda e(x)) > \alpha(1 + \lambda/3).$$

Secondly, suppose that $\varphi(x) = 1 - 3\alpha$. By (17), $\varphi(x - \Gamma(\alpha)) = \varphi(x) - \varphi(\Gamma(\alpha)) \leq 1 - 3\alpha - (1 - 2\alpha) = -\alpha$. It follows immediately that

$$\begin{aligned} \varphi(x + \lambda e(x)) &= \varphi(x) + \lambda\|x - \Gamma(\alpha)\|^{-1}\varphi(x - \Gamma(\alpha)) \\ &\leq \varphi(x) - \lambda\|x - \Gamma(\alpha)\|^{-1}\alpha < 1 - 3\alpha - \lambda 3^{-1}\alpha = 1 - 3\alpha(1 + \lambda/9) \end{aligned}$$

and consequently,

$$x + \lambda e(x) \notin Q_{\alpha(1+\lambda/9)}, \quad v(x + \lambda e(x)) > \alpha(1 + \lambda/9).$$

Thus, in both cases, $v(x + \lambda e(x)) > v(x)(1 + \lambda/9)$.

In virtue of (15), $v(y) = \|y\| - 1$, $\Gamma(v(y)) = 0_x$ whenever $\|y\| \geq 2$. For $y \in X$, define

$$f(y) = \begin{cases} 2e(y) & \text{if } \|y\| < 2 \\ y & \text{if } \|y\| \geq 2. \end{cases}$$

It is not hard to show that f is locally Lipschitzian. More precisely, Claim 1, Claim 2, (16), (18) and an easy computation imply that for each $k > 0$ there is a constant $M(k) > 0$ such that

$$\|f(y) - f(z)\| \leq M(k)\|y - z\| \quad \text{whenever } v(y), v(z) \geq k.$$

In particular, $f \in C^0(X, X)$.

We prove that $f \in \Phi$. Conditions (9) and (10) are trivially satisfied. Since f is locally Lipschitzian and satisfies inequality $\|f(y)\| \leq 2 + \|y\|$, $y \in X$, a well-known global version of the Picard-Lindelöf theorem (see e.g. [11, Theorem 2.8.1]) implies that f satisfies (11). It remains to prove (12). As an

immediate consequence of Claim 3, we have

$$v(y + \lambda f(y)) > v(y)(1 + 2\lambda/9) \quad \text{for all } y \in X, \quad \lambda > 0.$$

Therefore, by a simple comparison argument,

$$v(y(\tau, y_0)) \geq v(y_0) \exp(2\tau/9) \quad \text{for all } y_0 \in X, \quad \tau \geq 0.$$

Here of course, $y(\cdot, y_0)$ is the solution of (8) through $(0, y_0) \in \mathbf{R} \times X$. In particular, v is strictly increasing along the trajectories of (8) and, for all $y_0 \in X$, $v(y(\tau, y_0)) \rightarrow \infty$, $\|y(\tau, y_0)\| \rightarrow \infty$ as $\tau \rightarrow \infty$. (Similarly, for all $y_0 \in X$, $v(y(\tau, y_0)) \rightarrow 0$ as $\tau \rightarrow -\infty$. Roughly speaking, the empty set is globally repulsive.) Thus, $f \in \Phi$.

Observation. Let X be a Banach space and suppose we are given an $f \in \Phi(X)$. For $c > 0$, let $R_c = \{y(t, y_0) \in X \mid y_0 \in B(2), t \leq -c^{-1}\}$. Then the collection $\{R_c \mid c > 0\}$ satisfies (13). Moreover, R_c is a bounded closed subset of X . Hence it is natural to ask whether the previous method can be modified so as to be valid for infinite-dimensional reflexive spaces as well. The answer is probably affirmative but the technical details seem to be enormous. The basic problem is the lack of convexity. The existence of a collection $\{Q_c \subset X \mid c > 0\}$ with properties (13) and (14) yields that X is not reflexive. (Since convex closed sets are weakly closed, this is an immediate corollary of the Banach-Alaoglu theorem.)

4. Proof of Theorem 1. The reflexive case

Throughout this section, let $(X, \|\cdot\|)$ be an infinite-dimensional reflexive Banach space. In virtue of the Remark, there is no loss of generality in assuming that $\|\cdot\|$ is of class C^1 . Similarly, by the same Remark, the conditions of Lemma 2 are satisfied. Therefore, there exists a deleting C^1 diffeomorphism $h: X \rightarrow X \setminus \{0_x\}$, $w = h(x)$.

The ODE $D_t w = w$, $w \neq 0_x$ is transformed into

$$(19) \quad D_t x = e(x) := [D_w h^{-1}(h(x))]h(x), \quad x \in X.$$

Clearly (19) induces a dynamical system on X and, for each $z_0 \in X$, $x(t, z_0) = h^{-1}(h(z_0) \exp(t))$, $t \in \mathbf{R}$. Here of course, $x(\cdot, z_0)$ is the solution of (19) through $(0, z_0) \in \mathbf{R} \times X$. It follows immediately that (19) has no bounded trajectories. Finally, observe that $e(x) = x$ whenever $\|x\| \geq 1$. Thus, $e \in C^0(X, X)$ satisfies conditions (10), (11), (12). On the other hand, in general, condition (9) is not satisfied. (A simple analysis of the proof of the inverse function theorem shows that $\sup\{\|D_w h^{-1}(h(x))\| \mid \|x\| \leq 1\} = \infty$.) This difficulty can be overcome by Vinograd's reparametrization [3].

Let $a: (1, \infty) \rightarrow [1, 2]$ be a C^∞ function such that $a(r) = r$ whenever $r \in (1, 3/2)$ and $a(r) = 1$ whenever $r \in [2, \infty)$. For $y \in X$, define

$$\lambda(y) = \begin{cases} \|e(y)\| & \text{if } \|y\| \leq 1 \\ a(\|y\|) & \text{if } \|y\| > 1. \end{cases}$$

Observe that $\lambda \in C^0(X, \mathbf{R})$, $\lambda(y) > 0$ and consider the ODE

$$(20) \quad D_t y = f(y) := (\lambda(y))^{-1} e(y), \quad y \in X.$$

We prove that $f \in \Phi$. Note that $f \in C^0(X, X)$. (Moreover, $f \in C^{p-1}(X, X)$ provided that $h \in C^p(X, X)$ and that $\|\cdot\|$ is of class C^{p-1} , $p = 1, 2, 3, \dots, \infty$.) Conditions (9), (10) are trivially satisfied. For $z_0 \in X$, define

$$\tau^-(z_0) = \int_0^{-\infty} \lambda(x(s, z_0)) ds, \quad \tau^+(z_0) = \int_0^{\infty} \lambda(x(s, z_0)) ds.$$

It is well-known [3] that (20) defines a local dynamical system on X and that $y(\cdot, z_0)$, the solution of (20) through $(0, z_0) \in \mathbf{R} \times X$ is defined for $\tau \in (\tau^-(z_0), \tau^+(z_0))$. Further [3], for all $t \in \mathbf{R}$, there holds

$$x(t, z_0) = y(\tau, z_0), \quad \text{where } \tau = \int_0^t \lambda(x(s, z_0)) ds, \quad z_0 \in X.$$

It is left to prove that $\tau^-(z_0) = -\infty$, $\tau^+(z_0) = \infty$ for all $z_0 \in X$. Without loss of generality, we may assume that $\|z_0\| = 1$. Clearly $\tau^+(z_0) = \infty$. If $\tau^-(z_0) \neq -\infty$, then

$$\begin{aligned} \|x(t_2, z_0) - x(t_1, z_0)\| &= \left\| \int_{t_1}^{t_2} D_t x(s, z_0) ds \right\| = \left\| \int_{t_1}^{t_2} e(x(s, z_0)) ds \right\| \\ &\leq \int_{t_1}^{t_2} \|e(x(s, z_0))\| ds = \int_{t_1}^{t_2} \lambda(x(s, z_0)) ds \rightarrow 0 \end{aligned}$$

as $t_1 \leq t_2 < 0$, $t_2 \rightarrow -\infty$. Therefore, there exists an $\hat{x} \in X$ such that $x(t, z_0) \rightarrow \hat{x}$ as $t \rightarrow -\infty$. It follows immediately that \hat{x} is an equilibrium point of (19), a contradiction.

5. Related results

Modifying an example by Cellina [4, p. 1071], we have constructed [8] an $F \in \mathcal{F}(\ell_1)$. The construction is completely elementary and it is independent of techniques used in Section 3 and 4. In his remarkable paper, applying James' characterization of reflexivity [10], Cellina [4, p. 1069–1070] has shown that Peano's existence theorem fails in nonreflexive spaces. Section 3 of the present paper can be considered as a global version of Cellina's approach. We

note here that we were not able to find any relationship between deleting homeomorphisms and Godunov's general counterexample [9].

To the best of our knowledge, the present paper is the first application of negligibility theory to ordinary differential equations. (In a forthcoming paper, deleting homeomorphisms will be applied to study the geometry of the failure of Kneser's property in infinite dimension.) For applications of deleting homeomorphisms in topological dynamics (parallelizable dynamical systems with uniformly bounded trajectories in $c_0 = c_0(N)$; infinite-dimensional aspects of Coleman's conjecture), see [7].

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