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Asymptotic Behaviour of Solutions to Some Degenerate Parabolic Equations

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1. Introduction

This paper is concerned with a degenerate parabolic equation of the form:

$$u_t - \Delta g(u) = f \qquad \text{in } Q = I \times \Omega\,, \tag{1.1}$$

where I is an interval of $\boldsymbol{R}$ of the form $[t_0, +\infty)$ or $\boldsymbol{R}$; Ω is a bounded domain in $\boldsymbol{R}^N$ ($N \geq 1$) with sufficiently smooth boundary $\Gamma = \partial\Omega$; $g: \boldsymbol{R} \to \boldsymbol{R}$ is a given non-decreasing and Lipschitz continuous function; f is a given function on Q.

Equation (1.1) represents the enthalpy formulation of the Stefan problem, when

$$g(r) = \begin{cases} c_1(r-1) & \text{for } r \geq 1\,, \\ 0 & \text{for } 0 < r < 1\,, \\ c_2 r & \text{for } r \leq 0\,, \end{cases}$$

for some positive constants c_1, c_2. As far as the existence and uniqueness of the solution to the initial-boundary value problem for (1.1) are concerned, there are some well-known results in a more general setting (e.g. Kamenomostskaja [20], Friedman [12], Oleinik [27], Ladyzhenskaja-Solonnikov-Ural'ceva [26]). In this paper we are mainly interested in the asymptotic stability of solutions to (1.1) with the Neumann type boundary condition

$$\partial_\nu g(u) = h \qquad \text{on } \Sigma = I \times \Gamma\,, \tag{1.2}$$

where ∂_ν denotes the outward normal derivative on Γ and h is a given function on Σ. This question is studied by reformulating the problem (1.1)–(1.2) as a nonlinear evolution equation involving time-dependent subdifferential operators in a suitable Hilbert space. Such a technique was suggested to us by the work of Damlamian [8] who treated the mixed boundary conditions

$$g(u) = h_D \qquad \text{on } \Sigma_D = I \times \Gamma_D\,,$$

$$\partial_\nu g(u) = h_N \qquad \text{on } \Sigma_N = I \times \Gamma_N\,, \qquad \Gamma_N = \Gamma \backslash \Gamma_D\,,$$

under the assumption that $\text{meas}_\Gamma(\Gamma_D) > 0$. In the present situation, $\Gamma_D = \emptyset$ and we shall establish that problem (1.1)–(1.2) can be reformulated as a nonlinear evolution equation of the form

$$v'(t) + \partial\varphi^t(v(t)) = f^*(t)\,, \qquad t \in I\,, \tag{1.3}$$

in the dual space X^* of the Hilbert space $X = \{z \in H^1(\Omega); \int_\Omega z(x)dx = 0\}$ with norm

$$|z|_X = \left\{\int_\Omega |\nabla z(x)|^2 dx\right\}^{1/2},$$

where $\partial\varphi^t$ is the subdifferential of a convex function φ^t on X^*.

Once the problem is represented in the form (1.3), we can apply some general results from [3, 13, 22–24] on the investigation of asymptotics for abstract evolution equations to problem (1.1)–(1.2).

Throughout this paper we use the following notations:

(1) For a real Banach space V we denote by V^* the topological dual of V and $|\cdot|_V$ the norm in V. The duality pairing between V^* and V will be written as $(\cdot, \cdot)_{V^*,V}$. As a special case the inner product on a Hilbert space V will be denoted by $(\cdot, \cdot)_V$.

(2) The surface element on Γ is denoted by $d\sigma$ and the N-dimensional Lebesgue measure of Ω by $|\Omega|$. In addition we use the simplfied forms:

$$Y = H^1(\Omega)\,.$$

$$(u, v) = \int_\Omega u(x)v(x)dx \qquad \text{for } u,\ v \text{ in } L^2(\Omega)\,.$$

$$\langle f, v\rangle = (f, v)_{Y^*,Y} \qquad \text{for } f \text{ in } Y^* \text{ and } v \text{ in } Y.$$

$$a(u, v) = \int_\Omega \nabla u(x)\cdot\nabla v(x)dx \qquad \text{for } u,\ v \text{ in } Y.$$

(3) For a proper lower semicontinuous convex function $\varphi: V \to (-\infty, +\infty]$ where V is a real Hilbert space, we denote by $D(\varphi)$ the effective domain $\{z \in V; \varphi(z) < +\infty\}$ and by $\partial\varphi$ the subdifferential of φ, i.e., $\partial\varphi$ is a (generally multivalued) operator which assigns to each $z \in D(\varphi)$ the set $\partial\varphi(z)$ in V defined by

$$\partial\varphi(z) = \{z^* \in V; (z^*, v - z)_V \le \varphi(v) - \varphi(z) \text{ for all } v \in V\}\,.$$

The domain of $\partial\varphi$ is the set $D(\partial\varphi) = \{z \in V; \partial\varphi(z) \ne \emptyset\}$. For the general properties of subdifferential operators we refer to Brézis [6].

2. Weak solutions and the initial value problem

First of all we introduce a notion of weak solution for the problem (1.1)–(1.2). Throughout this work we assume that the function $g: \boldsymbol{R} \to \boldsymbol{R}$ satisfies the following conditions (2.1) and (2.2):

$$\text{(2.1)} \quad \begin{cases} g: \boldsymbol{R} \to \boldsymbol{R} \text{ is a non-decreasing and Lipschitz continuous function with} \\ \text{Lipschitz constant denoted by } \alpha . \end{cases}$$

$$\text{(2.2)} \quad \begin{cases} \text{There are some positive constants } \beta, \eta \text{ such that} \\ \qquad |g(r)| \geq \eta |r| - \beta \qquad \text{for all } r \in \boldsymbol{R} . \end{cases}$$

Definition 2.1. Let $J = [t_0, t_1]$ be a compact interval, $Q = (t_0, t_1) \times \Omega$, $\Sigma = (t_0, t_1) \times \Gamma$ and $f \in L^2(Q)$, $h \in L^2(\Sigma)$. Then a function $u: (t_0, t_1) \times \Omega \to \boldsymbol{R}$ is called a solution of (1.1)–(1.2) on J if $u \in L^\infty(J; L^2(\Omega))$, $g(u) \in L^2(J; Y)$ and the following variational equation is satisfied:

$$\text{(2.3)} \quad -\int_Q u\zeta_t dxdt + \int_Q \nabla g(u) \cdot \nabla \zeta dxdt = \int_\Sigma h\zeta d\sigma dt + \int_Q f\zeta dxdt , \qquad {}^\forall \zeta \in Z ,$$

where $Z = \{\zeta \in C^1(J; Y); \zeta(t_0, \cdot) = \zeta(t_1, \cdot) = 0\}$.

Definition 2.2. Let J' be any interval in $\boldsymbol{R}$ and $f \in L^2_{loc}(J'; L^2(\Omega))$, $h \in L^2_{loc}(J'; L^2(\Gamma))$. Then a function $u: J' \times \Omega \to \boldsymbol{R}$ is called a solution of (1.1)–(1.2) on J' if for every compact subinterval $J = [t_0, t_1]$ of J' the restriction of u to $(t_0, t_1) \times \Omega$ is a solution of (1.1)–(1.2) on J.

Remark 2.3. In Definition 2.1, it follows from (2.3) that for some constant $K \geq 0$

$$\left| \int_Q u\zeta_t dxdt \right| \leq K \left\{ \int_Q |\nabla g(u)|^2 dxdt + \int_\Sigma h^2 d\sigma dt + \int_Q f^2 dxdt \right\}^{1/2}$$
$$\times \left\{ \int_Q (\zeta^2 + |\nabla \zeta|^2) dxdt \right\}^{1/2} \qquad \text{for all } \zeta \in Z .$$

This shows in particular that the linear functional defined on Z by $\zeta \to \int_Q u\zeta_t dxdt$ is continuous in the topology of $L^2(J; Y)$. Taking account of the inclusions $Y \subset L^2(\Omega) \subset Y^*$ with compact and dense imbeddings, we infer that $u \in W^{1,2}(J; Y^*) \cap L^\infty(J; L^2(\Omega))$, in particular $u: J \to L^2(\Omega)$ is weakly continuous and moreover the function $(u(t), \zeta(t))$ is absolutely continuous in t on J with

$$\int_s^t \langle u'(\tau), \zeta(\tau) \rangle d\tau + \int_s^t (u(\tau), \zeta'(\tau)) d\tau = (u(t), \zeta(t)) - (u(s), \zeta(s))$$

for any $\zeta \in W^{1,2}(J; L^2(\Omega)) \cap L^2(J; Y)$ and any $s, t \in J$. It follows that (2.3) can be written in the following form

(2.4) $$\int_J \langle u', \zeta\rangle d\tau + \int_J a(g(u), \zeta)d\tau = \int_J (f, \zeta)d\tau + \int_\Sigma h\zeta d\sigma d\tau\,, \quad {}^\forall\zeta \in L^2(J; Y)\,.$$

Besides, (2.4) is equivalent to

(2.5) $$\langle u'(t), \xi\rangle + a(g(u(t)), \xi) = (f(t), \xi) + \int_\Gamma h(t, \sigma)\xi(\sigma)d\sigma\,, \qquad {}^\forall\xi \in Y, \quad \text{a.e. } t \in J\,.$$

It is then quite obvious to obtain

Proposition 2.4. *Let J, f, h be as in Definition 2.1 and consider u: $(t_0, t_1) \times \Omega \to \boldsymbol{R}$. Then u is a solution of (1.1)–(1.2) on J if and only if $u \in W^{1,2}(J; Y^*) \cap L^\infty(J; L^2(\Omega))$, $g(u) \in L^2(J; Y)$ and (2.5) is fulfilled.*

We now consider the initial value problem associated to the degenerate parabolic equation (1.1)–(1.2). This problem can be solved by the general methods of [2, 21]. In section 3 of this work we shall use the theory of time-dependent subdifferential operaors to establish directly the following existence and uniqueness result.

Proposition 2.5. *Let J, f, h be as in Definition 2.1. Then for any $u_0 \in L^2(\Omega)$ there exists a unique solution u of (1.1)–(1.2) on J such that $u(t_0) = u_0$ in the sense of Y^*. In addition, the solution u satisfies the following bound*

$$\sup_{t\in J}|u(t)|_{L^2(\Omega)} + |u'|_{L^2(J;Y^*)} \le K_0(|J|, |f|_{L^2(J;L^2(\Omega))}, |h|_{L^2(J;L^2(\Gamma))}, |u_0|_{L^2(\Omega)})\,,$$

where K_0 is a non-decreasing function in its four arguments.

Corollary 2.6. *Let J' be an interval in $\boldsymbol{R}$ of the form $[t_0, t_1]$ or $[t_0, t_1)$ with t_1 possibly infinite and $f \in L^2_{\mathrm{loc}}(J'; L^2(\Omega))$, $h \in L^2_{\mathrm{loc}}(J'; L^2(\Gamma))$ (i.e. $f \in L^2([t_0, t]; L^2(\Omega))$, $h \in L^2([t_0, t]; L^2(\Gamma))$ for all finite $t \in J'$). Then, for any $u_0 \in L^2(\Omega)$, there exists a unique solution u of (1.1)–(1.2) on J' such that $u \in W^{1,2}_{\mathrm{loc}}(J'; Y^*) \cap L^\infty_{\mathrm{loc}}(J'; L^2(\Omega))$, $g(u) \in L^2_{\mathrm{loc}}(J'; Y)$ and*

(2.6) $$u(t_0) = u_0 \text{ in } Y^*\,.$$

Definition 2.7. Under the hypotheses of Corollary 2.6, the solution u given by Corollary 2.6 is called the solution of (1.1)–(1.2)–(2.6), or alternatively the solution of (1.1)–(1.2) such that $u(t_0) = u_0$.

It should be noticed (cf. Example 7.1) that the set of all equilibrium solutions of (1.1)–(1.2) is in general non convex, when f and h are independent of t. This suggests to us that our problem cannot be treated in just the same setting as in Damlamian [8]. The following observation is our main motivation for the rather complicated construction of section 3.

Lemma 2.8. *Under the hypotheses of Corollary* 2.6, *the unique solution* u *of* (1.1)–(1.2)–(2.6) *satisfies*

$$\int_\Omega u(t,x)dx = \int_\Omega u_0(x)dx + \int_{t_0}^t \int_\Omega f(s,x)dxds + \int_{t_0}^t h(s,\sigma)d\sigma ds\,, \quad {}^\forall t \in J'\,. \tag{2.7}$$

Proof. Indeed, (2.7) is an immediate consequence of formula (2.5) applied with $\xi = 1$. By integrating over $[t_0, t]$ the result follows.

Remark 2.9. Our main interest in this paper is asymptotic behavior of $u(t)$ as $t \to +\infty$, when $J' = [t_0, +\infty)$. It follows immediately from (2.7) that in order for problem (1.1)–(1.2) to have at least one solution bounded in $L^1(\Omega)$ on J' it is necessary that f, h satisfy the following *non-resonance condition*

$$\int_{t_0}^t \int_\Omega f(s,x)dxds + \int_{t_0}^t \int_\Gamma h(s,\sigma)d\sigma ds \qquad \text{is bounded in } t \text{ on } J'\,. \tag{2.8}$$

Although (1.1)–(1.2) cannot be formulated as an evolution equation of monotone type in $L^2(\Omega)$, in section 4 we shall establish that the philosophy of Brézis-Haraux [7] and Haraux [14] is applicable, and more precisely the non-resonance condition (2.8) is exactly sufficient in order for all solutions of (1.1)–(1.2) on $J' = [t_0, +\infty)$ to be bounded in $L^2(\Omega)$ on J'.

3. A subdifferential formulation in Y^*

Throughout this section we assume that f and h satisfy

$$f \in L^2_{\text{loc}}(I; L^2(\Omega))\,, \qquad h \in L^2_{\text{loc}}(I; L^2(\Gamma))\,, \qquad \text{where } I = [t_0, +\infty)\,, \qquad t_0 \in \boldsymbol{R}\,.$$

It will be convenient to introduce the unique solution $h_0 \in L^2_{\text{loc}}(I; Y)$ of

$$a(h_0(t), \xi) + (h_0(t), \xi) = \int_\Gamma h(t,\sigma)\xi(\sigma)d\sigma\,, \qquad {}^\forall \xi \in Y\,, \qquad \text{a.e. } t \in I\,. \tag{3.1}$$

For this function h_0 and any given real number a_0 we define

$$a(t) = a_0 + \frac{1}{|\Omega|}\int_{t_0}^t \int_\Omega (f + h_0)dxd\tau\,, \qquad \forall t \in I\,. \tag{3.2}$$

From now on we shall use the following function spaces:

(i) $H = \{z \in L^2(\Omega); \int_\Omega z(x)dx = 0\}$ is a Hilbert space with norm $|z|_H$ and inner product $(u, v)_H$ deduced from the Hilbert space $L^2(\Omega)$, i.e. $|z|_H = |z|_{L^2(\Omega)}$ and $(u, v)_H = (u, v)$ for u, $v \in H$.

(ii) $X = Y \cap H$ $(= \{z \in Y; \int_\Omega z(x)dx = 0\})$ is a Banach space with norm

$|z|_X = |\nabla z|_{L^2(\Omega)}$ and we denote the duality between X^* and X by $\langle\cdot,\cdot\rangle_0$ i.e.

$$\langle f, v\rangle_0 = \langle f, v\rangle_{X^*,X} \qquad \text{for } f \in X^* \text{ and } v \in X .$$

We identify H with its dual H^* and therefore

$$X \subset H \subset X^* \text{ with compact and dense imbeddings.}$$

Also we introduce the linear mappings $P: L^2(\Omega) \to H$ and $F: X \to X^*$ as follows:

$$Pz = z - \frac{1}{|\Omega|}\int_\Omega z(x)dx, \qquad {}^\forall z \in L^2(\Omega);$$

$$F \text{ is the duality mapping from } X \text{ to } X^* .$$

By the definition of duality mapping, $F: X \to X^*$ is given by the formula

$$\langle Fw, z\rangle_0 = a(w, z), \qquad {}^\forall w,\ z \in X .$$

We have the following obvious properties:

$$\begin{cases} X = P[Y] \text{ and } P: Y \to X \subset Y \text{ is also the projection operator from} \\ Y \text{ to } X \text{ in the sense of the usual Hilbert structure on } Y; \end{cases}$$

(3.3) $$(Pw, z)_H = (w, z), \qquad {}^\forall w \in L^2(\Omega), \qquad {}^\forall z \in H;$$

(3.4) $$\langle Pw, z\rangle_0 = (Pw, z)_H, \qquad {}^\forall w \in L^2(\Omega), \qquad {}^\forall z \in X;$$

$$\begin{cases} X^* \text{ is a Hilbert space with inner product } (\cdot,\cdot)_* \text{ given by} \\ \ (w, z)_* = \langle w, F^{-1}z\rangle_0\ (= \langle z, F^{-1}w\rangle_0), \qquad {}^\forall w, z \in X^*; \end{cases}$$

(3.5) $$a(w, z) = \langle FPw, Pz\rangle_0, \qquad {}^\forall w,\ z \in Y.$$

Finally we introduce

$$G(s) := \int_0^s g(r)dr, \qquad {}^\forall s \in \boldsymbol{R},$$

and for each $t \in I$ we define a function $\varphi^t: X^* \to (-\infty, +\infty]$ by the formula

(3.6) $$\varphi^t(z) = \begin{cases} \displaystyle\int_\Omega G(z(x) + a(t))dx & \text{for } z \in H . \\ +\infty & \text{for } z \in X^*\setminus H . \end{cases}$$

Clearly φ^t is a proper, l.s.c. convex function on X^* and $D(\varphi^t) = H$ for each $t \in I$. Denoting by $\partial\varphi^t$ the subdifferential of φ^t in X^* and taking account of (2.1)–(2.2), we obtain the following lemma.

Lemma 3.1. *For each $t \in I$, $\partial\varphi^t$ is singlevalued in X^* and we have*

$$D(\partial\varphi^t) = \{z \in H; g(z + a(t)) \in Y\}$$

and

$$\partial\varphi^t(z) = FP[g(z + a(t))] \qquad \textit{for all } z \in D(\partial\varphi^t).$$

Proof. Let $z \in H$ and $z^* \in X^*$. If $z^* \in \partial\varphi^t(z)$, then for any $w \in H$, $(z^*, w - z)_* \leq \varphi^t(w) - \varphi^t(z)$ and this can be written as

$$(3.7)\quad (F^{-1}z^*, w - z)_H \leq \int_\Omega \{G(w(x) + a(t)) - G(z(x) + a(t))\}dx, \qquad {}^\forall w \in H.$$

In (3.7) we choose $w = z + \varepsilon v$, $\varepsilon > 0$, and dividing through by ε we obtain

$$(F^{-1}z^*, v)_H \leq \frac{1}{\varepsilon}\int_\Omega \{G(z(x) + a(t) + \varepsilon v(x)) - G(z(x) + a(t))\}dx \qquad \text{for all } v \in H.$$

Then letting $\varepsilon \to 0$ yields

$$(F^{-1}z^*, v)_H \leq \int_\Omega g(z(x) + a(t))v(x)dx = (P[g(z + a(t))], v)_H \qquad \text{for all } v \in H.$$

This implies in particular that $P[g(z + a(t))] = F^{-1}z^* \in X$ and therefore

$$g(z + a(t)) \in Y, \qquad z^* = FP[g(z + a(t))].$$

Conversely, let $z \in H$ with $g(z + a(t)) \in Y$ and $z^* = FP[g(z + a(t))]$. Then

$$\begin{aligned}(z^*, w - z)_* &= (F^{-1}z^*, w - z)_H = (P[g(z + a(t))], w - z)_H = (g(z + a(t)), w - z)\\ &\leq \int_\Omega \{G(w + a(t)) - G(z + a(t))\}dx\\ &= \varphi^t(w) - \varphi^t(z) \qquad \text{for all } w \in H.\end{aligned}$$

Therefore $z^* \in \partial\varphi^t(z)$ and Lemma 3.1 is completely proved.

In order to apply the subdifferential theory to our evolution problem we shall use the following lemma.

Lemma 3.2. *Let $\{\varphi^t\}_{t \in I}$ be the family of proper l.s.c. convex functions on X^* defined by* (3.6) *with $a(t)$ given by* (3.2). *Then for each compact interval $J = [t_0, t_1] \subset I$ there is a constant $K > 0$ such that*

$$(3.8)\quad |\varphi^s(z) - \varphi^t(z)| \leq K|a(s) - a(t)|\{1 + |z|_H\} \qquad \textit{for all } s,\ t \in J,\ z \in H,$$

where K depends only on J, g and the restriction of $a(\cdot)$ to J.

Proof. This is easily deduced from (3.6) with the help of (2.1), (2.2) and (3.2).

We now consider the evolution equation

(3.9) $$v'(t) + \partial\varphi^t(v(t)) = f^*(t), \qquad t \in J = [t_0, t_1],$$

where $f^* \in L^2_{\mathrm{loc}}(I; X^*)$. Under (3.8), the following result follows easily from Attouch-Damlamian [2] or Kenmochi [21]:

Lemma 3.3. *For any $v_0 \in H$ there exists a unique function v in $W^{1,2}(J; X^*) \cap L^\infty(J; H)$ which satisfies* (3.9) *in the sense of X^* a.e. on J and the initial condition*

(3.10) $$v(t_0) = v_0 \qquad \text{in } X^*.$$

In addition, v satisfies the following bound

(3.11) $$\sup_{t\in J}\{|v(t)|_{X^*} + |\varphi^t(v(t))|\} + |v'|_{L^2(J;X^*)} \le M(|J|, |a|_{W^{1,1}(J)}, |f^*|_{L^2(J;X^*)}, |\varphi^{t_0}(v_0)|),$$

where M is a nondecreasing function in its four arguments.

The relationship between the original problem (1.1)–(1.2) and equation (3.9) is now clarified as follows.

Proposition 3.4. *Let $J = [t_0, t_1]$ be a compact interval, $Q = (t_0, t_1) \times \Omega$, $\Sigma = (t_0, t_1) \times \Gamma$, $f \in L^2(Q)$, $h \in L^2(\Sigma)$ and $u_0 \in L^2(\Omega)$. Then a function $u: Q \to \boldsymbol{R}$ is a solution of* (1.1)–(1.2) *on J satisfying the initial condition $u(t_0) = u_0$, if and only if $v := u - a$ is the solution of* (3.9) *on J satisfying the initial condition $v(t_0) = Pu_0$, where $a(t)$ and φ^t are defined respectively by* (3.2) *and* (3.6), *and a_0, f^* are given by*

(3.12) $$a_0 = \frac{1}{|\Omega|}\int_\Omega u_0 dx, \quad f^*(t) = P[f(t) + h_0(t)] + FPh_0(t) \quad \text{for a.e. } t \in J.$$

Proof. First let u be a solution of (1.1)–(1.2) satisfying $u(t_0) = u_0$. Then it follows from (2.5) and (3.1) that for a.e. $t \in J$

(3.13) $$\langle u'(t), \xi\rangle + a(g(u(t)) - h_0(t), \xi) = (f(t) + h_0(t), \xi), \qquad {}^\forall\xi \in Y.$$

It now follows from Lemma 2.8 that $v(t) := u(t) - a(t) = Pu(t) \in H$ for all $t \in J$. In particular, for all $\xi \in X$ and a.e. $t \in J$ we have the identities

$$\langle u'(t), \xi\rangle = \langle v'(t), \xi\rangle + a'(t)\int_\Omega \xi(x)dx = \langle v'(t), \xi\rangle_0,$$

$$(f(t) + h_0(t), \xi) = \langle P[f(t) + h_0(t)], \xi\rangle_0,$$

$$\begin{aligned} a(g(u(t)) - h_0(t), \xi) &= a(g(v(t) + a(t)) - h_0(t), \xi) \\ &= \langle FP[g(v(t) + a(t)) - h_0(t)], \xi\rangle_0 \end{aligned}$$

and therefore from (3.13) we infer

$$\text{(3.14)}\quad \begin{cases} \langle v'(t), \xi\rangle_0 + \langle FP[g(v(t) + a(t))], \xi\rangle_0 = \langle P[f(t) + h_0(t)] + FPh_0(t), \xi\rangle_0 \\ \qquad\qquad\qquad\qquad \text{for all } \xi \in X \text{ and a.e. } t \in J\,. \end{cases}$$

Hence v is the solution of (3.9) on J satisfying $v(t_0) = Pu_0$, with $a(t)$, φ^t, a_0 and f^* as specified.

Conversely, if v is the solution of (3.9) on J satisfying the initial condition $v(t_0) = Pu_0$, with $a(t)$, φ^t, a_0 and f^* specified as above, then for $u := v + a$ and for all $\xi \in Y$ we have

$$\begin{aligned} \langle u'(t), \xi\rangle &= \langle v'(t), P\xi\rangle_0 + a'(t)\int_\Omega \xi(x)dx \\ &= -\langle FP[g(v(t) + a(t))], P\xi\rangle_0 + \langle P[f(t) + h_0(t)] + FPh_0(t), P\xi\rangle_0 \\ &\quad + \frac{1}{|\Omega|}\int_\Omega (f(t,x) + h_0(t,x))dx \int_\Omega \xi(x)dx \\ &= -a(g(u(t)), \xi) + a(h_0(t), \xi) + (f(t) + h_0(t), \xi) \\ &= -a(g(u(t)), \xi) + \int_\Gamma h(t,\sigma)\xi(\sigma)d\sigma + (f(t), \xi) \end{aligned}$$

for a.e. $t \in J$.

Therefore we obtain (2.5). Since $u(t_0) = v(t_0) + a(t_0) = u_0$ by definition, we conclude with the help of Proposition 2.4 that u is a solution of (1.1)–(1.2) on J satisfying $u(t_0) = u_0$.

As an immediate consequence of Proposition 3.4 we now give:

Proof of Proposition 2.5. As a consequence of Lemma 3.3, for any $v_0 \in H$ and any compact interval $J = [t_0, t_1] \subset I$ there exists a unique function $v \in W^{1,2}(J; X^*) \cap L^\infty(J; L^2(\Omega))$, with $v(t) \in H$ for all $t \in J$ and $g(v(t) + a(t)) \in Y$ for a.e. $t \in J$, which satisfies (3.14) and (3.10). The conclusion then follows immediately from Proposition 3.4.

4. A boundedness theorem in $L^2(\Omega)$

In this paragraph, we take $I = [t_0, +\infty)$ and we assume that f, h satisfy the following conditions:

$$\text{(4.1)}\quad f \in L^2_{\text{loc}}(I; L^2(\Omega)) \qquad \text{with } \sup\left\{\int_t^{t+1}\int_\Omega |f|^2 dx d\tau;\, t \in I\right\} := \mathbf{F}^2 < +\infty\,.$$

(4.2) $h \in L^2_{\mathrm{loc}}(I; L^2(\Gamma))$ with $\sup\left\{\int_t^{t+1}\int_\Gamma |h|^2 d\sigma d\tau;\ t \in I\right\} := \mathbf{H}^2 < +\infty$.

We shall give a direct proof of the following boundedness theorem.

Theorem 4.1. *Assume that f and h satisfy the boundedness conditions* (4.1)–(4.2) *in addition to the non-resonance condition* (2.8). *Then any solution u of* (1.1)–(1.2) *on I in the sense of Definition* 2.2 *satisfies*

(4.3) $$u: I \to L^2(\Omega) \text{ is bounded}.$$

Proof. First we define $w \in W^{1,2}_{\mathrm{loc}}(I; X)$ by the formula

(4.4) $$w(t) = F^{-1}Pu(t) = F^{-1}[u(t) - a(t)] \qquad \text{with } a(t) = \frac{1}{|\Omega|}\int_\Omega u(t, x)dx .$$

We prove the theorem in three steps (1), (2) and (3).

(1) As a preliminary step we show that

(4.5) $$w: I \to X \text{ is bounded}.$$

Indeed, by (3.3), (3.4) and (3.5) we have the obvious sequence of formulas:

$$\frac{d}{dt}\left[\frac{1}{2}|w(t)|_X^2\right] = a(w(t), w'(t)) = \langle Fw'(t), w(t)\rangle_0$$
$$= \langle u'(t) - a'(t), w(t)\rangle_0 = \langle u'(t), w(t)\rangle .$$

On the other hand, by applying (2.5) with $\xi = w(t)$ we find

$$\langle u'(t), w(t)\rangle + a(g(u(t)), w(t)) = \langle f^*(t), w(t)\rangle \qquad \text{for a.e. } t \in I ,$$

where f^* is given by (3.12). Now we have by (3.5), (3.3), (3.4) and (4.4)

$$a(g(u(t)), w(t)) = \langle u(t) - a(t), Pg(u(t))\rangle_0 = (u(t) - a(t), Pg(u(t)))_H$$
$$= (u(t) - a(t), g(u(t))) \geq \int_\Omega G(u(t, x))dx - \int_\Omega G(a(t))dx .$$

Therefore, for a.e. $t \in I$ we have

(4.6) $$\frac{d}{dt}\left[\frac{1}{2}|w(t)|_X^2\right] + \int_\Omega G(u(t, x))dx \leq |\Omega|\, G(a(t)) + \langle f^*(t), w(t)\rangle .$$

Here, as a consequence of the various hypotheses on f, g, h, note that

$$\int_\Omega G(u(t, x))dx \geq \eta_1 |u(t)|^2_{L^2(\Omega)} - K_1 \geq \eta_2 |u(t) - a(t)|_H^2 - K_2$$

and

$$|u(t) - a(t)|_H = |Fw(t)|_H \geq \eta_3 |Fw(t)|_{X^*} = \eta_3 |w(t)|_X ,$$

where η_1, η_2, η_3 and K_1, K_2 are suitable positive constants. On account of these inequalities, from (4.6) it is straightforward to deduce

$$\frac{d}{dt}\left[\frac{1}{2}|w(t)|_X^2\right] \leq -\delta |w(t)|_X^2 + \mathbf{H}_1(t) \qquad \text{for a.e. } t \in I , \tag{4.7}$$

where δ is a positive constant and $\mathbf{H}_1(t)$ is a positive locally integrable function on I such that

$$\int_t^{t+1} \mathbf{H}_1(s)ds \leq D(1 + \mathbf{F}^2 + \mathbf{H}^2) < +\infty \qquad \text{for all } t \in I , \tag{4.8}$$

where $D > 0$ is a positive constant. Then (4.5) follows at once from (4.7)–(4.8).

(2) By integrating both sides of (4.6) over $[t, t + 1]$, it follows from (4.5) and (4.6) that for some positive constant M we have the following inequality for $v := u - a$:

$$\int_t^{t+1} |v(s)|_H^2 ds \leq M \qquad \text{for all } t \in I , \tag{4.9}$$

which already strongly suggests that alleged boundedness property of $u(t)$.

(3) As a consequence of (4.9), for any integer $n \geq t_0$ we can find $\theta_n \in [n, n + 1]$ such that $|v(\theta_n)|_H^2 \leq M$. Then boundedness of $v: I \to H$ follows as a straightforward consequence of the estimate (3.11). Of course (4.3) then is obtained at once.

Remark 4.2. Since H is compactly imbedded in X^*, Theorem 4.1 implies in particular precompactness in X^* of the range of any positive trajectory of (3.9). Since an information will be quite useful to study the asymptotic behavior of solutions by means of the general methods developed in Haraux [15–17], Ishii [19] and Kenmochi-Otani [23, 24], when the functions f and h are almost periodic in t.

5. Periodic, almost periodic solutions and their stability

First of all we consider the case where f, h are both T-periodic for some positive number T, which means that they are defined and measurable respectively on $\boldsymbol{R} \times \Omega$ and $\boldsymbol{R} \times \Gamma$, and

(5.1) $f(t + T, x) = f(t, x)$ a.e. on $\boldsymbol{R} \times \Omega$; $h(t + T, x) = h(t, x)$ a.e. on $\boldsymbol{R} \times \Gamma$.

Definition 5.1. A function $u: \boldsymbol{R} \times \Omega \to \boldsymbol{R}$ is called a T-periodic solution of (1.1)–(1.2), if u is a solution of (1.1)–(1.2) on $\boldsymbol{R}$ in the sense of Definition

2.2 and u satisfies

$$u(t+T, x) = u(t, x) \qquad \text{a.e. on } \boldsymbol{R} \times \Omega .$$

Remark 5.2. In the sequel we always assume that

$$f \in L^2((0, T) \times \Omega); \qquad h \in L^2((0, T) \times \Gamma). \tag{5.2}$$

In these conditions a solution u of (1.1)–(1.2) on $\boldsymbol{R}$ is T-periodic if and only if we have

$$u(T) = u(0) \qquad \text{in the sense of } Y^* .$$

In section 6 we shall establish the following theorem.

Theorem 5.3. *Assume that f, h satisfy* (5.1) *and* (5.2). *Then there exists a T-periodic solution of* (1.1)–(1.2) *if and only if*

$$\int_0^T \int_\Omega f(s, x)dxds + \int_0^T \int_\Gamma h(s, \sigma)d\sigma ds = 0 . \tag{5.3}$$

In such a case we have the following results (1) $\sim$ (4):

(1) *For each $a_0 \in \boldsymbol{R}$, there exists a T-periodic solution u of* (1.1)–(1.2) *such that*

$$\frac{1}{|\Omega|}\int_\Omega u(0, x)dx = a_0 .$$

(2) *Let u be a solution of* (1.1)–(1.2) *on $\boldsymbol{R}$. Then u is T-periodic if and only if $u \in L^\infty(\boldsymbol{R}; L^2(\Omega))$.*

(3) *Let u_1, u_2 be two T-periodic solutions of* (1.1)–(1.2) *such that*

$$\int_\Omega u_1(0, x)dx = \int_\Omega u_2(0, x)dx .$$

Then we have

$$g(u_1) = g(u_2) \qquad \text{a.e. on } \boldsymbol{R} \times \Omega$$

and there exists a function $\zeta \in H$ such that

$$u_1(t, x) - u_2(t, x) = \zeta(x) \qquad \text{a.e. on } \Omega \text{ and for all } t \in \boldsymbol{R} .$$

(4) *For any solution u of* (1.1)–(1.2) *on $[t_0, +\infty)$, there exists a T-periodic solution ω of* (1.1)–(1.2) *such that*

$$u(t) - \omega(t) \in H \qquad \text{for all } t \geq t_0 , \tag{5.4}$$

$$u(t) - \omega(t) \to 0 \qquad \text{weakly in } L^2(\Omega) \text{ as } t \to +\infty . \tag{5.5}$$

In the same direction we have:

Theorem 5.4. *Assume that f is an S^2-almost periodic function from $\boldsymbol{R}$ into $L^2(\Omega)$ and h is an S^2-almost periodic function from $\boldsymbol{R}$ into $L^2(\Gamma)$. Then there exists a solution u of* (1.1)–(1.2) *on $\boldsymbol{R}$ which is bounded in $L^2(\Omega)$ on $\boldsymbol{R}$ if and only if*

$$(5.6) \qquad \int_0^t \int_\Omega f(s, x)dxds + \int_0^t \int_\Gamma h(s, \sigma)d\sigma ds \qquad \text{is bounded in } t \in \boldsymbol{R}.$$

In such a case we have the following results (1) ~ (4):

(1) *For each $a_0 \in \boldsymbol{R}$ there exists a solution u of* (1.1)–(1.2) *such that u is weakly almost periodic as a function from $\boldsymbol{R}$ into $L^2(\Omega)$ and $(1/|\Omega|)\int_\Omega u(0, x)dx = a_0$.*

(2) *Let u be a solution of* (1.1)–(1.2) *on $\boldsymbol{R}$. Then u is weakly almost periodic as a function from $\boldsymbol{R}$ into $L^2(\Omega)$ if and only if $u \in L^\infty(\boldsymbol{R}; L^2(\Omega))$.*

(3) *Let $u_1: \boldsymbol{R} \to L^2(\Omega)$ and $u_2: \boldsymbol{R} \to L^2(\Omega)$ be two weakly almost periodic solutions of* (1.1)–(1.2) *on $\boldsymbol{R}$ such that $\int_\Omega u_1(0, x)dx = \int_\Omega u_2(0, x)dx$. Then there exists $\zeta \in H$ such that $u_1(t) - u_2(t) = \zeta$ in H for all $t \in \boldsymbol{R}$ and $g(u_1) = g(u_2)$ a.e. on $\boldsymbol{R} \times \Omega$.*

(4) *For any solution u of* (1.1)–(1.2) *on $[t_0, +\infty)$, there exists a weakly almost periodic solution $\omega: \boldsymbol{R} \to L^2(\Omega)$ of* (1.1)–(1.2) *such that $u(t) - \omega(t) \in H$ for all $t \geq t_0$ and $u(t) - \omega(t) \to 0$ weakly in $L^2(\Omega)$ as $t \to +\infty$.*

6. Proofs of the results in section 5

The assertions of Theorems 5.3 and 5.4 are obtained as direct applications of the abstract results of Kenmochi-Otani [23, 24] concerning asymptotics as $t \to +\infty$ in the framework of problem (3.9).

For the moment, let us assume that f, h satisfy the conditions of Theorem 5.3. Let a_0, t_0 be two real numbers, $h_0: \boldsymbol{R} \to Y$ the function defined by (3.1) and $a: \boldsymbol{R} \to \boldsymbol{R}$ the function defined by (3.2). Let φ^t be the function on X^* defined by (3.6) for each $t \in \boldsymbol{R}$ and $f^* := P[f(t) + h_0(t)] + FPh_0(t)$ for a.e. $t \in \boldsymbol{R}$. By assumptions (5.1), (5.2) and (5.3), f^* is T-periodic on $\boldsymbol{R}$ as well as a, so that φ^t is T-periodic on $\boldsymbol{R}$, i.e. $\varphi^{t+T} = \varphi^t$ on X^* for all $t \in \boldsymbol{R}$. Next, by Proposition 3.4 and Theorem 4.1, there is a solution w of (3.9) on $I = [t_0, +\infty)$ such that

$$\{w(t); t \geq t_0\} \text{ is bounded in } L^2(\Omega) \text{ (hence precompact in } X^*).$$

Therefore it is possible to apply [23; Lemma 5 and Theorems 1, 2, 3] by Lemmas 3.1 and 3.2, and we obtain the following statements:

(i) There is a solution v of (3.9) on $\boldsymbol{R}$ with $v(\boldsymbol{R}) := \{v(t); t \in \boldsymbol{R}\}$ precompact in X^*.

(ii) For a solution v of (3.9) on $\boldsymbol{R}$, $v(\boldsymbol{R})$ is precompact in X^* if and only if v is a T-periodic function.

(iii) For any solution v of (3.9) on any inerval I of the form $[t_0, +\infty)$, there exists a T-periodic solution w of (3.9) on $\boldsymbol{R}$ such that $|v(t) - w(t)|_{X^*} \to 0$ as $t \to +\infty$.

Proof of Theorem 5.3. The first assertion and the points (1), (2) of the theorem follow easily as a consequence of the above facts (i), (ii) with the help of Lemma 2.8 and Proposition 3.4.

To obtain (3) it is sufficient to notice that as a standard consequence of the singlevalued character of $\partial\varphi^t$ (cf. [14], [24]) the difference $u_1(t, \cdot) - u_2(t, \cdot)$ is independent of t, and it follows that $g(u_1) - g(u_2)$ does not depend on x for each $t \in \boldsymbol{R}$. Because, by Lemma 2.8 there is $\zeta \in H$ such that $u_1(t) - u_2(t) = \zeta$ for all $t \in \boldsymbol{R}$, so that

$$\int_\Omega [g(u_1(t, x)) - g(u_2(t, x))][u_1(t, x) - u_2(t, x)]dx = 0 \qquad \text{for all } t \in \boldsymbol{R},$$

which shows that $g(u_1) = g(u_2)$ a.e. on $\boldsymbol{R} \times \Omega$, since g is monotone non-decreasing.

Finally we establish (4). Let u be any solution of (1.1)–(1.2) on $I = [t_0, +\infty)$ and let $v(t) = u(t) - a(t)$ with $a(\cdot)$ defined by (3.2) and $a_0 = (1/|\Omega|) \times \int_\Omega u(t_0, x)dx$. By Proposition 3.4, v is a solution of (3.9) on I. Using (iii), we can find a T-periodic solution w of (3.9) on $\boldsymbol{R}$ such that

$$|v(t) - w(t)|_{X^*} \to 0 \qquad \text{as } t \to +\infty . \tag{6.1}$$

Clearly, the function $\omega(t) := w(t) + a(t)$ is a T-periodic solution of (1.1)–(1.2) on $\boldsymbol{R}$ satisfying (5.4). Finally, since $u(I)$ and $\omega(\boldsymbol{R})$ are both bounded subsets of $L^2(\Omega)$, it follows from (6.1) that (5.4) holds. Thus we obtain (4).

Proof of Theorem 5.4. Let us now consider the almost periodic case. We assume that f, h satisfy the conditions of Theorem 5.4, and we use the same notations for $a(t)$, φ^t and $f^*(t)$ as in the periodic case. By the assumptions on f and h, f^* is an S^2-almost periodic function from $\boldsymbol{R}$ into X^* and as a consequence of (5.6), we also obtain that $a(\cdot)$ is an almost periodic function from $\boldsymbol{R}$ into $\boldsymbol{R}$ (cf. [1; Chapter 4, p. 53]) so that φ^t is almost periodic in $t \in \boldsymbol{R}$ in the sense of [24: section 6]. Just as in the periodic case, we now deduce from the general results of [24; Theorems 6.1, 7.1–7.3] that

(i) There is a solution v of (3.9) on $\boldsymbol{R}$ with $v(\boldsymbol{R})$ precompact in X^*.

(ii) For a solution v of (3.9) on $\boldsymbol{R}$, $v(\boldsymbol{R})$ is precompact in X^* if and only if v is almost periodic from $\boldsymbol{R}$ into X^*.

(iii) For any solution v of (3.9) on any interval of the form $[t_0, +\infty)$, there exists an almost periodic solution $w: \boldsymbol{R} \to X^*$ of (3.9) such that

$$|v(t) - w(t)|_{X^*} \to 0 \qquad \text{as } t \to +\infty .$$

Using these facts, we can deduce Theorem 5.4 in a way similar to the periodic case.

Remark 6.1. Since the dynamical process generated by (3.9) is defined on a time-independent set (actually X^*) here, it is also possible to use the main result of Ishii [19] in order to derive the first result of Theorem 5.4.

7. Additional remarks

In this section, we collect together some additional information on the solutions of (1.1)–(1.2).

(a) *The case of a bi-Lipschitz continuous function g*

When g is increasing and bi-Lipschitz continuous on $\boldsymbol{R}$, it is natural to deal with problem (1.1)–(1.2) in the space $L^2(\Omega)$. Indeed, under the condition

$$h \in W^{1,2}_{\mathrm{loc}}(I; L^2(\Gamma)), \qquad I = [t_0, +\infty),$$

it has been established in [21; Chapter 2] that problem (1.1)–(1.2)–(2.6) has a unique solution $u \in W^{1,2}_{\mathrm{loc}}(I; L^2(\Omega)) \cap L^\infty_{\mathrm{loc}}(I; Y)$ for each initial datum $u_0 \in Y$, and the solution u satisfies the following estimate:

$$\text{(7.1)} \qquad k_1 \int_t^{t+\tau} (s-t)|g(u)'(s)|^2_{L^2(\Omega)} ds + \tau |\nabla g(u(t+\tau))|^2_{L^2(\Omega)}$$
$$\le L(\tau, |g(u(t))|_{L^2(\Omega)}, |f|_{L^2(t,t+\tau; L^2(\Omega))}, |h|_{W^{1,2}(t,t+\tau; L^2(\Gamma))})$$

for any $\tau > 0$ and $t \ge t_0$, where L is a nondecreasing function in its arguments, and k_1 is a positive constant depending only on the Lipschitz constant of g. The uniqueness of u, which is established in the space $L^1(\Omega)$, is essentially due to Bénilan [4; Chapter 2].

Moreover, assume that

$$\text{(7.2)} \qquad \sup\{|h'|_{L^2(t,t+1; L^2(\Gamma))};\, t \ge t_0\} < +\infty$$

and

$$\text{(7.3)} \qquad \sup\{|f|_{L^2(t,t+1; L^2(\Omega))};\, t \ge t_0\} < +\infty .$$

Then we obtain from Theorem 4.1 that $u \in L^\infty(I; L^2(\Omega))$, hence

$$\text{(7.4)} \qquad g(u) \in L^\infty(I; L^2(\Omega)) .$$

Therefore, from (7.4) and (7.1) we deduce

(7.5) $$g(u) \in L^\infty(I; Y).$$

If g is bi-Lipschitz continuous on $\boldsymbol{R}$, then we conclude from (7.5) that

$$u \in L^\infty(I; Y).$$

In particular, all the convergence results of section 5 then occur in the weak topology of Y, hence in the strong topology of $L^2(\Omega)$. Also, since g is increasing, we have one and only one periodic (respectively, almost periodic) solution with prescribed initial mean-value.

(b) *Further boundedness properties*

Under the hypotheses of Theorem 4.1, it is rather easy to check that any solution u of (1.1)–(1.2) satisfies

(7.6) $$g(u) \in L^2_{\text{loc}}([t_0, +\infty); Y) \text{ with } \sup\left\{\int_t^{t+1}\int_\Omega |\nabla g(u)|^2 dx d\tau;\ t \geq t_0\right\} < +\infty .$$

Since $g(u) \in L^\infty([t_0, +\infty); L^2(\Omega))$ by Theorem 4.1, formally the bound (7.6) follows immediately by multiplying the equation by $g(u)$ and integrating over $(t, t+1) \times \Omega$. In addition, if (7.2) and (7.3) are fulfilled, then we obtain (7.5). In fact, both estimates (7.6) and (7.5) can be justified rigourously by approximating h by smooth functions and g by bi-Lipschitz continuous increasing functions and using the results of Damlamian [8; section 3] or Damlamian-Kenmochi [9; section 4]. In the general case when g is not invertible, this does not give us any precompactness property of trajectories in $L^2(\Omega)$. At least we obtain from assumption (2.2) that

$$u \in L^\infty([t_0, +\infty); L^p(\Omega)),$$

where $p = +\infty$ if $N = 1$, $1 \leq p < +\infty$ if $N = 2$, $1 \leq p \leq 2^* = 2N/(N-2)$ if $N > 2$.

(c) *The set of periodic solutions is in general non convex*

As already mentioned once and for all just above Lemma 2 in section 2, this circumstance precludes the possibility of studying all solutions of (1.1)–(1.2) in the same monotone framework without using the "fibration" procedure which relies essentially on formula (2.7). Even if f and h are time-independent and g is a bi-Lipschitz continuous function, the set of all periodic solutions is in general non-convex, as will be shown by the following example.

Example 7.1. Assume that g is given by the formula

$$g(r) = \begin{cases} c_1(r-1) & \text{for } r \geq 1, \\ 0 & \text{for } 0 < r < 1, \\ c_2 r & \text{for } r \leq 0, \end{cases}$$

for some positive constants c_1, c_2 and let $\Omega = \{x \in \boldsymbol{R}^N; |x| < 1\}$. It is readily verified that the two functions u_1, u_2 defined by

$$u_1(x) = |x|^2, \qquad u_2(x) = 2,$$

are two solutions of the problem

$$w \in H^2(\Omega), \qquad -\Delta g(w) = 0 \text{ in } \Omega, \qquad \partial_\nu g(w) = 0 \text{ on } \Gamma.$$

However the function $u = (u_1 + u_2)/2$ is not a solution of this problem, because $g(u(x)) = (c_1/2)|x|^2$ is not constant.

Actually the convexity of the set of stationary solutions can fail even when g is increasing and bi-Lipschitz continuous on $\boldsymbol{R}$, as shown by the slightly more complicated following example.

Example 7.2. Assume that g is given by the formula

$$g(r) = \begin{cases} c_2(r-1) + c_1 & \text{for } r \geq 1, \\ c_1 r & \text{for } 0 < r < 1, \\ \gamma(r) & \text{for } r \leq 0, \end{cases}$$

for some positive constants c_1, c_2 with $c_1 \neq c_2$ and some nondecreasing Lipschitz continuous function $\gamma(r)$ with $\gamma(0) = 0$. Let Ω be any bounded open subset of $\boldsymbol{R}^N$ with smooth boundary $\Gamma = \partial\Omega$ and consider a non constant smooth function φ on $\boldsymbol{R}^N$ such that for some $\varepsilon > 0$

$$0 \leq \varphi \leq 1 - \varepsilon \text{ on } \boldsymbol{R}^N, \qquad \partial_\nu \varphi = 0 \text{ on } \Gamma.$$

Then, $-\Delta g(\varphi) = -c_1 \Delta\varphi$ in Ω. On the other hand, if we set

$$\psi(x) := 1 + k\varphi(x) \qquad \text{with } k = c_1/c_2,$$

then obviously $\partial_\nu \psi = 0$ on Γ and we obtain

$$-\Delta g(\psi) = -c_1 \Delta\varphi \qquad \text{in } \Omega,$$

since $g(\psi(x)) = c_2 k\varphi(x) + c_1 = c_1(1 + \varphi(x))$. Therefore, φ and ψ are two solutions of the same problem:

$$w \in H^2(\Omega), \qquad -\Delta g(w) = -c_1 \Delta\varphi \text{ in } \Omega, \qquad \partial_\nu g(w) = 0 \text{ on } \Gamma.$$

Now we claim that for each $s > 0$ small enough the function $u = (1-s)\varphi + s\psi$ is not a solution of this problem. Indeed, if it were a solution, then the function $g(u) - g(\varphi)$ should be constant on Ω. On the other hand for each $s > 0$ small enough we have $u \leq 1$ in Ω, therefore

$$g(u(x)) - g(\varphi(x)) = c_1(u(x) - \varphi(x)) = sc_1(\psi(x) - \varphi(x)) = sc_1[1 + (k-1)\varphi(x)].$$

Since $c_1 \neq c_2$ (hence $k \neq 1$) and φ is not constant, this is impossible. Thus the set of stationary solutions is not convex. If for instance $\gamma(r) = cr$, $c > 0$, then g is increasing and bi-Lipschitz continuous on $\boldsymbol{R}$.

(d) *The difference of two arbitrary periodic solutions may in general depend on t*

Point (3) of Theorem 5.3 asserts that the difference of two T-periodic solutions of (1.1)–(1.2) having the same initial mean-value is independent of t. It is a natural question to ask whether this is also true for two arbitrary T-periodic solutions. The answer turns out to be negative even when g is increasing and bi-Lipschitz continuous on $\boldsymbol{R}$, as shown by the example below.

Example 7.3. Let Ω, g, φ be as in Example 7.2 with $c_1 \geq 0$, $c_2 > c_1$, and assume in addition that $\varphi(p) = 0$ for a certain $p \in \Omega$. Consider the time π-periodic function $\omega(t, x) := [\sin^2 t]\varphi(x)$ on $\boldsymbol{R} \times \Omega$ and the parabolic problem:

$$u_t - \Delta g(u) = f(t, x) \text{ in } \boldsymbol{R} \times \Omega\,, \qquad \partial_\nu g(u) = 0 \text{ on } \boldsymbol{R} \times \Gamma\,,$$

where $f(t, x) := 2[\sin t \cdot \cos t]\varphi(x) - c_1[\sin^2 t]\Delta\varphi(x)$ in $\boldsymbol{R} \times \Omega$. Then there exist infinitely many time π-periodic solutions of this problem whose difference with the particular solution $\omega(t, x)$ is not independent of t.

Proof. Clearly ω is a solution of the problem. If $w(t, x) := \omega(t, x) + \lambda(x)$ is a π-periodic solution of this problem, then automatically we have

$$g(w(t, x)) - g(\omega(t, x)) =: y(t) \qquad \text{a.e. on } \boldsymbol{R} \times \Omega\,.$$

By letting $x = p$ in this formula we get $y(t) = g(\lambda(p))$, therefore y is in fact independent of t. Also, by letting $t = 0$ in the formula we obtain

$$g(\lambda(x)) = g(\lambda(p)) := y_0 \qquad \text{a.e. on } \Omega\,. \tag{7.7}$$

Now, by Theorem 5.3, there are infinitely many π-periodic solutions with arbitrarily large positive integral on Ω at $t = 0$. For such a solution w we obtain, assuming it is of the above form, the inequality $y_0 > c_1$. Then the inequality $y_0 > c_1$ together with (7.7) implies that

$$\lambda(x) = \lambda^* \qquad \text{a.e. on } \Omega \text{ for some constant } \lambda^* > 1\,. \tag{7.8}$$

Therefore $w(t, x) = \omega(t, x) + \lambda^*$ a.e. on $\boldsymbol{R} \times \Omega$. This is easily seen to contradict the formula

$$g(w(t, x)) - g(\omega(t, x)) = y_0 \qquad \text{a.e. on } \boldsymbol{R} \times \Omega\,, \tag{7.9}$$

because we assumed $c_1 < c_2$ and λ^* can be taken so as to be larger than 1. Indeed, it follows from (7.8) and (7.9) that $y_0 = g(w(t, x)) - g(\omega(t, x)) = c_2(\omega(t, x) + \lambda^* - 1) + c_1 - c_1\omega(t, x) = (c_2 - c_1)\omega(t, x) + c_2(\lambda^* - 1) +$

$c_1 \neq$ const. on $\boldsymbol{R} \times \Omega$, which is a contradiction. We conclude that any π-periodic solution w, with sufficiently large integral $\int_{\Omega} w(0, x) dx$, must differ from ω by a function which is not independent of t.

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