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Total Stability Property in Limiting Equations of Integrodifferential Equations

By

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For the ordinary differential equation and the functional differential equation with infinite delay, it is known that the total stability of a bounded solution can be deduced from the total stability in a certain limiting equation which is obtained by employing the Bohr topology (cf. [4], [7]). However, this is not true when we use the compact open topology. For a counter example, see [1].

In this paper, we shall consider the relationships between the stability properties of solutions of the nonlinear integrodifferential equation and those in its limiting equations, and we shall obtain similar results to those in [4] and [7]. To do this, we consider stability properties in a certain sense. Under the condition that the bounded solution has some kind of stability, we shall obtain also some results on the existence of an almost periodic solution of an almost periodic integrodifferential equation.

First of all, we shall give the definition of the almost periodic function with parameters in the usual way (cf. [12]). Let $R^* = (-\infty, 0] \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ and let $F(t, s, x, y)$ be a function which is defined and is continuous for $t \in \boldsymbol{R}$, $s \in (-\infty, 0]$, $x \in \boldsymbol{R}^n$ and $y \in \boldsymbol{R}^n$.

Definition 1. $F(t, s, x, y)$ is said to be almost periodic in t uniformly for $(s, x, y) \in \boldsymbol{R}^*$ if for any $\varepsilon > 0$ and any compact set K^* in R^*, there exists an $L(\varepsilon, K^*) > 0$ such that any interval of length $L(\varepsilon, K^*)$ contains a τ for which

$$|F(t+\tau, s, x, y) - F(t, s, x, y)| \leqq \varepsilon$$

for all $t \in R$ and all $(s, x, y) \in K^*$.

For the properties of the almost periodic function with parameters, see [12].

BC denotes the vector space of bounded continuous functions mapping $(-\infty, 0]$ into $\boldsymbol{R}^n$, and for any ϕ, $\psi \in BC$, we set

$$\rho(\phi, \psi) = \sum_{j=1}^{\infty} \rho_j(\phi, \psi)/[2^j(1 + \rho_j(\phi, \psi))],$$

where $\rho_j(\phi, \psi) = \sup_{-j \leqq s \leqq 0} |\phi(s) - \psi(s)|$. Clearly, $\rho(\phi_n, \phi) \to 0$ as $n \to \infty$ if and only if $\phi_n(s) \to \phi(s)$ uniformly on any compact subset of $(-\infty, 0]$ as $n \to \infty$. We denote by (BC, ρ) the space of bounded continuous functions $\varphi: (-\infty, 0] \to \boldsymbol{R}^n$ with metric ρ.

We consider the nonlinear integrodifferential equation

$$(1) \qquad \dot{x}(t) = f(t, x(t)) + \int_{-\infty}^{0} F(t, s, x(t+s), x(t))\,ds + h(t, x_t),$$

where if x is a function defined on $(-\infty, a)$, x_t is defined by the relation $x_t(s) = x(t+s)$, $-\infty < s \leqq 0$.

We impose the following assumptions:

(A) $f: \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a continuous function and is almost periodic in t uniformly for $x \in \boldsymbol{R}^n$.

(B) $F(t, s, x, y)$ is defined and continuous for $t \in R$, $s \in (-\infty, 0]$, $x \in \boldsymbol{R}^n$ and $y \in \boldsymbol{R}^n$, and $F(t, s, x, y)$ is almost periodic in t uniformly for $(s, x, y) \in R^*$. Moreover, for any $r > 0$ there exists a function $m_r(s)$ such that $\int_{-\infty}^{0} m_r(s)\,ds < \infty$ and $|F(t, s, x, y)| \leqq m_r(s)$ for all $t \in \boldsymbol{R}$, $s \leqq 0$, $x \in \boldsymbol{R}^n$ and $y \in \boldsymbol{R}^n$ such that $|x| \leqq r$ and $|y| \leqq r$.

(C) $h(t, \varphi): \boldsymbol{R} \times (BC, \rho) \to \boldsymbol{R}^n$ is continuous in t and φ, and for any $r > 0$ there exists a continuous function $\beta_r(t)$ such that $\beta_r(t) \to 0$ as $t \to \infty$ and $|h(t, \varphi)| \leqq \beta_r(t)$, whenever $|\varphi(s)| \leqq r$ for all $s \in (-\infty, 0]$.

(D) Equation (1) has a bounded solution $u(t)$ defined on $[0, \infty)$ which passes through $(0, \phi^0)$, $\phi^0 \in BC$.

Under the above assumptions (A), (B) and (C), if $t_0 \in \boldsymbol{R}$ and $\phi \in BC$, there exists a solution of (1) which passes through (t_0, ϕ). Moreover, a solution $x(t)$ can be continuable up to $t = \infty$ if it remains in a compact set in $\boldsymbol{R}^n$, because $\dot{x}(t)$ is bounded as long as $x(t)$ remains in a compact set in $\boldsymbol{R}^n$.

Let K_0 be compact set in $\boldsymbol{R}^n$ such that $u(t) \in K_0$ for all $t \in \boldsymbol{R}$, where $u(t) = \phi^0(t)$ for $t < 0$.

Definition 2. The bounded solution $u(t)$ of (1) is said to be eventually totally (K_0, ρ)-stable, if for any $\varepsilon > 0$ there exist $\alpha(\varepsilon) \geqq 0$ and $\delta(\varepsilon) > 0$ such that if $t_0 \geqq \alpha(\varepsilon)$, $\rho(u_{t_0}, x_{t_0}) < \delta(\varepsilon)$ and $p(t)$ is any continuous function which satisfies $|p(t)| < \delta(\varepsilon)$ on $[t_0, \infty)$, then $\rho(u_t, x_t) < \varepsilon$ for all $t \geqq t_0$, where $x(t)$ is a solution of

$$(2) \qquad \dot{x}(t) = f(t, x(t)) + \int_{-\infty}^{0} F(t, s, x(t+s), x(t))\,ds + h(t, x_t) + p(t)$$

such that $x_{t_0}(s) \in K_0$ for all $s \leqq 0$. If we can choose $\alpha(\varepsilon) \equiv 0$, then $u(t)$ is said to be totally (K_0, ρ)-satble. In the case where $p(t) \equiv 0$, this gives the definition of the uniform (K_0, ρ)-stability of $u(t)$.

When we restrict solutions x to those which remain in K_0, that is, $x(t) \in K_0$

for all $t \geqq t_0$, we say that $u(t)$ is relatively eventually totally (K_0, ρ)-stable and so on.

Then we have the following theorem.

Theorem 1. *Under the assumptions* (A) *through* (D), *if the bounded solution* $u(t)$ of (1) *is relatively eventually totally* (K_0, ρ)*-stable, then* $u(t)$ *is asymptotically almost periodic.*

Proof. Let $\{t_k\}$ be a sequence such that $t_k \to \infty$ as $k \to \infty$. If we set $u^k(t) = u(t + t_k)$, $k = 1, 2, \cdots$, $u^k(t)$ is a solution of

$$(3) \quad \dot{x}(t) = f(t + t_k, x(t)) + \int_{-\infty}^{0} F(t + t_k, s, x(t + s), x(t))ds + h(t + t_k, x_t)$$

and $u^k(t)$ remains in K_0. Since $u(t)$ is relatively eventually totally (K_0, ρ)-stable, $u^k(t)$ is also relatively eventually totally (K_0, ρ)-stable with the same $(\delta(\cdot), \alpha(\cdot))$ as for $u(t)$.

For given $\varepsilon > 0$, there exists a positive integer $k_1(\varepsilon)$ such that $t_k \geqq \alpha(\varepsilon)$ if $k \geqq k_1(\varepsilon)$. Taking a subsequence if necessary, we can assume that $u^k(t)$ converges uniformly on any compact set in $(-\infty, 0]$ as $k \to \infty$. Therefore there exists a positive integer $k_2(\varepsilon)$ such that if $k, m \geqq k_2(\varepsilon)$, $\rho(u_0^k, u_0^m) < \delta(\varepsilon)$. Clearly $u^m(t) = u(t + t_m)$ is a solution of

$$(4) \quad \dot{x}(t) = f(t + t_k, x(t)) + \int_{-\infty}^{0} F(t + t_k, s, x(t + s), x(t))ds + h(t + t_k, x_t) + p(t)$$

and $u^m(t) \in K_0$ for all $t \in \boldsymbol{R}$, where

$$p(t) = f(t + t_m, u^m(t)) + \int_{-\infty}^{0} F(t + t_m, s, u^m(t + s), u^m(t))ds + h(t + t_m, u_t^m)$$
$$- f(t + t_k, u^m(t)) + \int_{-\infty}^{0} F(t + t_k, s, u^m(t + s), u^m(t))ds + h(t + t_k, u_t^m).$$

We shall show that there exists a positive integer $k_0(\varepsilon)$ such that if $k, m \geqq k_0(\varepsilon)$, $|p(t)| < \delta(\varepsilon)$ for $t \geqq 0$. There exists a $c > 0$ such that $|x| \leqq c$ for all $x \in K_0$. It is clear that $|u^k(t)| \leqq c$ and $|u^m(t)| \leqq c$ for all $t \in \boldsymbol{R}$. By assumption (B), there exists an $S = S(c, \varepsilon) > 0$ such that

$$\int_{-\infty}^{-S} |F(t + t_m, s, u^m(t + s), u^m(t))|ds \leqq \delta(\varepsilon)/5 \quad \text{for all} \quad t \in \boldsymbol{R}$$

and

$$\int_{-\infty}^{-S} |F(t + t_k, s, u^m(t + s), u^m(t))|ds \leqq \delta(\varepsilon)/5 \quad \text{for all} \quad t \in \boldsymbol{R}$$

Since $f(t, x)$ and $F(t, s, x, y)$ are almost periodic in t and $h(t, \varphi) \to 0$ as $t \to \infty$, for this S there exists a positive integer $k_0(\varepsilon) \geqq \max(k_1(\varepsilon), k_2(\varepsilon))$ such that if $k, m \geqq k_0(\varepsilon)$,

(5) $$|F(t+t_m, s, u^m(t+s), u^m(t)) - F(t+t_k, s, u^m(t+s), u^m(t))| < \delta(\varepsilon)/5S$$
$$\text{for all } t \in \boldsymbol{R} \text{ and } -S \leqq s \leqq 0,$$

(6) $$|f(t+t_m, u^m(t)) - f(t+t_k, u^m(t))| < \delta(\varepsilon)/5 \quad \text{for all} \quad t \in \boldsymbol{R},$$

and

(7) $$|h(t+t_m, u_t^m) - h(t+t_k, u_t^m)| < \delta(\varepsilon)/5 \quad \text{for all} \quad t \geqq 0.$$

Since we have

$$\left| \int_{-\infty}^0 F(t+t_m, s, u^m(t+s), u^m(t))\, ds - \int_{-\infty}^0 F(t+t_k, s, u^m(t+s), u^m(t))\, ds \right|$$
$$\leqq \int_{-\infty}^{-S} |F(t+t_m, s, u^m(t+s), u^m(t))|\, ds + \int_{-\infty}^{-S} |F(t+t_k, s, u^m(t+s), u^m(t))|\, ds$$
$$+ \int_{-S}^0 |F(t+t_m, s, u^m(t+s), u^m(t)) - F(t+t_k, s, u^m(t+s), u^m(t))|\, ds,$$

we obtain $|p(t)| < \delta(\varepsilon)$ for $t \geqq 0$ if $k, m \geqq k_0(\varepsilon)$. Since $u^m(t)$ is a solution of (4) which remains in K_0 and $u^k(t)$ is relatively eventually totally (K_0, ρ)-stable, we have $\rho(u_t^k, u_t^m) < \varepsilon$ for all $t \geqq 0$ if $k, m \geqq k_0(\varepsilon)$. This implies that if $k, m \geqq k_0(\varepsilon)$,

$$|u(t+t_k) - u(t+t_m)| \leqq \sup_{s \in [-1, 0]} |u(t+t_k+s) - u(t+t_m+s)| < 4\varepsilon$$

for all $\varepsilon \leqq 1/4$ and all $t \geqq 0$. Thus we see that for any sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$, there exists a subsequence $\{t_{k_j}\}$ of $\{t_k\}$ for which $u(t+t_{k_j})$ converges uniformly on $[0, \infty)$ as $j \to \infty$. This shows that $u(t)$ is asymptotically almost periodic in t.

Remark 1. In the case where $h(t, \varphi) \equiv 0$ in system (1), under the condition in Theorem 1, system (1) has an almost periodic solution. For a periodic system, Hamaya [6] has shown the existence of a periodic solution under the stability condition on the bounded solution, while Burton and others have obtained the existence theorems of a periodic solution under the condition that the solutions of the system are g-uniform bounded and g-uniform ultimately bounded (cf. [2]).

We donte by $\Omega(f, F)$ the set of all limit functions $(g(t, x), G(t, s, x, y))$ such that for some sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$, $f(t+t_k, x)$ converges to

$g(t, x)$ uniformly on $\boldsymbol{R} \times S$ for any compact subset S in $\boldsymbol{R}^n$ and $F(t + t_k, s, x, y)$ converges to $G(t, s, x, y)$ uniformly on $\boldsymbol{R} \times S^*$ for any compact subset S^* in R^*. Moreover, we denote by $(v, g, G) \in \Omega(u, f, F)$ when for the same sequence $\{t_k\}$, $u(t + t_k) \to v(t)$ uniformly on any compact subset in $\boldsymbol{R}$ as $k \to \infty$. Then a system

$$(8) \qquad \dot{x}(t) = g(t, x(t)) + \int_{-\infty}^{0} G(t, s, x(t+s), x(t))\, ds$$

is called a limiting equation of (1) when $(g, G) \in \Omega(f, F)$ and $v(t)$ is a solution of (8) when $(v, g, G) \in \Omega(u, f, F)$.

In the followings, we let K be the compact set such that $K = \overline{N(\varepsilon_0, K_0)}$ for some $\varepsilon_0 > 0$, where $\overline{N(\varepsilon_0, K_0)}$ denotes the closure of the ε_0-neighborhood $N(\varepsilon_0, K_0)$ of K_0.

Theorem 2. *Under the assumptions* (A) *through* (D), *assume that system* (1) *admits a limiting equation* (8) *whose solution* $v(t)$ *such that* $(v, g, G) \in \Omega(u, f, F)$ *is totally* (K_0, ρ)*-stable. Then* $u(t)$ *is eventually totally* (K_0, ρ)-stable.

Proof. Since $(v, g, G) \in \Omega(u, f, F)$, there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$, such that $f(t + t_k, x) \to g(t, x)$ uniformly on $\boldsymbol{R} \times K$, $F(t + t_k, s, x, y) \to G(t, s, x, y)$ uniformly on $\boldsymbol{R} \times S^* \times K \times K$ for any compact set S^* in $(-\infty, 0]$ and $u(t + t_k) \to v(t)$ uniformly on any compact set in $\boldsymbol{R}$ as $k \to \infty$. If we set $u^k(t) = u(t + t_k)$, $k = 1, 2, \cdots$, $u^k(t)$ is a solution of

$$(9) \quad \dot{x}(t) = f(t + t_k, x(t)) + \int_{-\infty}^{0} F(t + t_k, s, x(t+s), x(t))\, ds + h(t + t_k, x_t)$$

and $u_0^k(s) \in K_0$ for all $s \leqq 0$. There exists a $c > 0$ such that $|x| \leqq c$ for all $x \in K$. Let $x(\sigma)$ be a continuous function such that $x(\sigma) \in K$ for all $\sigma \leqq t$. By assumption (B), there exists an $S = S(c, \varepsilon) > 0$ such that

$$\int_{-\infty}^{-S} |F(t + t_k, s, x(t+s), x(t))|\, ds \leqq \delta(\delta(\varepsilon/2)/2)/5,$$

where $\delta(\cdot)$ is the number for the total (K_0, ρ)-stability of $v(t)$. Since $F(t + t_k, s, x, y) \to G(t, s, x, y)$, we have

$$\int_{-\infty}^{-S} |G(t, s, x(t+s), x(t))|\, ds \leqq \delta(\delta(\varepsilon/2)/2)/5$$

for the same S. Thus, by the same argument as in the proof of Theorem 1, there exists a positive integer $k_0(\varepsilon)$ such that if $k \geqq k_0(\varepsilon)$,

$$(10)\qquad |f(t+t_k,\ x(t))+\int_{-\infty}^{0} F(t+t_k,\ s,\ x(t+s),\ x(t))\,ds+h(t+t_k,\ x_t)$$
$$-g(t,\ x(t))-\int_{-\infty}^{0} G(t,\ s,\ x(t+s),\ x(t))\,ds|<\delta(\delta(\varepsilon/2)/2)$$

and

$$(11)\qquad \rho(u_0^k,\ v_0)<\delta(\delta(\varepsilon/2)/2).$$

Since $u^k(t)$ is a solution of (9), $u^k(t)$ is a solution of

$$\dot{x}(t)=g(t,\ x(t))+\int_{-\infty}^{0} G(t,\ s,\ x(t+s),\ x(t))\,ds+r(t),$$

where

$$r(t)=f(t+t_k,\ u^k(t))+\int_{-\infty}^{0} F(t+t_k,\ s,\ u^k(t+s),\ u^k(t))\,ds+h(t+t_k,\ u_t^k)$$
$$-g(t,\ u^k(t))-\int_{-\infty}^{0} G(t,\ s,\ u^k(t+s),\ u^k(t))\,ds$$

and $u_0^k(s)\in K_0$ for all $s\leqq 0$. Since $\rho(u_0^k,\ v_0)<\delta(\delta(\varepsilon/2)/2)$ by (11) and $|r(t)|<\delta(\delta(\varepsilon/2)/2)$ for $t\geqq 0$ by (10) if $k\geqq k_0(\varepsilon)$ and since $v(t)$ is totally $(K_0,\ \rho)$-stable, we have

$$(12)\qquad \rho(u_t^k,\ v_t)<\delta(\varepsilon/2)/2 \qquad \text{for all}\quad t\geqq 0.$$

We let $m=k_0(\varepsilon)$ and $\alpha(\varepsilon)=t_m$. We shall show that if $t_0\geqq\alpha(\varepsilon)$, $\rho(u_{t_0},\ y_{t_0})<\delta(\varepsilon/2)/2$ and $|p(t)|<\delta(\varepsilon/2)/2$ for $t\geqq t_0$, then $\rho(u_t,\ y_t)<\varepsilon$ for all $t\geqq t_0$, where $y(t)$ is a solution of

$$\dot{x}(t)=f(t,\ x(t))+\int_{-\infty}^{0} F(t,\ s,\ x(t+s),\ x(t))\,ds+h(t,\ x_t)+p(t)$$

such that $y_{t_0}(s)\in K_0$ for all $s\leqq 0$. Now we assume that $\rho(u_\sigma,\ y_\sigma)=\varepsilon$ for a $\sigma>t_0$ and $\rho(u_t,\ y_t)<\varepsilon$ for $t_0\leqq t<\sigma$. If we set $z(t)=y(t+t_m)$, $z(t)$ is a solution defined on $t_0-t_m\leqq t\leqq\sigma-t_m$ of

$$\dot{x}(t)=f(t+t_m,\ x(t))+\int_{-\infty}^{0} F(t+t_m,\ s,\ x(t+s),\ x(t))\,ds$$
$$+h(t+t_m,\ x_t)+p(t+t_m)$$

such that $z_{t_0-t_m}(s)=y_{t_0}(s)\in K_0$ for all $s\leqq 0$. Moreover, $z(t)$ is a solution of

$$(13)\qquad \dot{x}(t)=g(t,\ x(t))+\int_{-\infty}^{0} G(t,\ s,\ x(t+s),\ x(t))\,ds+q(t),$$

where

$$q(t) = f(t + t_m, z(t)) + \int_{-\infty}^{0} F(t + t_m, s, z(t + s), z(t))ds$$

$$+ h(t + t_m, z_t) + p(t + t_m) - g(t, z(t)) - \int_{-\infty}^{0} G(t, s, z(t + s), z(t))ds.$$

Since $|z(t)| \leqq c$ for $t \leqq \sigma - t_m$ and small $\varepsilon > 0$ and since $|p(t + t_m)| < \delta(\varepsilon/2)/2$ for $t \geqq t_0 - t_m$, we have $|q(t)| < \delta(\varepsilon/2)$ for $\sigma - t_m \geqq t \geqq t_0 - t_m$ by (10). Since $t_0 \geqq t_m$ and $\rho(v_{t_0 - t_m}, u_{t_0}) < \delta(\varepsilon/2)/2$ by (12) and $\rho(u_{t_0}, z_{t_0 - t_m}) = \rho(u_{t_0}, y_{t_0}) < \delta(\varepsilon/2)/2$, we have

$$\rho(v_{t_0 - t_m}, z_{t_0 - t_m}) \leqq \rho(v_{t_0 - t_m}, u_{t_0}) + \rho(u_{t_0}, z_{t_0 - t_m}) < \delta(\varepsilon/2).$$

Thus the total stability of $v(t)$ implies that

(14) $$\rho(v_{\sigma - t_m}, z_{\sigma - t_m}) < \varepsilon/2.$$

On the other hand, (12) implies that

(15) $$\rho(u_t, v_{t - t_m}) < \delta(\varepsilon/2)/2 \qquad \text{for} \quad t \geqq t_0.$$

Therefore, if $t_0 \geqq \alpha(\varepsilon)$, $\rho(u_{t_0}, y_{t_0}) < \delta(\varepsilon/2)/2$ and $|p(t)| < \delta(\varepsilon/2)/2$ for $t \geqq t_0$, then we have

$$\rho(u_\sigma, y_\sigma) \leqq \rho(u_\sigma, v_{\sigma - t_m}) + \rho(v_{\sigma - t_m}, z_{\sigma - t_m}) < \varepsilon.$$

This contradicts $\rho(u_\sigma, y_\sigma) = \varepsilon$. Thus $\rho(u_t, y_t) < \varepsilon$ for all $t \geqq t_0$, if $t_0 \geqq \alpha(\varepsilon)$, $\rho(u_{t_0}, y_{t_0}) < \delta^*(\varepsilon)$ and $|p(t)| < \delta^*(\varepsilon)$ for all $t \geqq t_0$, where $\delta^*(\varepsilon) = \delta(\varepsilon/2)/2$. This shows that $u(t)$ is eventually totally (K_0, ρ)-stable.

In order to obtain the total stability of $u(t)$, we need the continuous dependence in initial conditions. Thus we assume the uniqueness. Without the uniqueness, the best conclusion in Theorem 2 is the eventually total (K_0, ρ)-stability. For the functional differential equation with infinite delay, the uniqueness does not necessarily imply the continuous dependence even for the space (BC, ρ). For examples, see [3], [8]. However, we can show the continuous dependence with respect to K_0 and ρ.

Lemma 1. *Under the assumptions* (A) *through* (D) *($f(t, x)$ and $F(t, s, x, y)$ do not need to be almost periodic in t), suppose that the solution $u = u(t, 0, \phi^0)$, $\phi^0 \in BC$ of* (1) *is unique for the initial value problem. Let $T > 0$ be given. Then for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $t_0 \in [0, T)$, $\rho(u_{t_0}, x_{t_0}) < \delta$ and $p(t)$ is any continuous function which satisfies $|p(t)| \leqq \delta$ for $t_0 \leqq t \leqq T$, we have $\rho(u_t, x_t) < \varepsilon$ for all $t_0 \leqq t \leqq T$, where $x(t)$ is a solution through (t_0, x_{t_0}) of the system*

$$(16)\qquad \dot{x}(t)=f(t,\ x(t))+\int_{-\infty}^{0} F(t,\ s,\ x(t+s),\ x(t))\,ds+h(t,\ x_t)+p(t)$$

such that $x_{t_0}(s)\in K_0$ *for all* $s\leqq 0$.

Proof. Suppose that for some $\varepsilon>0$ there exists no δ that satisfies the condition in Lemma. Then for any integer $k>0$, there exist sequences $\{t_k\}$, $\{r_k\}$, $\{p_k\}$ and $\{x^k\}$ such that $0\leqq t_k<r_k<T$ and p_k: $\boldsymbol{R}\to\boldsymbol{R}^n$ is a continuous function satisfying $|p_k(t)|\leqq 1/k$ and that $\rho(u_{t_k},\ x^k_{t_k})<1/k$, $\rho(u_{r_k},\ x^k_{r_k})=\varepsilon$ and $\rho(u_t,\ x^k_t)<\varepsilon$ on $t_k\leqq t<r_k$, where $x^k(t)$ is a solution of

$$(17)\qquad \dot{x}(t)=f(t,\ x(t))+\int_{-\infty}^{0} F(t,\ s,\ x(t+s),\ x(t))ds+h(t,\ x_t)+p_k(t)$$

such that $x^k_{t_k}(s)\in K_0$ for all $s\leqq 0$. Then we can choose a sequence $\{s_k\}$ such that $t_k<s_k<r_k$, $\rho(u_{s_k},\ x^k_{s_k})=\varepsilon/2$ and $\rho(u_t,\ x^k_t)<\varepsilon/2$ on $t_k\leqq t<s_k$. Also we can assume that $t_k\to\tau$ and $s_k\to s^*$ as $k\to\infty$, where $0\leqq\tau<T$ and $0<s^*<T$. If we set $y^k(t)=x^k(t+t_k-\tau)$, then $y^k(t)$ is defined on $\tau\leqq t\leqq\tau+r_k-t_k$, $y^k(t)$ is a solution of

$$(18)\qquad \dot{x}(t)=f(t+t_k-\tau,\ x(t))+\int_{-\infty}^{0} F(t+t_k-\tau,\ s,\ x(t+s),\ x(t))\,ds$$
$$+\,h(t+t_k-\tau,\ x_t)+p_k(t+t_k-\tau)$$

and $y^k_\tau(s)=x^k_{t_k}(s)\in K_0$ for all $s\leqq 0$.

We can show that the sequence $\{y^k(t)\}$ is uniformly bounded and equicontinuous on $[\tau,\ s^*]$ for all large k. Then there exists a subsequence of $\{y^k(t)\}$ which converges to a continuous function $y(t)$ uniformly on $[\tau,\ s^*]$ as $k\to\infty$. We shall denote it by $\{y^k(t)\}$ again. Since $\rho(u_\tau,\ y^k_\tau)\leqq\rho(u_\tau,\ u_{t_k})+\rho(u_{t_k},\ x^k_{t_k})$ and since $\rho(u_{t_k},\ x^k_{t_k})<1/k$ and $t_k\to\tau$ as $k\to\infty$, we have $\rho(u_\tau,\ y^k_\tau)\to 0$ as $k\to\infty$. Therefore $y^k(\tau+s)\to u(\tau+s)$ uniformly on any compact set in $(-\infty,\ 0]$. On the other hand, $y^k(\tau)\to y(\tau)$ as $k\to\infty$, and hence we have $y(\tau)=u(\tau)$. Define $z(t)$ by

$$z(t)=\begin{cases} y(t) \text{ for } t\in[\tau,\ s^*], \\ u(t) \text{ for } t\in(-\infty,\ \tau]. \end{cases}$$

We shall show that $z(t)$ is a solution of (1) defined on $[\tau,\ s^*]$ such that $z_\tau=u_\tau$. For $t\in[\tau,\ s^*]$, it follows from (18) that

$$(19)\qquad y^k(t)=y^k(\tau)+\int_{\tau}^{t} f(v+t_k-\tau,\ y^k(v))\,dv$$
$$+\int_{\tau}^{t}\int_{-\infty}^{0} F(v+t_k-\tau,\ s,\ y^k(v+s),\ y^k(v))\,dsdv$$

$$+\int_\tau^t h(v+t_k-\tau,\ y_v^k)\,dv+\int_\tau^t p_k(v+t_k-\tau)\,dv.$$

Since there exists a $c>0$ such that $|y^k(t)|\leqq c$ and $|z(t)|\leqq c$ on $(-\infty, s^*]$, there exists an $S=S(\varepsilon, c)>0$ such that

$$\int_{-\infty}^{-S}|F(v+t_k-\tau,\ s,\ y^k(v+s),\ y^k(v))|\,ds<\varepsilon/3$$

and

$$\int_{-\infty}^{-S}|F(v,\ s,\ z(v+s),\ z(v))|\,ds<\varepsilon/3$$

for all $v\in[\tau, s^*]$. Moreover, if $k\geqq N(\varepsilon)$ for some positive integer $N(\varepsilon)$,

$$\int_{-S}^{0}|F(v+t_k-\tau,\ s,\ y^k(v+s),\ y^k(v))-F(v,\ s,\ z(v+s),\ z(v))|\,ds<\varepsilon/3,$$

because $F(v, s, x, y)$ is continuous in v, s, x and y and $y^k(v)\to z(v)$ uniformly on $[-S, s^*]$ and $t_k\to\tau$ as $k\to\infty$. Thus

$$\int_{-\infty}^{0}F(v+t_k-\tau,\ s,\ y^k(v+s),\ y^k(v))\,ds\longrightarrow\int_{-\infty}^{0}F(v,\ s,\ z(v+s),\ z(v))\,ds$$

as $k\to\infty$. By assumption (B) and Lebesgue's convergence theorem, we have

$$\int_\tau^t\int_{-\infty}^{0}F(v+t_k-\tau,\ s,\ y^k(v+s),\ y^k(v))\,ds\,dv$$

$$\longrightarrow\int_\tau^t\int_{-\infty}^{0}F(v,\ s,\ z(v+s),\ z(v))\,ds\,dv$$

as $k\to\infty$. Letting $k\to\infty$ in (19), we have

$$z(t)=z(\tau)+\int_\tau^t f(v,\ z(v))\,dv+\int_\tau^t\int_{-\infty}^{0}F(v,\ s,\ z(v+s),\ z(v))\,ds\,dv$$

$$+\int_\tau^t h(v,\ z_v)\,dv$$

for $t\in[\tau, s^*]$. This shows that $z(t)$ is a solution of (1) such that $u_\tau(s)=z_\tau(s)\in K_0$ for all $s\leqq 0$. Since $y^k(s_k+\tau-t_k+t)=x^k(s_k+t)$, we have $\varepsilon/2=\rho(u_{s_k}, x_{s_k}^k)=\rho(u_{s_k}, y_{s_k+\tau-t_k}^k)$. Thus we have $\rho(u_{s^*}, z_{s^*})=\varepsilon/2$. This contradicts the uniqueness of $u(t)$. This completes the proof.

Then we have the following theorem from Theorem 2 and this lemma.

Theorem 3. *Under the assumption in Theorem 2, if $u(t)$ is the unique solution of* (1) *through* $(0, \phi^0)$, *then $u(t)$ is relatively totally (K_0, ρ)-stable.*

Proof. By Theorem 2, $u(t)$ is eventually totally (K_0, ρ)-stable. For a given $\varepsilon > 0$, let $\delta^*(\varepsilon) > 0$ and $\alpha(\varepsilon) \geqq 0$ be the numbers for eventually total stability of $u(t)$. We can assume that $\alpha(\varepsilon) > 0$. By Lemma 1, corresponding to $\delta^*(\varepsilon) > 0$ and $\alpha(\varepsilon) > 0$, there exists a $\delta_0(\varepsilon) > 0$, $\delta_0(\varepsilon) < \delta^*(\varepsilon)$, such that if $t_0 \in [0, \alpha(\varepsilon))$, $p(t)$ is a continuous function satisfying $|p(t)| < \delta_0(\varepsilon)$ for $t \in [t_0, \alpha(\varepsilon)]$ and $\rho(u_{t_0}, x_{t_0}) < \delta_0(\varepsilon)$, then we have $\rho(u_t, x_t) < \delta^*(\varepsilon)$ for $t \in [t_0, \alpha(\varepsilon)]$, where $x(t)$ is a solution of (16) such that $x_{t_0}(s) \in K_0$ for all $s \leqq 0$. Now we shall show that if $t_0 \geqq 0$, $\rho(u_{t_0}, x_{t_0}) < \delta_0(\varepsilon)$ and $p(t)$ is any continuous function which satisfies $|p(t)| < \delta_0(\varepsilon)$ for $t \geqq t_0$, then we have $\rho(u_t, x_t) < \varepsilon$ for all $t \geqq t_0$, where $x(t)$ is a solution of (2) such that $x(t) \in K_0$ for all $t \geqq t_0$ and $x_{t_0}(s) \in K_0$ for all $s \leqq 0$. In the case where $t_0 < \alpha(\varepsilon)$, we have $\rho(u_{\alpha(\varepsilon)}, x_{\alpha(\varepsilon)}) < \delta^*(\varepsilon)$ and $x_{\alpha(\varepsilon)}(s) \in K_0$ for all $s \leqq 0$. Since $|p(t)| < \delta_0(\varepsilon) < \delta^*(\varepsilon)$ for $t \geqq \alpha(\varepsilon)$, $\rho(u_t, x_t) < \varepsilon$ for $t \geqq \alpha(\varepsilon)$ by eventually total stability of $u(t)$. Since $\rho(u_t, x_t) < \delta^*(\varepsilon)$ for $t \in [t_0, \alpha(\varepsilon)]$, we have $\rho(u_t, x_t) < \varepsilon$ for all $t \geqq t_0$. In the case where $t_0 \geqq \alpha(\varepsilon)$, it is clear from eventually total stability of $u(t)$ that if $\rho(u_{t_0}, x_{t_0}) < \delta_0(\varepsilon)$ and $|p(t)| < \delta_0(\varepsilon)$ for $t \geqq t_0$, then we have $\rho(u_t, x_t) < \varepsilon$ for $t \geqq t_0$, because $\delta_0(\varepsilon) < \delta^*(\varepsilon)$. This completes the proof.

Remark 2. In Theorem 3, if $v(t)$ is totally (K, ρ)-stable, we can conclude that $u(t)$ is also totally (K, ρ)-stable, where $K = \overline{N(\varepsilon_0, K_0)}$ for some $\varepsilon_0 > 0$.

To obtain the following results, we shall give here the definitions for several kinds of stability properties.

Definition 3. The bounded solution $u(t)$ of (1) is said to be uniformly (K_0, ρ)-stable in $\Omega(f, F)$, if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for any $t_0 \geqq 0$ and any $(v, g, G) \in \Omega(u, f, F)$, $\rho(v_{t_0}, x_{t_0}) < \delta(\varepsilon)$ implies $\rho(v_t, x_t) < \varepsilon$ for all $t \geqq t_0$, where $x(t)$ is a solution of (8) such that $x_{t_0}(s) \in K_0$ for all $s \leqq 0$.

Definition 4. The bounded solution $u(t)$ of (1) is said to be (K_0, ρ)-attracting in $\Omega(f, F)$, if there exists a $\delta_0 > 0$ such that for any $t_0 \geqq 0$ and any $(v, g, G) \in \Omega(u, f, F)$, $\rho(v_{t_0}, x_{t_0}) < \delta_0$ implies that $\rho(v_t, x_t) \to 0$ as $t \to \infty$, where $x(t)$ is a solution of (8) such that $x_{t_0}(s) \in K_0$ for all $s \leqq 0$.

Definition 5. The bounded solution $u(t)$ of (1) is said to be weakly uniformly asymptotically (K_0, ρ)-stable in $\Omega(f, F)$, if it is uniformly (K_0, ρ)-stable in $\Omega(f, F)$ and (K_0, ρ)-attracting in $\Omega(f, F)$.

As in Definition 2, when we restrict solution to those which remain in K_0, we say that $u(t)$ is relatively uniformly (K_0, ρ)-stable in $\Omega(f, F)$ and so on.

Let S be a compact subset in (BC, ρ) and let $\alpha > 0$ and $\beta > 0$ be given. Denote by $X(S, \alpha, \beta)$ the set

$$X(S, \alpha, \beta) = \{x_t \in (BC, \rho): t \geqq 0, \text{ where } x(\cdot) \text{ is a function such that } x_0 \in S, |x(s)| \leqq \alpha \quad \text{for } s \in [0, \infty) \text{ and } |x(s^1) - x(s^2)| \leqq \beta |s^1 - s^2| \quad \text{for } s^1, s^2 \in [0, \infty)\}.$$

Then we have the following lemma, which can be proved by the same argument as in the proof of Corollary 3.2 in [5] given by Hale and Kato.

Lemma 2. *$X(S, \alpha, \beta)$ is relatively compact in (BC, ρ).*

Then we have the following propositions, which are essentially the same as Proposition 4.1 and Theorem 4.1 in [9] for the functional differential equation on the axiomatic phase space. To make this paper concrete, we shall give the proofs.

Proposition 1. *Under the assumptions* (A) *through* (D), *if $u(t)$ is weakly uniformly asymptotically (K, ρ)-stable in $\Omega(f, F)$, then the bounded solution $u(t)$ of* (1) *is eventually totally* (K, ρ)-*stable. Moreover, if $u(t)$ is the unique solution of* (1) *through* $(0, \phi^0)$, *then $u(t)$ is totally (K, ρ)-stable. Here $K = \overline{N(\varepsilon_0, K_0)}$ for some $\varepsilon_0 > 0$.*

Proof. Suppose that $u(t)$ is not eventually totally (K, ρ)-stable. Then there exists a small $\varepsilon > 0$ and sequence $\{s_k\}$, $\{t_k\}$, $\{p_k\}$, $\{x^k\}$ such that $s_k \to \infty$ as $k \to \infty$, $0 < s_k < t_k$, $p_k: \boldsymbol{R} \to \boldsymbol{R}^n$ is a continuous function satisfying $|p_k(t)| < 1/k$ for $t \geqq s_k$ and that $\rho(u_{s_k}, x_{s_k}^k) < 1/k$,

$$\rho(u_{t_k}, x_{t_k}^k) = \varepsilon \text{ and } \rho(u_t, x_t^k) < \varepsilon \quad \text{on} \quad [s_k, t_k), \tag{20}$$

where $x^k(t)$ is a solution of

$$\dot{x}(t) = f(t, x(t)) + \int_{-\infty}^{0} F(t, s, x(t+s), x(t))\, ds + h(t, x_t) + p_k(t) \tag{21}$$

such that $x_{s_k}^k(s) \in K$ for all $s \leqq 0$. We can assume that $\varepsilon < \delta_0$, where δ_0 is the number for (K, ρ)-attracting in $\Omega(f, F)$. Moreover, there exists a sequence $\{\tau_k\}$ such that $s_k < \tau_k < t_k$,

$$\rho(u_{\tau_k}, x_{\tau_k}^k) = \delta(\varepsilon/2)/2 \tag{22}$$

and

$$\delta(\varepsilon/2)/2 \leqq \rho(u_t, x_t^k) \leqq \varepsilon \quad \text{for} \quad t \in [\tau_k, t_k], \tag{23}$$

where $\delta(\cdot)$ is the number for uniform (K, ρ)-stability in $\Omega(f, F)$.

Now we set $2r_k = t_k - \tau_k$. Suppose that $r_k \to \infty$ as $k \to \infty$. For the sequence $\{\tau_k + r_k\}$, taking a subsequence if necessary, there exists a $(v, g, G) \in \Omega(u, f, F)$. If we set $y^k(t) = x^k(t + \tau_k + r_k)$, then $y^k(t)$ is defined on $-r_k \leqq t \leqq r_k$ and $y^k(t)$ is a solution of

$$\text{(24)} \qquad \dot{x}(t) = f(t + \tau_k + r_k, x(t)) + \int_{-\infty}^{0} F(t + \tau_k + r_k, s, x(t+s), x(t))\,ds$$
$$+ h(t + \tau_k + r_k, x_t) + p_k(t + \tau_k + r_k)$$

such that $y^k_{-r_k}(s) = x^k_{\tau_k}(s) \in K$ for all $s \leqq 0$. Then we can show that, taking a subsequence if necessary, $y^k(t)$ converges to a solution $y(t)$ of

$$\dot{x}(t) = g(t, x(t)) + \int_{-\infty}^{0} G(t, s, x(t+s), x(t))\,ds$$

such that $y_0(s) \in K$ for $s \leqq 0$. Letting $k \to \infty$ in (23), we have $\delta(\varepsilon/2) \leqq \rho(v_t, y_t) \leqq \varepsilon$ on $t \geqq 0$. Since $\varepsilon < \delta_0$ and $u(t)$ is attracting in $\Omega(f, F)$, we have $\delta(\varepsilon/2)/2 \leqq \rho(v_t, y_t) \to 0$ as $t \to \infty$, which is a contradiction. Thus $r_k \not\to \infty$ as $k \to \infty$. Taking a subsequence again if necessary, we can assume that $r_k \to r < \infty$ and that for the sequence $\{\tau_k\}$ there exists a $(\hat{v}, \hat{g}, \hat{G}) \in \Omega(u, f, F)$. If we set $z^k(t) = x^k(t + \tau_k)$, then $z^k(t)$ is defined on $s_k - \tau_k \leqq t \leqq 2r_k$. We can assume that $z^k(t) \to z^*(t)$ uniformly on $[0, 2r]$ as $k \to \infty$ and $u_{s_k} \to w$ for some $w \in (BC, \rho)$. Since $\rho(u_{s_k}, x^k_{s_k}) < 1/k$, we have $x^k_{s_k} \to w$. Therefore the closure S of $\{x^k_{s_k}\}$ is a compact set in (BC, ρ). Moreover, there are positive constants c and L^* such that $|x^k(t)| \leqq c$ and $|\dot{x}^k(t)| \leqq L^*$ for all $t \in [s_k, t_k]$. Let X_0 be the closure of $X(S, c, L^*)$. Since $x^k_{\tau_k} \in X_0$, taking a subsequence, $x^k_{\tau_k} \to x_0$ as $k \to \infty$ for some $x_0 \in X_0$. On the other hand, $z^k(0) \to z^*(0)$ and $z^k(0) = x^k_{\tau_k}(0) \to x_0(0)$ as $k \to \infty$. Thus $x_0(0) = z^*(0)$. Define $z(t)$ by

$$z(t) = \begin{cases} z^*(t) & \text{for } t \in [0, 2r], \\ x_0(t) & \text{for } t \in (-\infty, 0]. \end{cases}$$

Then $z(t)$ is a solution of

$$\dot{x}(t) = \hat{g}(t, x(t)) + \int_{-\infty}^{0} \hat{G}(t, s, x(t+s), x(t))\,ds,$$

which is defined on $[0, 2r]$ and $z_0(s) \in K$ for all $s \leqq 0$. Letting $k \to \infty$ in (22) and (20), we have $\rho(\hat{v}_0, z_0) = \delta(\varepsilon/2)/2 < \delta(\varepsilon/2)$ and $\rho(\hat{v}_{2r}, z_{2r}) = \varepsilon$. But $u(t)$ is uniformly (K, ρ)-stable in $\Omega(f, F)$. Thus $\rho(\hat{v}_0, z_0) < \delta(\varepsilon/2)$ yields $\rho(\hat{v}_{2r}, z_{2r}) < \varepsilon/2$, which contradicts $\rho(\hat{v}_{2r}, z_{2r}) = \varepsilon$. This shows that $u(t)$ is eventually totally (K, ρ)-stable. The remainder can be proved by the same argument as in the last part of the proof of Theorem 3, using the uniqueness of $u(t)$ for initial value problem.

Remark 3. If the bounded solution $u(t)$ is relatively weakly uniformly asymptotically (K_0, ρ)-stable in $\Omega(f, F)$, then $u(t)$ is relatively eventually totally (K_0, ρ)-stable.

Definition 6. The bounded solution $u(t)$ of (1) is said to be totally asymptotically (K_0, ρ)-stable, if it is totally (K_0, ρ)-stable and if there exists a $\delta_0 > 0$ and for any $\varepsilon > 0$ there exists an $\eta(\varepsilon) > 0$ and a $T(\varepsilon) > 0$ such that if $t_0 \geqq 0$, $\rho(u_{t_0}, x_{t_0}) < \delta_0$ and $p(t)$ is any continuous function which satisfies $|p(t)| < \eta(\varepsilon)$ for $t \geqq t_0$, then $\rho(u_t, x_t) < \varepsilon$ for all $t \geqq t_0 + T(\varepsilon)$, where $x(t)$ is a solution of

$$(25) \qquad \dot{x}(t) = f(t, x(t)) + \int_{-\infty}^{0} F(t, s, x(t+s), x(t))ds + h(t, x_t) + \mathrm{p}(t)$$

such that $x_{t_0}(s) \in K_0$ for all $s \leqq 0$. In particular, in the case where $p(t) \equiv 0$, this gives the definition of the uniformly asymptotic (K_0, ρ)-stability of $u(t)$.

Proposition 2. *Under the assumptions* (A) *through* (D), *if* $u(t)$ *is* (K, ρ)-*attracting in* $\Omega(f, F)$ *and if* $u(t)$ *is totally* (K, ρ)-*stable, then it is totally asymptotically* (K, ρ)-*stable.*

Proof. Let δ_0 be the number for (K, ρ)-attracting of $u(t)$ in $\Omega(f, F)$, and set $\delta_0^* = \delta(\delta_0/2)$, where $\delta(\cdot)$ is the number for the (K, ρ)-total stability of $u(t)$. For this δ_0^* and for any $\varepsilon > 0$, we shall show that there exist $T(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ which satisfy the condition in Definition 6. Suppose that this is not the case. Then there exists an $\varepsilon > 0$, $\varepsilon \leqq \delta_0/4$ and sequences $\{s_k\}$, $\{t_k\}$, $\{p_k\}$, $\{x^k\}$ such that $s_k \geqq 0$, $t_k \geqq s_k + 2k$, $\rho(u_{s_k}, x^k_{s_k}) < \delta_0^*$ and

$$(26) \qquad \rho(u_{t_k}, x^k_{t_k}) \geqq \varepsilon \qquad \text{for all} \quad k = 1, 2, \cdots,$$

where $p_k \colon \boldsymbol{R} \to \boldsymbol{R}^n$ is a continuous function satisfying $|p_k(t)| < \min(1/k, \delta_0^*)$ for $t \geqq s_k$ and $x^k(t)$ is a solution of

$$\dot{x}(t) = f(t, x(t)) + \int_{-\infty}^{0} F(t, s, x(t+s), x(t))ds + h(t, x_t) + p_k(t)$$

such that $x^k_{s_k}(s) \in K$ for all $s \leqq 0$. Since $\rho(u_{s_k}, x^k_{s_k}) < \delta(\delta_0/2)$ and $|p_k(t)| < \delta(\delta_0/2)$ for $t \geqq s_k$, we have

$$(27) \qquad \rho(u_t, x^k_t) < \delta_0/2 \text{ for all } t \geqq s_k \quad \text{and} \quad k = 1, 2, \cdots,$$

because $u(t)$ is totally (K, ρ)-stable. There exists an integer $k_0 = k_0(\varepsilon) > 0$ such that if $k \geqq k_0$, we have $|p_k(t)| < 1/k < \delta(\varepsilon)$ for $t \geqq s_k$. If $\rho(u_t, x^k_t) < \delta(\varepsilon)$ for some $t \in [s_k + k, s_k + 2k]$, $\rho(u_t, x^k_t) < \varepsilon$ for $t \geqq s_k + 2k$, since $u(t)$ is (K, ρ)-stable. This contradicts (26), because $t_k \geqq s_k + 2k$. Thus we have

$$(28) \qquad \rho(u_t, x^k_t) \geqq \delta(\varepsilon) \qquad \text{for all} \quad t \in [s_k + k, s_k + 2k]$$

if $k \geqq k_0$.

For the sequence $\{s_k + k\}$, taking a subsequence if necessary, there exists a $(v, g, G) \in \Omega(u, f, F)$. If we set $y^k(t) = x^k(t + s_k + k)$, then $y^k(t)$ is defined on $-k \leqq t \leqq k$. We can show that there exists a subsequence of $\{y^k(t)\}$, which we denote by $\{y^k(t)\}$ again, and a contiuous function $y(t)$ such that $y^k(t) \to y(t)$ uniformly on any compact subset of $\boldsymbol{R}$ as $k \to \infty$. By the same argument as in the proof of Lemma 1, we can see that $y(t)$ is a solution of

$$\dot{x}(t) = g(t, x(t)) + \int_{-\infty}^{0} G(t, s, x(t+s), x(t))\, ds$$

such that $y_0(s) \in K$ for all $s \leqq 0$. It follows from (27) and (28) that $\delta(\varepsilon) \leqq \rho(u_{t+s_k+k}, y_t^k) < \delta_0/2$ for all $t \in [0, k]$ if $k \geqq k_0$. Letting $k \to \infty$, we have $\delta(\varepsilon) \leqq \rho(v_t, y_t) \leqq \delta_0/2$ for all $t \geqq 0$. But $u(t)$ is (K, ρ)-attracting in $\Omega(f, F)$. Thus $\rho(v_t, y_t) \to 0$ as $t \to \infty$. This contradicts $\rho(v_t, y_t) \geqq \delta(\varepsilon)$, which shows that $u(t)$ is totally asymptotically (K, ρ)-stable.

We say that system (1) is regular, if the solutions of every limiting equation of (1) are unique for the initial value problem.

The following lemma can be proved in the same argument as in the proof of Lemma 7 in [11], so we shall omit the proof.

Lemma 3. *Under the assumptions* (A) *through* (D), *suppose that system* (1) *is regular. If the bounded solution $u(t)$ of* (1) *is uniformly (K_0, ρ)-stable, then $u(t)$ is uniformly (K_0, ρ)-stable in $\Omega(f, F)$. Moreover, if $u(t)$ is uniformly asymptotically (K_0, ρ)-stable, then $u(t)$ is also uniformly asymptotically (K_0, ρ)-stable in $\Omega(f, F)$.*

Remark 4. It should be noticed here that the relatively uniformly (K_0, ρ)-stability is not inherited and so for the relatively uniformly asymptotic (K_0, ρ)-stability.

Applying the above results, we have the following result, which corresponds to D'Anna's result for the ordinary differential equation and to the result by Hino and Yoshizawa for the functional differential equation with infinite delay.

Theorem 4. *Under the assumptions* (A) *through* (D), *suppose that system* (1) *is regular and $u(t)$ is the unique solution of* (1) *through* $(0, \phi^0)$. *If system* (1) *admits a limiting equation* (8) *whose solution $v(t)$, where $(v, g, G) \in \Omega(u, f, F)$, is uniformly asymptotically (K, ρ)-stable, then $u(t)$ is totally asymptotically (K, ρ)-stable, where $K = \overline{N(\varepsilon_0, K_0)}$ for some $\varepsilon_0 > 0$.*

Proof. Since $\Omega(g, G) = \Omega(f, F)$ and system (1) is regular, system (8) is also regular. Applying Lemma 3 to $v(t)$, we can see that $v(t)$ is uniformly asymptotically (K, ρ)-stable in $\Omega(g, G)$, and hence $v(t)$ is weakly uniformly asymptotically (K, ρ)-stable in $\Omega(g, G)$. Also $v(t)$ is the unique solution through $(0, v_0)$ of (8). Applying Proposition 1 to $v(t)$, we can see that $v(t)$ is totally (K, ρ)-stable. Since $u(t)$ is unique, it follows from Theorem 3 that $u(t)$ is totally (K, ρ)-stable. Thus, by Theorem 1, $u(t)$ is asymptotically almost periodic in t, and hence $\Omega(u, f, F) = \Omega(v, g, G)$. Since $v(t)$ is weakly uniformly asymptotically (K, ρ)-stable in $\Omega(g, G)$, $u(t)$ is (K, ρ)-attracting in $\Omega(f, F)$. Thus it follows from Proposition 2 that $u(t)$ is totally asymptotically (K, ρ)-stable.

Remark 5. In Theorem 4, if $v(t)$ is uniformly asymptotically (K_0, ρ)-stable, we can conclude only that $u(t)$ is relatively totally asymptotically (K_0, ρ)-stable.

Finally, we shall show that Theorem 1 and Remark 3 to Proposition 1 can be applied to the existence of a strictly positive almost periodic solution of a prey-predator equation

$$\text{(E)} \qquad \begin{cases} \dot{x}_1(t) = x_1(t)[b_1(t) - a_1(t)x_1(t) - c_2(t)\displaystyle\int_{-\infty}^{t} K_2(t-u)x_2(u)\,du] \\ \dot{x}_2(t) = x_2(t)[-b_2(t) - a_2(t)x_2(t) + c_1(t)\displaystyle\int_{-\infty}^{t} K_1(t-u)x_1(u)\,du], \end{cases}$$

where $a_i(t)$, $b_i(t)$ and $c_i(t)$ $(i = 1, 2)$ are continuous and almost periodic in t, and K_i: $[0, \infty) \to [0, \infty)$ $(i = 1, 2)$ denote delay kernels such that

$$\int_0^\infty K_i(s)\,ds = 1 \ \text{ and } \int_0^\infty sk_i(s)\,ds < \infty \quad (i = 1, 2).$$

Definition 7. A set K_0 in $\boldsymbol{R}^n$ is said to be invariant for system (1) if, for any initial date (t_0, ϕ) with $t_0 \geqq 0$ and $\phi \in BC$ such that $\phi(s) \in K_0$ for all $s \leqq 0$, the solution $x(t, t_0, \phi)$ of (1) through (t_0, ϕ) remains in K_0 for all $t \geqq t_0$.

Then we have the following lemma.

Lemma 4. *If system* (1) *is regular and if a compact set* K_0 *in* $\boldsymbol{R}^n$ *is invariant for system* (1), *then* K_0 *is invariant also for every limiting equation of system* (1).

Proof. Let

$$\text{(30)} \qquad \dot{x}(t) = g(t, x(t)) + \int_{-\infty}^{0} G(t, s, x(t+s), x(t))\,ds$$

be a limiting equation of (1). Since $(g, G) \in \Omega(f, F)$, there exists a sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$ and that $f(t + t_k, x) \to g(t, x)$ uniformly on $\boldsymbol{R} \times K_0$,

$F(t+t_k, s, x, y) \to G(t, s, x, y)$ uniformly on $\boldsymbol{R} \times S^* \times K_0 \times K_0$ for any compact set S^* in $(-\infty, 0]$ as $k \to \infty$. Let $t_0 \geqq 0$, $\psi \in BC$ such that $\psi(s) \in K_0$ for all $s \leqq 0$, and let $y(t)$ be a solution of (30) through (t_0, ψ). Let $x^k(t)$ be the solution of (1) through (t_0+t_k, ψ). Then $x^k_{t_0+t_k}(s) = \psi(s) \in K_0$ for all $s \leqq 0$ and $x^k(t)$ is defined on $t \geqq t_0 + t_k$. Since K_0 is invariant for system (1), $x^k(t) \in K_0$ for all $t \geqq t_0 + t_k$. If we set $z^k(t) = x^k(t+t_k)$, $k = 1, 2, \cdots$, then $z^k(t)$ is defined on $t \geqq t_0$ and is a solution of

$$\text{(31)} \qquad \dot{x}(t) = f(t+t_k, x(t)) + \int_{-\infty}^{0} F(t+t_k, s, x(t+s), x(t))\,ds + h(t+t_k, x_t)$$

such that $z^k_{t_0}(s) = x^k_{t_0+t_k}(s) = \psi(s) \in K_0$ for all $s \leqq 0$. Since $x^k(t) \in K_0$ for all $t \geqq t_0 + t_k$, $z^k(t) \in K_0$ for all $t \geqq t_0$. Since the sequence $\{z^k(t)\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$ and $z^k_{t_0} = \psi$, $\{z^k(t)\}$ can be assumed to converge to the solution $y(t)$ of (30) through (t_0, ψ) uniformly on any compact set $[t_0, \infty)$, because $y(t)$ is the unique solution through (t_0, ψ). Therefore, $y(t) \in K_0$ for all $t \geqq t_0$ since $z^k(t) \in K_0$ for all $t \geqq t_0$ and K_0 is compact. This shows that K_0 is invariant for limiting equation (30).

In system (E), setting $a_i = \inf_{t\in\boldsymbol{R}} a_i(t)$, $A_i = \sup_{t\in\boldsymbol{R}} a_i(t)$, $b_i = \inf_{t\in\boldsymbol{R}} b_i(t)$, $B_i = \sup_{t\in\boldsymbol{R}} b_i(t)$, $c_i = \inf_{t\in\boldsymbol{R}} c_i(t)$, and $C_i = \sup_{t\in\boldsymbol{R}} c_i(t)$ $(i = 1, 2)$, we now assume that

(i) $a_i > 0$, $b_i > 0$ $(i = 1, 2)$ and $c_1 > 0$, $c_2 \geqq 0$,
(j) $b_1 > B_1C_1C_2/a_1a_2$ and $B_2 < b_1c_1/A_1 - B_1c_1C_1C_2/a_1a_2A_1$,
(k) there exists a positive constant m such that $a_i > C_i + m$ $(i = 1, 2)$.

If we set

$$\alpha_1 = B_1/a_1,\ \alpha_2 = B_1C_1/a_1a_2,\ \beta_1 = b_1/A_1 - B_1C_1C_2/a_1a_2A_1$$

and

$$\beta_2 = b_1c_1/A_1A_2 - B_2/A_2 - B_1c_1C_1C_2/a_1a_2A_1A_2,$$

then $0 < \beta_i < \alpha_i$ for each $i = 1, 2$. If $u(t) = (u_1(t), u_2(t))$ is a solution of (E) through $(0, \phi)$ such that $\beta_i \leqq \phi_i(s) \leqq \alpha_i$ for all $s \leqq 0$, then we have $\beta_i \leqq u_i(t) \leqq \alpha_i$, $(i = 1, 2)$, for all $t \geqq 0$. Let K_0 be a bounded closed set in $\boldsymbol{R}^2$ such that

$$K_0 = \{(x_1, x_2) \in \boldsymbol{R}^2;\ \beta_i \leqq x_i \leqq \alpha_i \text{ for } i = 1, 2\}.$$

Then K_0 is invariant for system (E), and hence K_0 is invariant for its limiting equations by Lemma 4. Now we consider a Liapunov functional

$$\text{(32)} \qquad V(t, u(\cdot), x(\cdot)) = \sum_{i=1}^{2} \{|\log u_i(t) - \log x_i(t)|$$

$$+\int_0^\infty K_i(s)\int_{t-s}^t c_i(s+v)|u_i(v)-x_i(v)|dvds\},$$

where x is a solution of (E) which remains in K_0. Calculating the derivative, we have

$$\dot V(t,\ u(\cdot),\ x(\cdot)) \leqq -m\sum_{i=1}^2 |u_i(t)-x_i(t)|. \tag{33}$$

Thus $\sum_{i=1}^2 |u_i(t)-x_i(t)| \to 0$ as $t\to\infty$, and hence $\rho(u_t, x_t)\to 0$ as $t\to\infty$, because the function $\sum_{i=1}^2 |u_i(t)-x_i(t)|$ is uniformly continuous on $[0, \infty)$. Moreover, we can show that $u(t)$ is uniformly (K_0, ρ)-stable, by the same argument as in [10]. By Lemma 3, $u(t)$ is uniformly (K_0, ρ)-stable in $\Omega(E)$. By using a similar Liapunov functional to (32), we can show that $u(t)$ is (K_0, ρ)-attracting in $\Omega(E)$. Therefore $u(t)$ is weakly uniformly asymptotically (K_0, ρ)-stable in $\Omega(E)$. Thus, by Remark 3 to Proposition 1, $u(t)$ is eventually totally (K_0, ρ)-stable, because K_0 is invariant. Therefore it follows from Theorem 1 that system (E) has an almost periodic solution $p(t)$ such that $\beta_i \leqq p_i(t) \leqq \alpha_i$, $(i=1,2)$, for all $t\in \boldsymbol{R}$.

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