

On the Rate of Convergence of Solutions for the Singular Perturbations of Gradient Obstacle Problems

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§1. Introduction

In this paper we are concerned with the singular perturbations of obstacle problems with gradient constraint.

For any $\varepsilon > 0$, we consider the following nonlinear elliptic partial differential equation (PDE);

$$(1.1)_\varepsilon \quad \begin{cases} \max\{L_\varepsilon u_\varepsilon - f, |Du_\varepsilon| - g\} = 0 & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where L_ε is a linear second order elliptic operator defined in a bounded domain $\Omega \subset \mathbb{R}^N$;

$$(1.2) \quad L_\varepsilon u = -\varepsilon^2 a_{ij} u_{x_i x_j} + \varepsilon b_i u_{x_i} + cu,$$

Du denotes the gradient of u , and f, g are nonnegative functions in Ω . Here and in the sequel we use the usual summation convention on repeated indices. Our main purpose is to get an estimate on the rate of convergence of the solution u_ε of (1.1) $_\varepsilon$ to the solution u_0 of the first order PDE;

$$(1.1)_0 \quad \begin{cases} \max\{cu_0 - f, |Du_0| - g\} = 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

As to the equation (1.1) $_\varepsilon$, L.C. Evans [1] has treated for the first time. Relaxing the restrictions in [1], H. Ishii and S. Koike [4] proved the existence of solutions in $W^{2,\infty}(\Omega)$ and the uniqueness in the class $W_{loc}^{2,r}(\Omega) \cap C(\bar{\Omega})$ with $r > N$. Using the notion of viscosity solutions by P.-L. Lions [8], the second author [11] has shown the existence and uniqueness of viscosity solution in the class $C^1(\Omega) \cap C(\bar{\Omega})$.

On the other hand, the estimates on the singular perturbation problems usually depend on complicated PDE or probabilistic techniques (for example, S.R.S. Varadhan [10] and M.I. Freidlin and A.D. Wentzel [3]). But recently H. Ishii and S. Koike [5] pointed out that there are cases in which one can handle the problems by means of viscosity solution methods. They applied the

comparison principle of viscosity solutions to prove the convergence of solutions for singular perturbation problems. Moreover, using these methods S. Koike [7] has obtained the rate of convergence of solutions u_ε of obstacle problems $\max\{L_\varepsilon u_\varepsilon - f, u - \varphi\} = 0$ to $u_0 = \min\{f/c, \varphi\}$.

In Section 2 we state our assumptions and show the existence and uniqueness of viscosity solutions of (1.1) $_\varepsilon$ and (1.1) $_0$. Section 3 is devoted to the proof of our main result. Our argument is based upon those in [5] and [7]. In Section 4 we mention some extensions of our main result. First we show the rate of convergence of solutions of the Hamilton-Jacobi-Bellman (HJB) equation with gradient constraint. Next we treat an elliptic equation with gradient constraint whose principal part is a fully nonlinear operator. These arguments are totally similar to that in Section 2.

We conclude this introduction by recalling the notion of viscosity solutions. As to their properties, we refer the reader to H. Ishii and P.-L. Lions [6], P.-L. Lions [8], [9], and references therein. For any function $u: \Omega \rightarrow \mathbf{R}$, we define the function $u^*: \bar{\Omega} \rightarrow \bar{\mathbf{R}} \equiv \mathbf{R} \cup \{-\infty, \infty\}$ by

$$(1.3)^* \quad u^*(x) = \limsup_{y \in \Omega, y \rightarrow x} u(y),$$

and the function $u_*: \bar{\Omega} \rightarrow \bar{\mathbf{R}}$ by

$$(1.3)_* \quad u_*(x) = \liminf_{y \in \Omega, y \rightarrow x} u(y).$$

We give the definition of viscosity solutions for the following fully nonlinear PDEs;

$$(1.4) \quad F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega,$$

where $F: \bar{\Omega} \times \mathbf{R} \times \mathbf{R}^N \times S^N \rightarrow \mathbf{R}$ is continuous, nonincreasing with respect to D^2u and S^N denotes the set of all $N \times N$ symmetric matrices.

Definition. Let u be a function defined on Ω .

- (1) u is a viscosity subsolution of (1.4) provided $u^*(x) < \infty$ in Ω and for any $\varphi \in C^2(\Omega)$, if $u^* - \varphi$ attains a local maximum at $x_0 \in \Omega$, then

$$F(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

- (2) u is a viscosity supersolution of (1.4) provided $u_*(x) > -\infty$ in Ω and for any $\varphi \in C^2(\Omega)$, if $u_* - \varphi$ attains a local minimum at $x_0 \in \Omega$, then

$$F(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0.$$

- (3) u is a viscosity solution of (1.4) provided u is both a viscosity subsolution

and a supersolution of (1.4).

Remark. In the case that $F(x, u, Du, D^2u) = G(x, u, Du)$ i.e., in the case of first order PDEs, we can replace $C^2(\Omega)$ in (1) or (2) with $C^1(\Omega)$.

§2. Preliminaries

In this section we shall state our assumptions and shall show the existence and uniqueness of viscosity solutions of (1.1)_ε and (1.1)₀ satisfying the Dirichlet boundary condition. We make the following assumptions;

- (A.1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.
- (A.2) For the matrix $(a_{ij}(x))$, there exists a nonnegative matrix $(\sigma_{ij}(x))$ such that

$$(a_{ij}) = {}^t(\sigma_{ij})(\sigma_{ij}), \text{ with } \sigma_{ij} \in C^{0,\alpha}(\bar{\Omega}) \ (\alpha > 1/2, \ i, j = 1, \dots, N)$$

where tA is the transposed matrix of A .

- (A.3) $b_i \in C(\bar{\Omega}) \ (i = 1, \dots, N)$.
- (A.4) $c \in W^{1,\infty}(\bar{\Omega}), \ c \geq c_0$ on $\bar{\Omega}$ for some $c_0 > 0$.
- (A.5) $f \in W^{1,\infty}(\bar{\Omega}), \ f \geq 0$ on $\bar{\Omega}$.
- (A.6) $g \in W^{1,\infty}(\bar{\Omega}), \ g \geq \theta$ on $\bar{\Omega}$ for some $\theta > 0$.

We denote by $K_c, K_f,$ and K_g the Lipschitz constants for $c, f,$ and $g,$ respectively.

Concerning the existence and uniqueness of viscosity solutions of (1.1)_ε and (1.1)₀ satisfying the Dirichlet boundary condition, we have the following Theorem.

- Theorem 1.** (1) For each $\varepsilon > 0,$ there exists a unique viscosity solution u_ε of (1.1)_ε in $W^{1,\infty}(\bar{\Omega})$ satisfying $u_\varepsilon = 0$ on $\partial\Omega$.
 (2) There exists a unique viscosity solution u_0 of (1.1)₀ in $W^{1,\infty}(\bar{\Omega})$ satisfying $u_0 = 0$ on $\partial\Omega$.

Proof. The uniqueness follows from the comparison principle due to H. Ishii and P.-L. Lions [6].

Next, we prove the existence of viscosity solutions. We note that

$$(2.1) \quad w_1(x) \equiv 0 \quad (x \in \bar{\Omega})$$

is a viscosity subsolution of (1.1)_ε and (1.1)₀. On the other hand,

$$(2.2) \quad w_2(x) = \inf_{y \in \partial\Omega} L(x, y) \quad (x \in \bar{\Omega}),$$

where $L(x, y)$ is defined by

$$L(x, y) = \inf_{\xi \in \mathcal{A}} \int_0^t g(\xi(s)) ds,$$

$$\mathcal{A} = \left\{ \xi \in C[0, t] \mid \xi(0) = x, \xi(t) = y \in \partial\Omega, \right. \\ \left. \xi(s) \in \bar{\Omega} \ (0 \leq s \leq t), \left| \frac{d\xi}{ds} \right| \leq 1 \text{ a.e. } s \in [0, t] \right\},$$

is a viscosity solution of $|Du| - g = 0$ in Ω (see P.-L. Lions [9]). Hence w_2 is a viscosity supersolution of $(1.1)_\varepsilon$ and $(1.1)_0$. Thus, by Perron's method ([6], Proposition II.1) there exist viscosity solutions u_ε and $u_0 \in C(\bar{\Omega})$ of $(1.1)_\varepsilon$, $(1.1)_0$, respectively, satisfying the Dirichlet boundary condition and

$$(2.3) \quad 0 \leq u_\varepsilon, u_0 \leq w_2 \quad \text{in } \bar{\Omega}.$$

Moreover u_ε and u_0 are viscosity subsolutions of $|Du| - g = 0$ in Ω by the form of equations $(1.1)_\varepsilon$ and $(1.1)_0$. Hence u_ε and u_0 are Lipschitz continuous on $\bar{\Omega}$ by M. G. Crandall and P.-L. Lions [2], Corollary I.15. Therefore the proof is completed. \square

Remark. Since g is a bounded constraint for the gradient of u_ε , the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ are equi-Lipschitz continuous on $\bar{\Omega}$. By (2.3) $\{u_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded. In what follows we denote by K Lipschitz constant of u_ε and u_0 .

§3. Main result

In this section we shall prove our main result.

Theorem 2. *Under the assumptions (A.1)–(A.6), let u_ε, u_0 be viscosity solutions of $(1.1)_\varepsilon$ and $(1.1)_0$, respectively satisfying the Dirichlet boundary condition. Then there exist $\mu > 0$ and $\varepsilon_0 > 0$ such that*

$$\|u_\varepsilon - u_0\| \leq \mu\varepsilon \quad \text{for any } \varepsilon \in (0, \varepsilon_0),$$

where $\|\cdot\|$ denotes the sup norm in $C(\bar{\Omega})$.

Proof. It is sufficient to prove $u_\varepsilon - u_0 \leq \mu\varepsilon$ on $\bar{\Omega}$ because the remaining inequality can be proved similarly. We take ε_0 such that

$$\varepsilon_0 < \frac{\theta}{K_g K}$$

and for any fixed $\varepsilon \in (0, \varepsilon_0)$, define

$$\Phi_\varepsilon(x, y) = \rho u_\varepsilon(x) - u_0(y) - \frac{|x - y|^2}{\varepsilon} - \mu\varepsilon \quad \text{on } \overline{\Omega \times \Omega},$$

where $\rho = 1 - K_g K \lambda \varepsilon / \theta$ with a constant $\lambda > 1$ and $\mu > 0$ is a constant to be determined later. Let $\bar{x}, \bar{y} \in \bar{\Omega}$ be such that

$$\Phi_\varepsilon(\bar{x}, \bar{y}) = \max_{\overline{\Omega \times \Omega}} \Phi_\varepsilon(x, y).$$

It is easily verified by taking λ sufficiently close to 1, that

$$(3.1) \quad \lambda - 1 < \rho < 1.$$

(For example, if $\varepsilon_0 = \theta / 3K_g K$, we can take $\lambda = 3/2$.) Noting $\Phi_\varepsilon(\bar{x}, \bar{x}) \leq \Phi_\varepsilon(\bar{x}, \bar{y})$, we have

$$\frac{|\bar{x} - \bar{y}|^2}{\varepsilon} \leq u_0(\bar{x}) - u_0(\bar{y}).$$

Since u_0 is Lipschitz continuous, we get

$$(3.2) \quad |\bar{x} - \bar{y}| \leq K\varepsilon.$$

We consider the following three cases.

(I) The case $\bar{x}, \bar{y} \in \Omega$.

The function

$$x \rightarrow u_\varepsilon(x) - \frac{1}{\rho} \left\{ u_0(\bar{y}) + \frac{|x - \bar{y}|^2}{\varepsilon} + \mu\varepsilon \right\}$$

takes the maximum at \bar{x} . Since u_ε is a viscosity subsolution of (1.1)_ε, we have

$$(3.3) \quad \max \left\{ -\frac{2\varepsilon}{\rho} \sum_{i=1}^N a_{ii}(\bar{x}) + \frac{2}{\rho} b_i(\bar{x})(\bar{x}_i - \bar{y}_i) + c(\bar{x})u_\varepsilon(\bar{x}) - f(\bar{x}), \right. \\ \left. \frac{2}{\rho\varepsilon} |\bar{x} - \bar{y}| - g(\bar{x}) \right\} \leq 0.$$

Similarly, the function

$$y \rightarrow u_0(y) - \left\{ \rho u_\varepsilon(\bar{x}) - \frac{|\bar{x} - y|^2}{\varepsilon} - \mu\varepsilon \right\}$$

takes the minimum at \bar{y} . Using the fact that u_0 is a viscosity supersolution of (1.1)₀, we have

$$(3.4) \quad \max \left\{ c(\bar{y})u_0(\bar{y}) - f(\bar{y}), \frac{2}{\varepsilon} |\bar{x} - \bar{y}| - g(\bar{y}) \right\} \geq 0.$$

We claim that $2|\bar{x} - \bar{y}|/\varepsilon - \bar{g}(\bar{y}) < 0$ in (3.4). To prove the inequality by contradiction, suppose that $2|\bar{x} - \bar{y}|/\varepsilon - g(\bar{y}) \geq 0$ in (3.4). Since $2|\bar{x} - \bar{y}|/\rho\varepsilon - g(\bar{x}) \leq 0$ by (3.3), we get

$$g(\bar{y}) \leq \frac{2}{\varepsilon}|\bar{x} - \bar{y}| \leq \rho g(\bar{x}).$$

Hence (A.6) and (3.2) imply

$$(1 - \rho)\theta \leq (1 - \rho)g(\bar{y}) \leq \rho(g(\bar{x}) - g(\bar{y})) < K_g|\bar{x} - \bar{y}| \leq K_g K \varepsilon.$$

Therefore, we have

$$1 - \rho \leq \frac{K_g K}{\theta} \varepsilon$$

which contradicts the choice of λ . Thus we obtain the claim. Hence we have from (3.4),

$$(3.5) \quad c(\bar{y})u_0(\bar{y}) - f(\bar{y}) \geq 0.$$

Note that (3.3) implies

$$(3.6) \quad -\frac{2}{\rho}\varepsilon \sum_{i=1}^N a_{ii}(\bar{x}) + \frac{2}{\rho}b_i(\bar{x})(\bar{x}_i - \bar{y}_i) + c(\bar{x})u_\varepsilon(\bar{x}) - f(\bar{x}) \leq 0.$$

Subtracting (3.5) from (3.6) and using (3.1), (3.2), (A.2), (A.3) and (A.5), we get

$$\begin{aligned} & c(\bar{x})u_\varepsilon(\bar{x}) - c(\bar{y})u_0(\bar{y}) \\ & \leq \frac{2}{\rho}\varepsilon \sum_{i=1}^N a_{ii}(\bar{x}) - \frac{2}{\rho}b_i(\bar{x})(\bar{x}_i - \bar{y}_i) + f(\bar{x}) - f(\bar{y}) \\ & \leq C\varepsilon + C|\bar{x} - \bar{y}| + K_f|\bar{x} - \bar{y}| \leq C\varepsilon. \end{aligned}$$

Here and hereafter C denotes various constants depending only on known constants. Moreover, the Lipschitz continuity of c implies

$$c(\bar{x})\{u_\varepsilon(\bar{x}) - u_0(\bar{y})\} - K_c|\bar{x} - \bar{y}|u_0(\bar{y}) \leq C\varepsilon.$$

Then we get

$$c(\bar{x})\{u_\varepsilon(\bar{x}) - u_0(\bar{y})\} \leq (C + K_c K u_0(\bar{y}))\varepsilon \leq C\varepsilon.$$

Since $c \geq c_0$ in $\bar{\Omega}$, we have $u_\varepsilon(\bar{x}) - u_0(\bar{y}) \leq C\varepsilon$. Hence we obtain

$$\Phi_\varepsilon(\bar{x}, \bar{y}) < u_\varepsilon(\bar{x}) - u_0(\bar{y}) - \mu\varepsilon \leq (C - \mu)\varepsilon.$$

Now we choose μ large enough to get $\Phi_\varepsilon(\bar{x}, \bar{y}) \leq 0$ and thus

$$\rho u_\varepsilon(x) - u_0(x) - \mu\varepsilon \leq 0 \quad \text{for all } x \in \bar{\Omega}.$$

Hence

$$u_\varepsilon(x) - u_0(x) \leq \left(\mu + \frac{K_g K}{\theta} \lambda u_\varepsilon(x) \right) \varepsilon \leq \mu' \varepsilon$$

for sufficiently large $\mu' > 0$, and any $x \in \bar{\Omega}$.

(II) The case $\bar{x} \in \partial\Omega$.

Since the Dirichlet condition of (1.1) $_\varepsilon$ and (2.3) imply

$$\Phi_\varepsilon(\bar{x}, \bar{y}) = -u_0(\bar{y}) - \frac{|\bar{x} - \bar{y}|^2}{\varepsilon} - \mu\varepsilon \leq 0$$

for any $\mu > 0$, we can argue the remainder similar to (I).

(III) The case $\bar{y} \in \partial\Omega$.

By the Dirichlet condition of (1.1) $_\varepsilon$ and (1.1) $_0$, and the Lipschitz continuity of u_ε , we have

$$\begin{aligned} \Phi_\varepsilon(\bar{x}, \bar{y}) &= \rho u_\varepsilon(\bar{x}) - \frac{|\bar{x} - \bar{y}|^2}{\varepsilon} - \mu\varepsilon < u_\varepsilon(\bar{x}) - u_\varepsilon(\bar{y}) - \mu\varepsilon \\ &\leq (K^2 - \mu)\varepsilon. \end{aligned}$$

Therefore we get $\Phi_\varepsilon(\bar{x}, \bar{y}) \leq 0$ for $\mu \geq K^2$. The remainder is also similar to (I). From (I)-(III), if we choose $\mu > 0$ sufficiently large, then we obtain

$$u_\varepsilon(x) - u_0(x) \leq \mu\varepsilon \quad \text{for all } x \in \bar{\Omega}. \quad \square$$

§4. Some extensions

In this section we mention two extensions of Theorem 2.

4.1 HJB equation with gradient constraint.

Consider the Hamilton-Jacobi-Bellman equation with gradient constraint;

$$(4.1)_\varepsilon \quad \begin{cases} \max\{L_\varepsilon^1 u_\varepsilon - f^1, \dots, L_\varepsilon^m u_\varepsilon - f^m, |Du_\varepsilon| - g\} = 0 & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where L_ε^p , ($p = 1, \dots, m$), are linear elliptic operators defined in a bounded domain $\Omega \subset \mathbb{R}^N$;

$$(4.2) \quad L_\varepsilon^p u = -\varepsilon^2 a_{ij}^p u_{x_i x_j} + \varepsilon b_i^p u_{x_i} + c^p u,$$

and f^p, g are nonnegative functions in $\bar{\Omega}$ ($p = 1, \dots, m$). The second author [11] proved the existence and uniqueness of viscosity solution when the coefficients of L_ε^p, f^p , and g are in the class $C^2(\bar{\Omega})$. However, in this section we

make the same assumptions on them as (A.2)-(A.6). We show that the rate of convergence of the solutions u_ε of (4.1) $_\varepsilon$ to the solution u_0 of the following first order PDE;

$$(4.1)_0 \quad \begin{cases} \max\{c^1 u_0 - f^1, \dots, c^m u_0 - f^m, |Du_0| - g\} = 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

can be obtained by the same method as in Theorem 2.

Since w_1 in (2.1) and w_2 in (2.2) are viscosity sub- and supersolutions, respectively, of (4.1) $_\varepsilon$ and (4.1) $_0$ with the Dirichlet boundary condition, there exist the viscosity solutions $u_\varepsilon \in W^{1,\infty}(\bar{\Omega})$ of (4.1) $_\varepsilon$ and $u_0 \in W^{1,\infty}(\bar{\Omega})$ of (4.1) $_0$ which satisfy the Dirichlet boundary condition. The uniqueness of the solution follows from H. Ishii and P.-L. Lions [6].

Theorem 3. *Under the assumptions mentioned above, there exist $\mu > 0$ and $\varepsilon_0 > 0$ such that*

$$\|u_\varepsilon - u_0\| \leq \mu\varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_0),$$

where $\|\cdot\|$ denotes the sup norm in $C(\bar{\Omega})$.

Proof. Since the proof of the theorem is quite similar to the proof of Theorem 2, we only point out the difference.

Choosing the same ε_0 , consider the same function $\Phi_\varepsilon(x, y)$ and $\bar{x}, \bar{y} \in \bar{\Omega}$ as in Theorem 2. After we get (3.2), we also consider the three cases. The cases (II), (III) can be treated in the same way as in the proof of Theorem 2. For the case (I), let $\bar{x}, \bar{y} \in \Omega$.

Since the function

$$x \rightarrow u_\varepsilon(x) - \frac{1}{\rho} \left\{ u_0(\bar{y}) + \frac{|x - \bar{y}|^2}{\varepsilon} + \mu\varepsilon \right\}$$

takes the maximum at \bar{x} and u_ε is a viscosity subsolution of (4.1) $_\varepsilon$, we have

$$(4.3) \quad \max \left\{ -\frac{2\varepsilon}{\rho} \sum_{i=1}^N a_{ii}^1(\bar{x}) + \frac{2}{\rho} b_i^1(\bar{x})(\bar{x}_i - \bar{y}_i) + c^1(\bar{x})u_\varepsilon(\bar{x}) - f^1(\bar{x}), \right. \\ \dots, -\frac{2\varepsilon}{\rho} \sum_{i=1}^N a_{ii}^m(\bar{x}) + \frac{2}{\rho} b_i^m(\bar{x})(\bar{x}_i - \bar{y}_i) + c^m(\bar{x})u_\varepsilon(\bar{x}) - f^m(\bar{x}), \\ \left. \frac{2}{\rho\varepsilon} |\bar{x} - \bar{y}| - g(\bar{x}) \right\} \leq 0.$$

Similarly, noting that the function

$$y \rightarrow u_0(y) - \left\{ \rho u_\varepsilon(\bar{x}) - \frac{|\bar{x} - y|^2}{\varepsilon} - \mu\varepsilon \right\}$$

takes the minimum at \bar{y} and u_0 is a viscosity supersolution of (4.1)₀, we have

$$(4.4) \max \left\{ c^1(\bar{y})u_0(\bar{y}) - f^1(\bar{y}), \dots, c^m(\bar{y})u_0(\bar{y}) - f^m(\bar{y}), \frac{2}{\varepsilon}|\bar{x} - \bar{y}| - g(\bar{y}) \right\} \geq 0.$$

Repeating the same argument as in the proof of Theorem 2, we have $2|\bar{x} - \bar{y}|/\varepsilon - g(\bar{y}) < 0$. Hence we can find $p' = p'(\varepsilon)$ such that

$$c^{p'}(\bar{y})u_0(\bar{y}) - f^{p'}(\bar{y}) \geq 0$$

in (4.4).

On the other hand, (4.3) implies

$$-\frac{2\varepsilon}{\rho} \sum_{i=1}^N a_{ii}^{p'}(\bar{x}) + \frac{2}{\rho} b_i^{p'}(\bar{x})(\bar{x}_i - \bar{y}_i) + c^{p'}(\bar{x})u_\varepsilon(\bar{x}) - f^{p'}(\bar{x}) \leq 0.$$

Therefore, by the same calculation as in Theorem 2, we get $u_\varepsilon(\bar{x}) - u_0(\bar{y}) \leq C\varepsilon$. The remainder is totally similar to Theorem 2. Thus we have completed the proof. \square

4.2 General fully nonlinear operator case.

Consider the elliptic equation with gradient constraint whose principal part is a fully nonlinear operator;

$$(4.5)_\varepsilon \quad \begin{cases} \max\{F_\varepsilon(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon), |Du_\varepsilon| - g\} = 0 & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

and the first order PDE;

$$(4.5)_0 \quad \begin{cases} \max\{F_0(x, u_0), |Du_0| - g\} = 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

where F is a fully nonlinear elliptic operator such that $F_\varepsilon(x, u, Du, D^2u) = F(x, u, \varepsilon Du, \varepsilon^2 D^2u)$, $F_0(x, u) = F(x, u, 0, 0)$ and nonincreasing with respect to D^2u . We make the following assumptions on F .

$$(F.1) \quad \begin{aligned} F(x, u, p, A) &\in C(\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^N \times S^N), \text{ and} \\ F(x, 0, 0, 0) &\leq 0 \quad \text{for any } x \in \Omega, \end{aligned}$$

where S^N denotes the set of all $N \times N$ symmetric matrices. For each $R > 0$, there exist $\nu_R, K_R > 0$ such that

$$(F.2) \quad \begin{aligned} F(x, u, p, A) &\geq F(x, v, p, A) + \nu_R(u - v) \\ \text{for any } x \in \bar{\Omega}, \quad &-R \leq v \leq u \leq R, p \in \mathbf{R}, A \in S^N, \end{aligned}$$

$$(F.3) \quad |F(x, u, p, A) - F(y, u, p, A)| \leq K_R|x - y|$$

for any $x, y \in \bar{\Omega}$, $|u| \leq R$, $|p| \leq R$, $A \in S^N$,

$$(F.4) \quad |F(x, u, p, A) - F(x, u, q, A)| \leq K_R |p - q|$$

for any $x \in \bar{\Omega}$, $|u| \leq R$, $|p|, |q| \leq R$, $A \in S^N$,

$$(F.5) \quad |F(x, u, p, A) - F(x, u, p, B)| \leq K_R |A - B|$$

for any $x \in \bar{\Omega}$, $|u| \leq R$, $|p| \leq R$, $A, B \in S^N$,

and $|A| = (\sum_{i,j=1}^N |a_{ij}|^2)^{1/2}$ when $A = (a_{ij})$.

Under these assumptions, there exist the unique viscosity solutions $u_\varepsilon \in W^{1,\infty}(\bar{\Omega})$ of (4.5)_ε and $u_0 \in W^{1,\infty}(\bar{\Omega})$ of (4.5)₀ which satisfy the Dirichlet boundary condition since w_1 in (2.1) and w_2 in (2.2) are viscosity sub- and supersolutions, respectively, of (4.5)_ε and (4.5)₀ with the Dirichlet boundary condition. See M.G. Crandall and P.-L. Lions [2], H. Ishii and P.-L. Lions [6], and P.-L. Lions [9].

Theorem 4. *Under the assumptions (A.1), (A.6) and (F.1)-(F.5), there exist $\mu > 0$ and $\varepsilon_0 > 0$ such that*

$$\|u_\varepsilon - u_0\| \leq \mu\varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_0),$$

where $\|\cdot\|$ denotes the sup norm in $C(\bar{\Omega})$.

Proof. First we note that, as in the Remark in Section 2, there exists $R > 0$ such that $\|u_\varepsilon\|_{W^{1,\infty}}, \|u_0\|_{W^{1,\infty}} \leq R$ for all $\varepsilon > 0$.

As in Theorem 3, we only point out the different argument from Theorem 2.

Choose the same ε_0 , and consider the same function $\Phi_\varepsilon(x, y)$ and the same \bar{x}, \bar{y} as in the proof of Theorem 2. After we get (3.2), we also consider the three cases. It is sufficient to consider the case (I). We may assume that $u_\varepsilon(\bar{x}) - u_0(\bar{y}) \geq 0$, because if otherwise, we obtain $\Phi_\varepsilon(\bar{x}, \bar{y}) \leq 0$ for all $\mu > 0$ and the remainder is similar to Theorem 2.

The function

$$x \rightarrow u_\varepsilon(x) - \frac{1}{\rho} \left\{ u_0(\bar{y}) + \frac{|x - \bar{y}|^2}{\varepsilon} + \mu\varepsilon \right\}$$

takes the maximum at \bar{x} . Then since u_ε is a viscosity subsolution of (4.5)_ε, we have

$$(4.6) \quad \max \left\{ F_\varepsilon \left(\bar{x}, u_\varepsilon(\bar{x}), \frac{2}{\rho\varepsilon}(\bar{x} - \bar{y}), \frac{2}{\rho\varepsilon}I \right), \frac{2}{\rho\varepsilon}|\bar{x} - \bar{y}| - g(\bar{x}) \right\} \leq 0,$$

where I denotes the identity matrix. Similarly the function

$$y \rightarrow u_0(y) - \left\{ \rho u_\varepsilon(\bar{x}) - \frac{|\bar{x} - y|^2}{\varepsilon} - \mu\varepsilon \right\}$$

takes the minimum at \bar{y} . Using the fact that u_0 is a viscosity supersolution of (4.5)₀, we have

$$(4.7) \quad \max \left\{ F_0(\bar{y}, u_0(\bar{y})), \frac{2}{\varepsilon} |\bar{x} - \bar{y}| - g(\bar{y}) \right\} \geq 0.$$

By the same argument as in the proof of Theorem 2, we have $2|\bar{x} - \bar{y}|/\varepsilon - g(\bar{y}) < 0$. Hence we have in (4.7)

$$F_0(\bar{y}, u_0(\bar{y})) \geq 0.$$

On the other hand, (4.6) implies

$$F_\varepsilon \left(\bar{x}, u_\varepsilon(\bar{x}), \frac{2}{\rho\varepsilon} (\bar{x} - \bar{y}), \frac{2}{\rho\varepsilon} I \right) \leq 0.$$

Therefore we obtain by the same calculation as in [6], Theorem II.1,

$$\begin{aligned} 0 &\leq F_0(\bar{y}, u_0(\bar{y})) - F_\varepsilon \left(\bar{x}, u_\varepsilon(\bar{x}), \frac{2}{\rho\varepsilon} (\bar{x} - \bar{y}), \frac{2}{\rho\varepsilon} I \right) \\ &= F(\bar{y}, u_0(\bar{y}), 0, 0) - F \left(\bar{x}, u_\varepsilon(\bar{x}), \frac{2}{\rho} (\bar{x} - \bar{y}), \frac{2\varepsilon}{\rho} I \right) \\ &= F(\bar{y}, u_0(\bar{y}), 0, 0) - F(\bar{x}, u_0(\bar{y}), 0, 0) + F(\bar{x}, u_0(\bar{y}), 0, 0) \\ &\quad - F \left(\bar{x}, u_\varepsilon(\bar{x}), \frac{2}{\rho} (\bar{x} - \bar{y}), 0 \right) + F \left(\bar{x}, u_\varepsilon(\bar{x}), \frac{2}{\rho} (\bar{x} - \bar{y}), 0 \right) \\ &\quad - F \left(\bar{x}, u_\varepsilon(\bar{x}), \frac{2}{\rho} (\bar{x} - \bar{y}), \frac{2\varepsilon}{\rho} I \right) \\ &\leq K_R |\bar{x} - \bar{y}| - v_R (u_\varepsilon(\bar{x}) - u_0(\bar{y})) + K_R \frac{2}{\rho} |\bar{x} - \bar{y}| + K_R \frac{2\varepsilon}{\rho} |I|. \end{aligned}$$

Hence (3.1) and (3.2) imply $u_\varepsilon(\bar{x}) - u_0(\bar{y}) \leq C\varepsilon$. The remainder is totally similar to the proof of Theorem 2. Thus we have completed the proof. \square

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References

- [1] Evans, L. C., A second order elliptic equation with gradient constraint, *Comm. Partial Differential Equations*, **4**(1979), 552–572; Correction, *ibid.*, **4**(1979), 1199.
- [2] Crandall, M.G. and Lions, P.-L., Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.*, **277**(1983), 1–42.
- [3] Freidlin, M.I. and Wentzel, A.D., *Random Perturbations of Dynamical Systems*, Springer, Berlin/Heidelberg/New York/Tokyo, 1984.
- [4] Ishii, H. and Koike, S., Boundary regularity and uniqueness for an elliptic equation with gradient constraint, *Comm. Partial Differential Equations*, **8**(1983), 317–346.
- [5] ———, Remarks on elliptic singular perturbation problems, to appear in *Appl. Math. Optim.*
- [6] Ishii, H. and Lions, P.-L., Viscosity solutions of fully nonlinear second-order partial differential equations, *J. Differential Equations*, **83**(1990), 26–78.
- [7] Koike, S., On the rate of convergence of solutions in singular perturbation problems, to appear in *J. Math. Anal. Appl.*
- [8] Lions, P.-L., Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, Part II. Viscosity solutions and uniqueness, *Comm. Partial Differential Equations*, **8**(1983), 1229–1276.
- [9] ———, *Generalized solutions of Hamilton-Jacobi equations*, Pitman, Boston, 1982.
- [10] Varadhan, S.R.S., On the behavior of the fundamental solution of the heat equation with variable coefficients, *Comm. Pure Appl. Math.*, **20**(1967), 431–455.
- [11] Yamada, N., The Hamilton-Jacobi-Bellman equation with a gradient constraint, *J. Differential Equations*, **71**(1988), 185–199.

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