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Integral Inequalities of Bihari-Type and Applications

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1. Introduction

Differential and integral inequalities play a vital role in the study of existence, uniqueness, boundedness, stability and other qualitative properties of solutions of differential and integral equations ([1], [4], [7–8], [10], [14], [21–22]). This is the case of the attractive Gronwall-Bellman inequality [1]. Bihari's inequality is the most important nonlinear generalization of the Gronwall-Bellman inequality. Several semi-linear integral inequalities of Gronwall-Bellman-Bihari type have been obtained ([5], [12–13], [15]). However, at present, the result for integral inequalities with several nonlinear summands are only partial ([18], [19]).

The aim of this paper is to show a different approach for the "resolution" of some integral inequalities with several nonlinear terms which include the polynomial expressions and to apply these results to the problem of existence of polynomial asymptotic solutions for an n-th order nonlinear differential equation.

In section 2, we "resolve" nonlinear integral inequalities with several nonlinearities as, for instance,

$$u(t) \le c + \sum_{i=1}^{n} \int_a^t \lambda_1(s)\omega_i(u(s))\,ds \qquad \text{(see Ths. 1 and 2)}$$

or

$$u(t) \le h(t) + f(t)\sum_{i=1}^{n} f_i(t) \int_a^t \lambda_i(s)\omega_i(u(s))\,ds$$

(see Th. 3, Cor. 1 and 2)

or

$$u(t) \le c + \int_a^t \lambda_1(s)\omega_1(u(s))\,ds + \int_a^t \lambda_2(s)\omega_2\left(\int_a^s \lambda_3(\tau)\omega_3(u(\tau))\,d\tau\right)ds$$

(see Th. 4)

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etc.

These integral inequalities, as Bihari's or Gronwall-Bellman's inequalities [1], [4], [7], [8] are fundamental in the study of extension, stability and asymptotic behavior of the solutions of the differential equations ([6], [14], [15]).

In section 3 we apply some integral inequalities to obtain conditions for a nonlinear scalar equation

$$y^{(n)} + f(t, y, y' \dots, y^{(n-1)}) = 0 \quad \text{(see Ths. 5 and 6)}$$

to have solutions which behave like solutions of $x^{(n)} = 0$ as $t \to \infty$.

2. Nonlinear integral inequations

In this section nonlinear integral inequalities with several nonlinearities are obtained. We begin with two nonlinearities:

Lemma 1. *Let* ω_1, $i = 1, 2$ *be continuous and nondecreasing functions on* $[0, \infty)$ *and positive on* $(0, \infty)$ *such that* $\omega = \omega_2/\omega_1$ *is nondecreasing on* $(0, \infty)$ (*see remark* 1). *Let* $u, \lambda_i, i = 1, 2,$ *be continuous and nonnegative functions on* $[a, b]$ *and* $c > 0$ *a constant. If*

$$u(t) \le c + \int_a^t \lambda_1(s)\omega_1(u(s))\,ds + \int_a^t \lambda_2(s)\omega_2(u(s))\,ds \tag{1}$$

then, for $t \in [a, b_1]$,

$$u(t) \le W_2^{-1}\left[W_2(c_1) + \int_a^t \lambda_2(s)\,ds\right], \tag{2}$$

where

$$W_1(u) = \int_c^u \frac{dz}{\omega_1(z)},\ u > 0;\ W_2(u) = \int_{u_0}^u \frac{dz}{\omega_2(z)},\ u > 0,\ u_0 > 0 \tag{3}$$

and b_1 *is the largest number such that* $b_1 \ge a$ *and for* $i = 1, 2$:

$$\|\lambda_i\|_{b_1} := \int_a^{b_1} \lambda_i(s)\,ds \le \int_{c_{i-1}}^{\infty} \frac{dz}{\omega_i(z)},\quad c_0 = c, \tag{4}$$
$$c_1 = W_1^{-1}(\|\lambda_1\|_{b_1}).$$

Proof. Let $z_1(t) = \int_a^t \lambda_1(s)\omega_1(u(s))\,ds$, $z_2(t) + c + \int_a^t \lambda_2(s)\omega_2(u(s))\,ds$. Then

$$z_1' + z_2' = \lambda_1\omega_1(u) + \lambda_2\omega_2(u) \le \lambda_1\omega_1(z_1 + z_2) + \lambda_2\omega_2(z_1 + z_2).$$

Thus, $z = z_1 + z_2$ satisfies

$$\frac{d}{dt}(W_1(z)) = \frac{z'}{\omega_1(z)} \leq \lambda_1 + \lambda_2\omega(z),$$

where $\omega = \omega_2/\omega_1$. Hence, by integrating on $[a, t]$, $t \leq b_1$, we have

$$W_1(z(t)) \leq W_1(z(a)) + \int_a^t \lambda_1(s)\,ds + \int_a^t \lambda_2(s)\omega(z(s))\,ds$$

$$\leq \tilde{c} + \int_a^t \lambda_2(s)\omega(z(s))\,ds, \quad \tilde{c} = \int_a^{b_1} \lambda_1(s)\,ds.$$

Then, putting $v(t) = W_1(z(t))$, we obtain

$$v(t) \leq \tilde{c} + \int_a^t \lambda_2(s)\omega(W_1^{-1}(v(s)))\,ds, \quad \tilde{c} = \int_a^{b_1} \lambda_1(s)\,ds$$

and, using the Bihari's inequality [2], we get for $t \in [a, b_1]$

$$v(t) \leq \tilde{W}^{-1}\left[\tilde{W}(\tilde{c}) + \int_a^t \lambda_2(s)\,ds\right],$$

where

$$\tilde{W}(u) = \int_{\tilde{u}_0}^u \frac{dz}{\omega(W^{-1}(z))}, \quad \tilde{u}_0 = W_1(u_0)$$

Since $u \leq z = W_1^{-1}(v)$ and $\tilde{W} = W_2 \circ W_1^{-1}$ (the composition between W_2 and W_1^{-1}) the last inequality implies (2).

Notation. Let $A \subset \boldsymbol{R}$ be a set. For $\omega_1, \omega_2 : A \to \boldsymbol{R} - \{0\}$ two functions, we will denote $\omega_1 \propto \omega_2$ if ω_2/ω_1 is nondecreasing on A.

The relation $\propto$ is transitive, reflexive, and "almost" anti-symmetric because $\omega_1 \propto \omega_2$ and $\omega_2 \propto \omega_1$ implies $\omega_2 = c\,\omega_1$, c constant.

Remark 1. If in Lemma 1, $\omega_2 \propto \omega_1$ instead of $\omega_1 \propto \omega_2$ then the Lemma can be applied by permuting the roles of ω_1 and ω_2, and λ_2 and λ_1. In this case, the conclusion is

$$u(t) \leq W_1^{-1}[W_1(c_1) + \int_a^t \lambda_1(s)\,ds], \quad c_1 = W_2^{-1}[W_2(c) + \int_a^{b_1} \lambda_2(s)\,ds].$$

Obviously, Lemma 1 extends Bihari's inequality [2] to two nonlinearities. Really, we can obtain an inequality with any finite number of nonlinearities. The main result is the following:

Theorem 1. *Suppose the following two hypotheses.*

(H) *The functions* ω_i, $i = 1, \dots, n$ *are continuous and nondecreasing on* $[0, \infty)$ *and positive on* $(0, \infty)$ *such that* $\omega_1 \propto \omega_2 \propto \cdots \propto \omega_n$. *(see Remark 2).*

(H_1) *The functions* u, $\{\lambda_i\}_{i=1}^n$ *are continuous and nonnegative on* $I = [a, b]$ *and* c *is a positive constant.*

If

$$u(t) \le c + \sum_{i=1}^{n} \int_a^t \lambda_i(s)\omega_i(u(s))\,ds, \quad t \in [a, b] \tag{5}$$

then

$$u(t) \le W_n^{-1}\,[W_n(c_{n-1}) + \int_a^t \lambda_n(s)\,ds] \tag{6}$$

for $t \in [a, b_1]$, *where*

1)

$$W_k(u) = \int_{u_k}^{u} \frac{dz}{\omega_k(z)}, \quad u > 0, \quad u_k > 0, \quad k = 1, \dots, n \tag{7}$$

and W_k^{-1} *is the inverse function of* W_k.

2) *The constants* c_k *are given by* $c_0 = c$ *and*

$$c_k = W_k^{-1}\,[W_k(c_{k-1}) + \|\lambda_k\|_{b_1}], \quad k = 1, \dots, n-1. \tag{8}$$

3) *The number* $b_1 \in [a, b]$ *is the largest number such that*

$$\|\lambda_k\|_{b_1} := \int_a^{b_1} \lambda_k(s)ds \le \int_{c_{k-1}}^{\infty} \frac{dz}{\omega_k(z)}, \quad k = 1, \dots, n. \tag{9}$$

Proof. First, we note that, by (9), the constants c_k, $k = 1, \dots, n$ are well-defined. Now, the proof is by induction on n. Suppose that the result is valid for $n = m$. Let

$$u(t) \le c + \sum_{i=1}^{m+1} \int_a^t \lambda_i(s)\omega_i(u(s))\,ds, \quad t \in [a, b].$$

Put $v_i(t) = \int_a^t \lambda_i(s)\omega_i(u(s))\,ds$, $1 \le i \le m$, $v_{m+1}(t) = c + \int_a^t \lambda_{m+1}(s)\omega_{m+1}(u(s))\,ds$ and $z = \sum_{i=1}^{m+1} v_i$. Then $u \le z$ and

$$z' = \sum_{i=1}^{m+1} \lambda_i\omega_i(u) \le \sum_{i=1}^{m+1} \lambda_i\omega_i(z).$$

Hence

$$W_1(z)' = \frac{z'}{\omega_1(z)} \leq \lambda_1 + \sum_{i=2}^{m+1} \lambda_i \omega_{i,1}(z), \quad \omega_{i,1} = \omega_i/\omega_1$$

and, by integrating on $[a, t]$,

$$W_1(z(t)) \leq W_1(c) + \|\lambda_1\|_{b_1} + \sum_{i=2}^{m+1} \int_a^t \lambda_i(s)\omega_{i,1}(z(s))\,ds$$

$$= \tilde{c} + \sum_{i=1}^{m} \int_a^t \lambda_{i+1}(s)\omega_{i+1,1}(z(s))\,ds, \quad \tilde{c} = W_1(c) + \|\lambda_1\|_{b_1}.$$

Thus, $v_1 = W_1(z)$ satisfies

$$v_1(t) \leq \tilde{c} + \sum_{i=1}^{m} \int_a^t \lambda_{i+1}(s)(\omega_{i+1,1} \circ W_1^{-1})(v_1(s))\,ds, \tag{10}$$

where $\omega_{i+1,1} = \omega_{i+1}/\omega_1$ and $\tilde{c} = W_1(c) + \|\lambda_1\|_{b_1}$.

Define

$$\tilde{W}_{k+1}(u) = \int_{\tilde{u}_{k+1}}^{u} \frac{dz}{\omega_{k+1,1}(W_1^{-1}(z))}, \quad \tilde{u}_{k+1} = W_1(u_{k+1}), \quad k = 1, \ldots, m$$

Then, by induction, applied to λ_{i+1}, $\omega_{i+1} \circ W_1^{-1}$ and W_{i+1} instead of λ_i, ω_i and W_i, from (6) we have for $t \in [a, b_0]$

$$v_1(t) \leq \tilde{W}_{m+1}^{-1}\,[\tilde{W}_{m+1}(\tilde{c}_{m-1}) + \int_a^t \lambda_{m+1}(s)\,ds], \tag{11}$$

where $\tilde{c}_0 = \tilde{c} = W_1(c) + \|\lambda_1\|_{b_1}$,

$$\tilde{c}_k = \tilde{W}_k^{-1}\,[\tilde{W}_k(\tilde{c}_{k-1}) + \|\lambda_{k+1}\|_{b_0}], \quad k = 1, \ldots, m-1 \tag{12}$$

and b_0 is the largest number $b_0 \geq a$ such that

$$\|\lambda_{k+1}\|_{b_0} := \int_a^{b_0} \lambda_{k+1}(s)\,ds \leq \int_{\tilde{c}_{k-1}}^{\infty} \frac{dz}{\omega_{k+1,1}(W_1^{-1}(z))}. \tag{13}$$

But $\tilde{W}_{k+1} = W_{k+1} \circ W_1^{-1}$ (the composition between W_{k+1} and W_1^{-1}), $u \leq z = W_1^{-1}(v_1)$ and (11) implies

$$u \leq W_1^{-1}(v_1) \leq W_1^{-1}\,[W_1 \circ W_{m+1}^{-1}\,[W_{m+1}(W_1^{-1}(\tilde{c}_{m-1})) + \int_a^t \lambda_{m+1}(s)\,ds]]$$

(14)

$$= W_{m+1}^{-1}\,[W_{m+1}(\tilde{\tilde{c}}_{m-1}) + \int_a^t \lambda_{m+1}(s)\,ds],$$

where $\tilde{\tilde{c}}_{m-1} = W_1^{-1}(\tilde{c}_{m-1})$.

Now, denoting $\tilde{\tilde{c}}_k = W_1^{-1}(\tilde{c}_k)$, $k = 1, \dots, m-1$, by (12) we have $\tilde{\tilde{c}}_0 = W_1^{-1}[W_1(c) + \|\lambda_1\|_{b_1}]$ and

$$\tilde{\tilde{c}}_k = W_{k+1}^{-1}(\tilde{\tilde{c}}_{k-1}) + \|\lambda_{k+1}\|_{b_0}]. \tag{15}$$

Finally, since from (12) it is easy to verify that $b_0 = b_1$, then (15) implies $\tilde{\tilde{c}}_k = c_{k+1}$. In fact, $\tilde{\tilde{c}}_1 = W_2^{-1}[W_2(\tilde{c}_0) + \|\lambda_2\|_{b_1}] = W_2^{-1}[W_2(W_1^{-1}[W_1(c) + \|\lambda_1\|_{b_1}]) + \|\lambda_2\|_{b_1}] = c_2$ and if $\tilde{\tilde{c}}_m = c_{m+1}$ then, by (15),

$$\begin{aligned}\tilde{\tilde{c}}_{m+1} &= W_{m+1}^{-1}[W_{m+1}(\tilde{\tilde{c}}_m) + \|\lambda_{m+2}\|_{b_1}]\\ &= W_{m+1}^{-1}[W_{m+1}(c_{m+1}) + \|\lambda_{m+2}\|_{b_1}] = c_{m+2}.\end{aligned}$$

Thus, by (14),

$$u(t) \le W_{m+1}^{-1}[W_{m+1}(c_m) + \int_a^t \lambda_{m+1}(s)\,ds],$$

which proves the results for $n = m+1$. Now, the proof is complete.

Remark 2. On account of Remark 1 we have that Theorem 1 as well as the theorems that follow, can be applied when the "order" $\propto$ among ω_i, $i = 1, \dots, n$ has been precised.

Remark 3. If the inequality (9) is strict for $k = n$, then $c_n < \infty$ and $u(t) \le c_n$ for $t \in [a, b_1]$ (c_n given by (8) with $k = n$). Moreover, we have $c_0 \le c_1 \le \cdots \le c_n$ and $c_{i-1} < c_i$ if $\|\lambda_i\|_{b_1} \neq 0$. In fact, we have $c_{i-1} = W_i^{-1}(W_i(c_{i-1})) \le W_i^{-1}[W_i(c_{i-1}) + \|\lambda_i\|_{b_1}] = c_i$. Furthermore, $c_{i-1} = \infty$ implies $\|\lambda_i\|_{b_1} = 0$ and then $b_1 = a$ if $\lambda_i \neq 0$. Thus, (9) follows if

$$\int_a^{b_1} \lambda_k(s)\,ds \le \int_{c_{n-1}}^{\infty} \omega_k(z)^{-1}\,dz, \quad k = 1, \dots, n \tag{9'}$$

whose verification is easier than (9).

Remark 4. For any $k = 1, \dots, n$, let

$$\phi_k = \prod_{i=0}^{k-1} g_{k-i} = g_k \circ g_{k-1} \circ \cdots \circ g_1, \tag{16}$$

where

$$g_i(u) = W_i^{-1}[W_i(u) + \alpha_i], \quad \alpha_i = \int_a^{b_1} \lambda_i(s)\,ds, \quad i = 1, \dots, n.$$

The functions ϕ_k possess the following properties

i) Any ϕ_k, $k = 1, \dots, n$, is a continuous, positive and nondecreasing function on $(0, \infty)$, whose inverse function is given by

$$\phi_k^{-1} = \prod_{i=1}^{k} g_i^{-1}, \quad g_i^{-1}(v) = W_i^{-1}[W_i(v) - \alpha_i], \quad \alpha_i = \int_a^{b_1} \lambda_i(s)\, ds. \tag{17}$$

ii) Each ϕ_k (and g_k) does not depend on the choice of W_i, $i = 1, \dots, n$, that is, if we change u_i by $\tilde{u}_i$ in (7):

$$\tilde{W}_i(u) = \int_{\tilde{u}_i}^{u} \frac{dz}{\omega_i(z)}, \quad \tilde{u}_i > 0, \quad u > 0, \quad i = 1, \dots, n$$

then ϕ_k (and g_k) remains the same

$$\phi_k = \prod_{i=0}^{k-1} \tilde{g}_{k-i}, \; \tilde{g}_i(u) = \tilde{W}_i^{-1}[\tilde{W}_i(u) + \alpha_i], \; \alpha_i = \int_a^{b_1} \lambda_i(s)\, ds.$$

In fact, for any $k = 1, \dots, n$, we have $g_k = \tilde{g}_k$ because $\tilde{W}_i = W_i + \delta_i$, δ_i constant, $i = 1, \dots, n$.

iii) For any $k = 1, \dots, n$, the constants c_k in (8) are given by

$$c_k = \phi_k(c) \tag{18}$$

Thus, the result of Theorem 1, namely the constants c_k, $k = 1, \dots, n$, and (6) do not depend on the choice of W_i, $i = 1, \dots, n$.

(iv) If $W_i(0^+) = -\infty$, $i = 1, \dots, n$ then, for any $k = 1, \dots, n$, $\phi_k(u) \to 0$ as $u \to 0$.

Remark 5. Although (6) is not the best estimate, it ensures that u and all integrals in (5) are bounded, which is the most important and useful fact (see section 3 and [10], [15]).

Usually, if $\omega_0(u) = u$, $\omega_0 \propto \omega_i$ for all i then in Theorem 1 we could use $\omega_0 \propto \omega_1 \propto \cdots \propto \omega_n$, but, in this case, it is possible to establish a more precise version.

For that, we recall (Dannan [5]) that a function $\omega\colon [0, \infty) \to [0, \infty)$ belongs to the class H if

i) $\omega(u)$ is nondecreasing and continuous for $u \geq 0$ and positive for $u > 0$.

ii) There exists a function r (multiplier function), continuous on $[0, \infty)$ with $\omega(\alpha u) \leq r(\alpha)\omega(u)$ for $\alpha > 0$, $u \geq 0$.

Theorem 2. *Assume that u and λ_i, $i = 0, 1, \dots, n$ are continuous and nonnegative functions on $[a, b]$, $\omega_i \in H$, $i = 1, \dots, n$ with corresponding multiplier functions r_i, $i = 1, \dots, n$ such that $\omega_1 \propto \omega_2 \cdots \propto \omega_n$ and c is a positive constant. If*

$$u(t) \leq c + \int_a^t \lambda_0(s)u(s)ds + \sum_{i=1}^{n} \int_a^t \lambda_i(s)\omega_i(u(s))\,ds, \quad t \in [a, b]$$

then for all $t \in [a, b_1]$

$$u(t) \leq E(t) \cdot W_n^{-1}\left[W_n(c_{n-1}) + \int_a^t \lambda_n(s) \cdot \frac{r_n(E(s))}{E(s)}\,ds\right],$$

where the notations are the same as in Theorem 1 *by replacing* λ_i *by* $\lambda_i \cdot r_i(E)/E$ *with* $E(t) = \exp(\int_a^t \lambda_o(s)ds)$.

Proof. Put $z_0(t) + c + \int_a^t \lambda_0(s)u(s)ds$, $z_i(t) = \int_a^t \lambda_i(s)\omega_i(u(s))ds$, $i = 1, 2, \ldots, n$. Then $z = \sum_{i=0}^{n} z_i$ verifies

$$z' \leq \lambda_0 z + \sum_{i=1}^{n} \lambda_i \omega_i(z)$$

or, by denoting $E(t) = \exp(\int_a^t \lambda_o(s)ds)$,

$$(z/E)' \leq \sum_{i=1}^{n} (\lambda_i/E)\omega_i(z) \leq \sum_{i=1}^{n} (\lambda_i/E) \cdot r_i(E)\omega_i(z/E)$$

because $\omega_i \in H$, $i = 1, \ldots, n$.

Therefore, by integrating on $[a, t]$, the function $v = z/E$ verifies

$$v(t) \leq c + \sum_{i=1}^{n} \int_a^t \frac{\lambda_i(s) r_i(E(s))}{E(s)} \omega_i(v(s))\,ds$$

and the result follows by applying Theorem 1.

Theorem 3. *Let* u, λ_i, ω_i, $i = 1, \ldots, n$ *be as in Theorem* 1 *and suppose that* $\omega_i \in H$ *with corresponding multiplier function* r_i, $i = 1, \ldots, n$ *and* $h > 0$ *is a continuous and nondecreasing function on* $[a, b]$. *If*

$$u(t) \leq h(t) + \sum_{i=1}^{n} \int_a^t \lambda_i(s)\omega_i(u(s))\,ds, \quad t \in [a, b] \tag{19}$$

then, for $t \in [a, b_1]$,

$$u(t) \leq h(t)\, W_n^{-1}\, [W_n(c_{n-1}) + \int_a^t \lambda_n(s) r_n(h(s))\,ds]$$

where the notations are the same as in Theorem 1, *by replacing* λ_i *by* $\lambda_i \cdot r_i(h)$ *and* $c_o = 1$.

Proof. From (19) and the fact that $\omega_i \in H$ with corresponding multiplier function r_i, $i = 1, \ldots, n$, the function $z = u/h$ satisfies

$$z(t) \le 1 + \sum_{i=1}^{n} \int_a^t \lambda_i(s) r_i(h(s)) \omega_i(z(s))\, ds$$

and the result follows from Theorem 1.

Now, by proceeding in a similar way as in Dannan [5] Cor. 1, we obtain

Corollary 1. *Let u, λ_i, ω_i, $i = 1, \dots, n$ and h be as in the last Theorem and suppose $f_i(t)$, $i = 1, \dots, n$ are nonnegative, continuous and nondecreasing functions on $[a, b]$. If*

$$u(t) \le h(t) + \sum_{i=1}^{n} f_i(t) \int_a^t \lambda_i(s) \omega_i(u(s))\, ds, \quad t \in [a, b]$$

then, for $t \in [a, b_1]$,

$$u(t) \le h(t) W_n^{-1} [W_n(c_{n-1}) + f_n(t) \int_a^t \lambda_n(s) r_n(h(s))\, ds],$$

where the notations are the same as in Theorem 1, *by replacing λ_i by $f_i(b_1) \cdot \lambda_i \cdot r_i(h)$, $1 \le i \le n$ and $c_0 = 1$.*

Theorem 3 and Corollary 1 extend, respectively, Theorem 1 and Corollary 1 in [5]

Corollary 2. *Let u, λ_i, ω_i, r_i, $i = 1, \dots, n$ be as in Theorem* 1 *and f_i, $i = 1, \dots, n$ as in Corollary* 1. *Suppose that h and f are continuous and positive functions on $[a, b]$ such that $f \propto h$. If*

$$u(t) \le h(t) + f(t) \sum_{i=1}^{n} f_i(t) \int_a^t \lambda_i(s) \omega_i(u(s))\, ds, \quad t \in [a, b] \tag{20}$$

then, for $t \in [a, b_1]$,

$$u(t) \le \frac{h(t)}{f(t)} W_n^{-1} [W_n(c_{n-1}) + \int_a^t \lambda_n(s) r_n(f(s)) r_n(h(s)/f(s))\, ds],$$

where the notations are the same as in Theorem 1 *by replacing λ_i by $f_i(b_1) \cdot \lambda_i \cdot r_i(f) \cdot r_i(h/f)$, $i = 1, \dots, n$ and $c_0 = 1$.*

Proof. Since $\omega_i \in H$, $i = 1, \dots, n$, by (20), the function $z = u/f$ satisfies

$$z(t) \le h(t)/f(t) + \sum_{i=1}^{n} f_i(t) \int_a^t \lambda_i(s) r_i(f(s)) \omega_i(z(s))\, ds$$

and the result follows from Corollary 1.

Now, we obtain some nonlinear integral inequalities which are very useful in the theory of functional equations (see [8]).

The method can be applied to inequalities of type

$$u(t) \leq c + \sum_{0}^{n} \int_a^t \lambda_i(s)\omega_i[T_i(u(s))]\,ds,$$

where T_i are the operators

$$T_i(u) = T_i^1 \circ T_1^2 \circ \cdots \circ T_i^{i-1}(u), \quad 1 \leq i \leq n, \quad T_0 = \text{identity}$$

and

$$T_i^k(u)(t) = \int_a^t \lambda_i^k(s)\,\omega_i^k(u(s))\,ds, \quad i = 1, \dots. n, \quad k = 1, \dots, i-1.$$

For illustrating the method, we establish the result for $n = 1$ only:

Theorem 4. *Let u, λ_i, $i = 1, 2, 3$, ω_i, $i = 1, 2, 3$, and c be as in Theorem 1. If*

$$u(t) \leq c + \int_a^t \lambda_1(s)\omega_1(u(s))\,ds + \int_a^t \lambda_2(s)\omega_2\left(\int_a^s \lambda_3(\tau)\omega_3(u(\tau))\,d\tau\right)ds$$

then, for $t \in [a, b_1]$,

$$u(t) \leq W_3^{-1}\left[W_3(c_2) + \int_a^t \lambda_3(s)\,ds\right]$$

where the notations are the same as in Theorem 1.

Proof. Let $z_1(t) = c + \int_a^t \lambda_1(s)\omega_1(u(s))\,ds$, $z_3(t) = \int_a^t \lambda_3(\tau)\omega_3(u(\tau))\,d\tau$ and $z_2(t) = \int_a^t \lambda_2(s)\omega_2(z_3(s))\,ds$. Then

$$(z_1 + z_2 + z_3)' = \lambda_1\omega_1(u) + \lambda_2\omega_2(z_3) + \lambda_3\omega_3(u) \leq \sum_{i=1}^{3} \lambda_i\omega_i(z_1 + z_2 + z_3).$$

Thus, by integrating, $z = z_1 + z_2 + z_3$ satisfies

$$z(t) \leq c + \sum_{i=1}^{3} \int_a^t \lambda_i(s)\omega_i(z(s))\,ds$$

and the result follows from Theorem 1.

This theorem extends Pachpatte's results [11, 12, 13].

3. Application

In this section we apply some results of section 2 to study the asymptotic behaviour of solutions of the equation

$$y^{(n)} + f(t, y, y', \dots, y^{(n-1)}) = 0 \tag{21}$$

Theorem 5. *Let* $f = f(t, y_1, \dots, y_n)$ *be a continuous function on* $[1, \infty) \times \boldsymbol{R}^n$ *such that*

$$|f(t, y_1, \dots, y_n)| \le \sum_{i=1}^{n} \lambda_i(t)\omega_i(|y_i|/t^{n-i}), \tag{22}$$

where

(H) *The functions* ω_i, $i = 1, \dots, n$ *are continuous and nondecreasing on* $[0, \infty)$ *and positive on* $(0, \infty)$ *such that* $\omega_1 \propto \omega_2 \propto \cdots \propto \omega_n$ *(see Remark* 2), *and*

(H_2) λ_i, $i = 1, \dots, n$ *are nonnegative, continuous and integrable functions on* $[1, \infty)$.

We also suppose that there exists a positive constant c *such that*

$$\begin{aligned} &\int_1^\infty \lambda_i(s)\,ds \le \int_{\phi(c)}^\infty \frac{dz}{\omega_i(z)}, \quad i = 1, \dots, n-1, \\ &\int_1^\infty \lambda_n(s)\,ds < \int_{\phi(c)}^\infty \frac{dz}{\omega_n(z)} \end{aligned} \tag{23}$$

where $\phi = \phi_{n-1}$ *is the continuous, positive and nondecreasing function on* $(0, \infty)$ *given by* (16).

Then all solutions of (21) *such that* $\sum_{i=1}^n |y^{(n-i)}(1)| \le c$ *are defined on all* $[1, \infty)$ *and they satisfy*

$$y = \sum_{i=0}^{n-1} \delta_i t^i + o(t^{n-1}) \tag{24}$$

where δ_i, $i = 1, \dots, n$ *are constants. Moreover, the limit*

$$\lim_{t\to\infty} \int_1^t f(s, y(s), y'(s), \dots, y^{(n-1)}(s))\,ds = \delta$$

always exists and $(n-1)!\,\delta_{n-1} = y^{(n-1)}(1) - \delta$. *Thus* $\delta_{n-1} = 0$ *iff* $\delta = y^{(n-1)}(1)$, *in particular,* $\delta_{n-1} \ne 0$ *if* $y^{(n-1)}(1)$ *is small or big enough.*

Proof. The formula of variation of constants in (21) gives

$$y(t) = \sum_{i=1}^{n} c_i(t-1)^{i-1} - \int_1^t \frac{(t-s)^{n-1}}{(n-1)!}\, g(s)\,ds,$$

$$y'(t) = \sum_{i=2}^{n} (i-1)c_i(t-1)^{i-2} - \int_1^t \frac{(t-s)^{n-2}}{(n-2)!}\, g(s)\,ds,$$

$$\vdots$$

$$y^{(n-1)}(t) = (n-1)!\,c_n - \int_1^t g(s)\,ds,$$

where $g(s) = f(s, y(s), y'(s), \dots, y^{(n-1)}(s))$. Then,

$$|y(t)| \le (\sum_{i=1}^{n} |c_i|)\, t^{n-1} + t^{n-1} \int_1^t |g(s)|\, ds$$

or, by (22),

$$|y(t)|/t^{n-1} \le \sum_{i=1}^{n} |c_i| + \sum_{i=1}^{n} \int_1^t \lambda_i(s)\omega_i(|y^{(n-i)}(s)|/s^{i-1})\, ds$$
$$\le c + \sum_{i=1}^{n} \int_1^t \lambda_i(s)\omega_i(|y^{(n-i)}(s)|/s^{i-1})\, ds$$

because $\sum_{i=1}^{n} |c_i| \le \sum_{i=1}^{n} |y^{(n-i)}(1)| \le c$. Similarly, for any $k = 0, 1, \dots, n-1$

$$|y^{(k)}(t)|/t^{n-(k+1)} \le c + \sum_{i=1}^{n} \int_1^t \lambda_i(s)\omega_i\, [|y^{(n-i)}(s)|/s^{i-1}]\, ds.$$

Hence, if z denotes the right member in the last inequality, we have

$$z' \le \sum_{i=1}^{n} \lambda_i \omega_i(z), \quad z(1) = c$$

or, by integrating on $[1, t]$,

$$z(t) \le c + \sum_{i=1}^{n} \int_1^t \lambda_i(s)\omega_i(z(s))\, ds.$$

By Remarks 3 and 4, the hypotheses of Theorem 1 are satisfied with $b_1 = \infty$. Then, for $t \in [1, \infty)$

$$z(t) \le W_n^{-1}\, [W_n(\phi(c)) + \int_a^t \lambda_n(s)\, ds].$$

Since $\lambda_k \in L_1([1, \infty))$, $k = 1, \dots, n$, by (23), there exists a constant $K > 0$ such that

$$z(t) \le K \qquad \text{for} \quad t \ge 1.$$

Moreover, $g \in L_1([1, \infty))$. This implies $\lim_{t \to \infty} y^{(n-1)}(t) = a_{n-1}$ exists and by L'HOPITAL's rule $\lim_{t \to \infty} y^{(n-k-1)}(t)/t^k = a_{n-1}/k!$. Then $y^{(n-2)}(t) = a_{n-2} + a_{n-1}t + o(t)$, $y^{(n-3)}(t) = a_{n-3} + a_{n-2}t + a_{n-1}t^2/2 + o(t^2), \dots,$ and if a_i, $i = 0, 1, \dots, n-1$ are constants

$$y(t) = a_0 + a_1 t + \frac{a_2 t^2}{2} + \cdots + \frac{a_{n-1} t^{n-1}}{(n-1)!} + o(t^{n-1}) \text{ as } t \to \infty.$$

Thus, the proof is complete.

Theorem 5 extends some results of Kusano-Trench [17] and Medina-Pinto [9].

Corollary 3. *Suppose that in Theorem* 5 *the functions* ω_1, $i = 1, \dots, n$, *satisfy*

$$\int_1^\infty \frac{dz}{\omega_i(z)} = \infty, \quad i = 1, \dots, n. \tag{25}$$

Then all solutions y *of* (21) *satisfy* (24). *In particular, that is the case, if*

$$\limsup_{z\to\infty} \frac{\omega_i(z)}{z} < \infty, \quad i = 1, \dots, n.$$

By Remark 4, we have:

Corollary 4. *If in Theorem* 5 *the functions* ω_i, $i = 1, \dots, n$ *satisfy*

$$\int_{0^+}^1 \frac{dz}{\omega_i(z)} = \infty, \quad i = 1, \dots, n,$$

then there exists a constant c *which verifies* (23) *and all solutions* y *of* (21) *such that* $\sum_{i=1}^n |y^{(n-i)}(1)| \le c$ *satisfy* (24). *In particular, that is the case if*

$$\limsup_{z\to 0^+} \frac{\omega_i(z)}{z} > 0, \quad i = 1, \dots, n.$$

Remark 6. When (25) does not hold for all i, say for $i \in I \subset \{1, \dots, n\}$ and if for such $i \in I$, $x = x_i$ is the solution of the equation

$$\int_{\phi(x)}^\infty \frac{dz}{\omega_i(z)} = \alpha_i, \quad \alpha_i = \int_1^\infty \lambda_i(s)\, ds \tag{26}$$

then $c = \min\{x_i / i \in I\}$ satisfies (23) whenever $c < x_n$. The solution of (26) is given by

$$x_i = \phi^{-1} \circ W_i^{-1}[W_i(\infty) - \alpha_i], \ \alpha_i = \int_1^\infty \lambda_i(s)\, ds, \quad i = 1, \dots, n \tag{27}$$

where ϕ^{-1} is given by (17). This formula has the meaning $x_i = \infty$ when $W_i(\infty) = \infty$. Thus

$$c = \min\{x_i | 1 \le i \le n\} = \min\{x_i | i \in I\} \tag{28}$$

satisfies (23) if $c < x_n$. If $\min\{x_i / 1 \le i \le n\} = x_n$, then the result in Theorem 5 is true for all $c < x_n$ (see Example 1). Thus, in (23) it is possible to take c

$= c(x_1, x_2, \dots, x_n)$ where, for *any* $\varepsilon > 0$ ($\varepsilon < x_n$),

$$c(x_1, x_2, \dots, x_n) = \begin{cases} \min\{x_i | 1 \le i \le n\} & \text{if } x_n > \min\{x_i | 1 \le i \le n\} \\ x_n - \varepsilon, & \text{in other case.} \end{cases}$$

Example 1. The solutions of

$$y' = y^2/t^2, \quad y(1) = y_0,$$

are

$$y(t, y_0) = \frac{t y_0}{y_0 - y_0 t + t}$$

They are defined on all $[1, \infty)$ iff $y_0 \le 1$ and for $y_0 < 1$ they behave like constants as $t \to \infty$, but for $y_0 = 1$ the solution is not bounded. This shows that in Theorem 5 it is not possible to take $c = x_n$ (here $n = 1$), i.e. (24) can not be true if in (23) we have the equality for $i = n$. Moreover, this example shows that the solution of (21) can not be defined over all of $[1, \infty)$ if $\sum_{i=1}^{n} |y^{(n-i)}(1)| > c$.

Theorem 6. *Let $f = f(t, y_1, y_2)$ be a continuous function on $[1, \infty) \times \boldsymbol{R}^2$ such that*

$$|f(t, y_1, y_2)| \le \sum_{i=1}^{2} \lambda_i(t)\omega_i(|y_i|/t^{2-i}),$$

where

(H) *The functions ω_i, $i = 1, 2$, are continuous and nondecreasing on $[0, \infty)$ and positive on $(0, \infty)$ such that either $\omega_1 \propto \omega_2$ or $\omega_2 \propto \omega_1$, and*

(H_2) *The functions λ_i, $i = 1, 2$, are nonnegative, continuous and integrable on $[1, \infty)$.*

We also assume that there exists a positive constant c such that

$$\alpha_i = \int_1^\infty \lambda_i(s)\,ds \le \int_{\phi(c)}^\infty \frac{dz}{\omega_i(z)}, \quad i = 1, 2, \tag{29}$$

where $\phi(u) = W_k^{-1}[W_k(u) + \alpha_k]$, if $\omega_k \propto \omega_j$, $k, j = 1, 2$ and the inequality in (29) *is strict for $i = j$.*

Then all solutions of

$$y'' + f(t, y, y') = 0$$

such that $|y(1)| + |y'(1)| \le c$ are defined over all of $[1, \infty)$ and they satisfy

$$y = a + bt + o(t),$$

where a and b are constants. Moreover, the limit

$$\lim_{t\to\infty}\int_1^t f(s, y(s), y'(s))\,ds = \alpha$$

always exists and $b = y'(1) - \alpha$. *Thus* $b \neq 0$ *iff* $\alpha \neq y'(1)$; *in particular*, $b \neq 0$ *if* $y'(1)$ *is either small or big enough.*

This theorem extends the results of Cohen [3] and Tong [16].

Theorem 6 generalizes also Theorem 4 [5] which considers $\omega_2 =$ identity and asks further condition on ω_1. Moreover, the verification of our integral conditions are simpler (see examples 2 and 3).

Example 2. Consider the equation

$$y'' + \lambda(t)y' + \alpha e^{-t^2+y} = 0, \quad \lambda\in L_1([1, \infty)), \quad 0 < \alpha \leq 1. \tag{30}$$

Here $f(t, y_1, y_2) = \alpha e^{-t^2}\cdot e^{y_1} + \lambda(t)y_2$. Hence $|f| \leq |\lambda(t)||y_2| + \alpha e^{-t^2/2}\cdot e^{(y_1/t)^2/2}$ because $tv \leq (t^2 + v^2)/2$ and $\exp y_1 = \exp[t(y_1/t)] \leq \exp t^2/2\cdot\exp(y_1/t)^2/2$. We take $\omega_1(u) = e^{u^2/2}$, $\omega_2(u) = u$, $\lambda_1(t) = \alpha e^{-t^2/2}$, $\lambda_2(t) = |\lambda(t)|$. Since

$$\alpha\int_1^\infty e^{-t^2/2}\,dt \leq \int_1^\infty \frac{dz}{\omega_1(z)}$$

if $\alpha \leq 1$, all hypotheses of Theorem 6 are satisfied and each solution y of (30) such that $|y(1)| + |y'(1)| < 1$ verifies

$$y = a + bt + o(t) \quad \text{as} \quad t \longrightarrow \infty,$$

where a and b are constants. However, theorem 4 [5] can not be applied because ω_1 does not verify

$$\omega_1(bu) \leq r_1(b)\omega_1(u), \quad b \geq 1, \; u \geq 0$$

for $b > 1$. Kusano-Trench's results [17] can not be applied either.

Example 3. Consider the equation

$$y'' + \lambda(t)g_1(y, y')(y')^k + \beta(t)g_2(y, y')y^m = 0, \tag{31}$$

where λ, $\beta\cdot t^m\in L_1([1, \infty))$, $g_i = g_i(y_1, y_2)$ are bounded functins on $\boldsymbol{R}^2$ such that $|g_i(y_1, y_2)| \leq K_i$, $i = 1, 2$, $K_i > 0$ constants and $m, k > 0$. Here

$$f(t, y_1, y_2) = \beta(t)g_2(y_1, y_2)y_1^m + \lambda(t)g_1(y_1, y_2)y_2^k \quad \text{and}$$

$$\begin{aligned}|f(t, y_1, y_2)| &\leq K_2|\beta(t)|t^m|y_1/t|^m + K_1|\lambda(t))||y_2|^k \\ &= \lambda_1(t)\omega_1(|y_1|/t) + \lambda_2(t)\omega_2(|y_2|),\end{aligned}$$

where $\lambda_1(t) = K_2|\beta(t)|t^m$, $\omega_1(u) = u^m$, $\lambda_2(t) = K_1|\lambda(t)|$ and $\omega_2(u) = u^k$.

First, if $k, m \leq 1$ then Corollary 3 can be applied obtaining that all solutions y of (31) verify

$$y = a + bt + o(t) \quad \text{as} \quad t \longrightarrow \infty, \quad a, b, \text{ constants.} \tag{32}$$

Now we study the case $1 \leq k \leq m$. Let $k = 1 < m$. Then $\phi(u) = W_2^{-1}[W_2(u) + \alpha_2]$ because $\omega_2 \propto \omega_1$ and

$$\phi^{-1}(v) = W_2^{-1}[W_2(v) - \alpha_2] = e^{-\alpha_2}v, \quad \alpha_2 = \int_1^\infty \lambda_2(s)\,ds.$$

Hence, by (27),

$$x_1 = \phi^{-1} \circ [W_1^{-1}[W_1(\infty) - \alpha_1]] = e^{-\alpha_2} W_1^{-1}(-\alpha_1) \quad (W_1(\infty) = 0)$$

$$x_2 = \phi^{-1} \circ [W_2^{-1}[W_2(\infty) - \alpha_2]] = \infty$$

and from Remark 6

$$c = e^{-\alpha_2}((m-1)\alpha_1)^{1/(1-m)}, \quad \alpha_i = \int_1^\infty \lambda_i(s)\,ds, \quad i = 1, 2.$$

satisfies (23).

Thus, by Corollary 4, if $k = 1 < m$, then any solution y of (31) such that $|y(1)| + |y'(1)| \leq e^{-\alpha_2}((m-1)\alpha_1)^{-1/(m-1)}$ satisfies (32) where $b \neq 0$ if $|y'(1)|$ is sufficiently small.

On the other hand if $1 < k \leq m$, i.e. $\omega_1(u) = u^m$, $\omega_2(u) = u^k$, then $\phi(u) = W_2^{-1}[W_2(u) + \alpha_2]$

$$x_1 = W_2^{-1}[W_2(-\alpha_1) - \alpha_2], \quad x_2 = W_2^{-1}[-2\alpha_2] \quad (W_2(\infty) = 0).$$

Then Corollary 4, implies that for $1 < k \leq m$, any solution y of (31) with $|y(1)| + |y'(1)| \leq c(x_1, x_2)$ satisfies (32) where $b \neq 0$ if $|y'(1)|$ is small enough. For instance, for $m = k > 1$ and $\varepsilon > 0$

$$c(x_1, x_2) = \begin{cases} (2(k-1)\alpha_2)^{-1/(k-1)} - \varepsilon & \text{if} \quad \alpha_2 \geq \alpha_1 \\ ((k-1)(\alpha_1 + \alpha_2))^{-1/(k-1)} & \text{if } \alpha_1 \geq \alpha_2 . \end{cases}$$

The analysis for $1 \leq m \leq k$ is the same.

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