Funkcialaj Ekvacioj, 33 (1990) 127–138

On Existence of Periodic Solutions for a Class of Quasilinear Non-Coercive Problems

By

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1. Introduction

We study nonlinear boundary value problems in $n+1$ dimensional space $(x_1, \dots, x_n, y)$ of the type

$$(1.1)\qquad \begin{aligned} u_y - \sum_{k=1}^{\ell} a_k(x) D^{\alpha_k} u &= g(x, D^{\alpha_1}u, \dots, D^{\alpha_\ell}u) \qquad y = 1\,, \\ \Delta u &= f(x, y, u, Du) \qquad 0 < y < 1\,, \\ u &= 0 \qquad y = 0\,, \end{aligned}$$

where multi-index $\alpha_k = (\alpha_{k1}, \dots, \alpha_{kn}, 0)$, the given functions f and g and the unknown function u are 2π periodic in each variable x_i. In [2–4] we had studied the case $n = 2$, and the boundary condition $u_y - F u_{xx} = g(x, u)$ at $y = 1$. When the constant $F < 0$, the problem comes from three-dimensional water wave theory in the absence of surface tension, see [8]. We are interested in the problem primarily since it represents a model non-coercive problem (i.e. the Lopatinski-Schapiro condition fails at $y = 1$, see [3], and hence one cannot use the standard elliptic theory). In [2–4] we had considered only the case $F > 0$, as in the physical case $F < 0$ one has a difficulty caused by presence of small divisors.

In this paper we extend the results of [2, 3] in two directions. In section 3 we consider non-coercive problems of type (1.1) with nonlinear boundary conditions of arbitrarily high order. We introduce a notion of dominating derivatives, which plays a role similar to that of the derivatives of the highest order in the coercive case. We state conditions allowing establishment of a priori estimates, and prove existence results for nonlinear problems.

In Section 4, we consider the case $F < 0$, which leads to small divisors. To see the difficulty, let us consider the problem (4.7) with $f = 0$. Look for a solution in the form $u = \sum_{j,k=-\infty}^{\infty} u_{jk}(y) e^{ijx+ikz}$, then $u_{jk}(y) = c_{jk} \sinh \sqrt{j^2 + k^2}\, y$. If $g(x, z) = \sum_{j,k=-\infty}^{\infty} g_{jk} e^{ijx+ikz}$, then to satisfy the boundary condition at $y = 1$ we must solve $c_{jk}(\sqrt{j^2+k^2} \cosh \sqrt{j^2+k^2} - F j^2 \sinh \sqrt{j^2+k^2}) = g_{jk}$, which involves division by possibly arbitrarily small numbers.

We distinguish between two and three dimensional cases. For $n = 2$, it turns out there are really no "small divisors", i.e. for $F = F_j \equiv (\coth j)/j$, $j = 1, 2, \ldots$, we have zero divisors, while for $F \neq F_j$ divisors are bounded away from zero. We are then able to derive a priori estimates and establish existence results for nonlinear problems. For $n = 3$ the situation is more involved. We restrict ourselves to rational F, and discover that for $F > 1/2$ it is again either zero divisors or divisors bounded away from zero, while for $F \leqslant 1/2$ one can get arbitrarily small divisors. The a priori estimates which we derive for $F > 1/2$ allow us to prove existence for the linear problem, which is nontrivial in the presence of everywhere dense set of zero divisors.

2. Notation and the preliminary results

We consider functions of $n + 1$ variables $(x_1, \ldots, x_n, y)$ which are 2π periodic in each variable x_i, and $0 \leqslant y \leqslant 1$. By V we denote the domain $0 \leqslant x_i \leqslant 2\pi$, $i = 1, \ldots, n$, $0 \leqslant y \leqslant 1$; its boundary we denote by ∂V, and the top $(y = 1)$ part of the boundary by V_t. By $D^\alpha u$ we understand the derivative corresponding to the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$, $|\alpha| = \alpha_1 + \cdots + \alpha_{n+1} \geqslant 0$. Also,

$$u_i = \partial u/\partial x_i\,, \qquad u_{ij} = \partial^2 u/\partial x_i \partial x_j\,.$$

We shall denote

$$\int f = \int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^1 f(x_1, \ldots, x_n, y) dx_1 \ldots dx_n dy\,,$$

$$\int_t f = \int_0^{2\pi} \cdots \int_0^{2\pi} f(x_1, \ldots, x_n, 1) dx_1 \ldots dx_n\,.$$

By $\|\cdot\|_m$ we denote the norm in the Sobolev space $H^m(V)$, and by $\overline{\|\cdot\|}_m$ the one in $H^m(V_t)$. We shall also need the norms

$$|f|_N = \sum_{|\alpha| \leqslant N} |D^\alpha f|_{L^\infty(V)}\,, \qquad N = \text{integer} \geqslant 0\,.$$

All irrelevant positive constants independent of unknown functions we denote by c; $Du = \nabla u$, $i = 1, \ldots, n + 1$.

We shall need the following relations between our norms, see [3].

Lemma 2.1. *Assume that $v \in H^{m+1}(V)$. For any integer $m \geqslant 0$ and any $\varepsilon > 0$ one can find a constant $c(\varepsilon)$ so that*

(i) $\overline{\|v\|}_m \leqslant \|v\|_{m+1}$

(ii) $\|v\|_m \leqslant \varepsilon \|v\|_{m+1} + c(\varepsilon) \|v\|_0$

(iii) $\overline{\|v\|}_m \leqslant \varepsilon \|v\|_{m+1} + c(\varepsilon) \|v\|_0$.

The following lemma is taken essentially from [5].

Lemma 2.2. *Suppose that the functions* $w_1, \ldots, w_s \in C^m(V)$ *or* $C^m(V_t)$. *Suppose that* $\phi = \phi(w_1, \ldots, w_s)$ *possesses continuous derivatives up to order* $m \geqslant 1$ *bounded by c on* $\max_i |w_i| < 1$. *Then*

$$\text{(i)} \qquad \|\phi(w_1, \ldots, w_s)\|_m \leqslant c(\|w\|_m + 1) \quad \textit{for } \max_i \|w_i\|_{L^\infty} < 1 .$$

(*We denote* $\|w\|_m = \max_i \|w_i\|_m$). *If in addition we assume* $\phi(0, \ldots, 0) = 0$, $m \geqslant 1$ *then*

$$\text{(ii)} \qquad \|\phi(w_1, \ldots, w_s)\|_m = \delta(\|w\|_m) ,$$

where $\delta(t) \to 0$ *as* $t \to 0$.

Remark. The lemma is also true for functions $\phi = \phi(x, y, w_1, \ldots, w_s)$ with $\phi \in C^m$ in all variables. Conclusion (i) is as before, and for (ii) the corresponding assumption is $\phi(x, y, 0, \ldots, 0) = 0$ for all $(x, y) \in V$ or V_t.

Lemma 2.3. *Let* $\alpha_1, \ldots, \alpha_\ell$ *be some collection of multi-indices,* $k_0 = \max_{1 \leqslant k \leqslant l} |\alpha_k|$. *Consider the subset* G^m *of functions in* $H^m(V)$ *such that*

$$|||u|||_m \equiv \|u\|_m + \sum_{k=1}^{\ell} \overline{\|D^{\alpha_k} u\|_{m-1}} < \infty .$$

Then G^m *with norm* $|||\cdot|||_m$ *is a Banach space, provided that* $m \geqslant k_0 + [(n+1)/2] + 1$.

Proof. To prove completeness, let $\{u^r\}$ be a Cauchy sequence in G^m, i.e. $\|u^r - u^p\|_m + \sum_{k=1}^{\ell} \overline{\|D^{\alpha_k} u^r - D^{\alpha_k} u^p\|_{m-1}} \to 0$, as $r, p \to \infty$. Since $H^m(V)$ and $H^{m-1}(V_t)$ are Banach spaces, $u^r \to u$ in $H^m(V)$, and $D^{\alpha_k} u^r \to v_{\alpha_k}$ in $H^{m-1}(V_t)$. It remains to show that $v_{\alpha_k} = D^{\alpha_k} u(x, 1)$. Indeed, both functions are continuous and $\overline{\|v_{\alpha_k} - D^{\alpha_k} u\|_0} \leqslant \overline{\|v_{\alpha_k} - D^{\alpha_k} u^r\|_{m-1}} + \overline{\|D^{\alpha_k} u^r - D^{\alpha_k} u\|_{m-k_0}} \to 0$ as $r \to \infty$.

3. A priori estimates and existence results

Consider the problem (non-coercive in general)

$$(3.1) \qquad \begin{aligned} u_y + \sum_{k=1}^{\ell} r_k(x) D^{\alpha_k} u &= g(x) & y &= 1 , \\ \Delta u &= f(x, y) & 0 < y &< 1 , \\ u &= 0 & y &= 0 . \end{aligned}$$

Here $u = u(x, y)$, $x = (x_1, \ldots, x_n)$, $0 \leqslant y \leqslant 1$, D^{α_k} denotes derivatives in x

variables only, $|\alpha_k| = \alpha_{k1} + \cdots + \alpha_{kn}$; $r_k(x) = a_k + \rho_k(x)$, $a_k = \text{const}$, $\ell = \text{integer} \geqslant 1$. Throughout this section u, f, g, ρ_k are assumed to be 2π periodic in each x_i. We shall study solvability and derive a priori estimates for the problem (3.1) without restricting $\max|\alpha_k|$, the order of the boundary operator, and without requiring it to be coercive. Then we consider nonlinear problems.

Definition. We say that derivative $D^\beta u$ is subordinate to $D^\alpha u$ if $\alpha_i \geqslant \beta_i$, $i = 1, \ldots, n$, and $\beta_i \neq 0$ if $\alpha_i \neq 0$. We say that derivative $D^{\alpha_k} u$ is a dominating derivative in a set $S = \{D^{\alpha_1} u, \ldots, D^{\alpha_\ell} u\}$, if it is not subordinate to any other derivative in that set.

Clearly, there can be several dominating derivatives in a set, and of different orders.

Lemma 3.1. *For the problem* (3.1) *assume that* $(-1)^{|\alpha_k|/2} a_k \geqslant 0$ *for all even* $|\alpha_k|$, *and either* $(-1)^{(|\alpha_k|+1)/2} a_k \geqslant 0$ *holds for all odd* $|\alpha_k|$, *or the opposite inequality does. The above inequalites are assumed to be strict for all k corresponding to the dominating derivatives in the set S. Then for* $\max_k |\rho_k|_m$ *sufficiently small the following a priori estimate holds* (m = *integer* $\geqslant 0$)

$$\|u\|_{m+1} + \sum_{k=1}^{\ell} \overline{\|D^{\alpha_k} u\|}_m \leqslant c\left(\sum_{k=1}^{\ell} \|D^{\alpha_k} f\|_m + \|f\|_m + \|g\|_m\right). \tag{3.2}$$

Proof. We begin by assuming that $\rho_k(x) \equiv 0$ for all k. Let $u = \sum_j u_j(y) e^{ij\cdot x}$, $f = \sum_j f_j(y) e^{ij\cdot x}$, $g = \sum_j g_j e^{ij\cdot x}$, $\rho = \sqrt{j_1^2 + \cdots + j_n^2}$. Substituting these into (3.1) and suppressing the multi-index j, we get:

$$u'(1) + \sum_{k=1}^{\ell} a_k (ij)^{\alpha_k} u(1) = g\,, \tag{3.3a}$$

$$u''(y) - \rho^2 u = f(y) \qquad 0 < y < 1\,, \tag{3.3b}$$

$$u(0) = 0\,. \tag{3.3c}$$

For $\rho \neq 0$ solution of (3.3b) and (3.3c) is

$$u(y) = \gamma \sinh \rho y + \frac{1}{\rho}\int_0^y f(t) \sinh \rho(y-t)\,dt\,. \tag{3.4}$$

To find γ we substitute this into (3.3a)

$$\gamma\left(\rho \cosh \rho + \sum_{k=1}^{l} a_k (ij)^{\alpha_k} \sinh \rho\right) + \int_0^1 f(t)\left[\cosh \rho(1-t) + \frac{1}{\rho}\left(\sum_{k=1}^{l} a_k (ij)^{\alpha_k}\right) \sinh \rho(1-t)\right] dt = g\,. \tag{3.5}$$

Denote $A = \rho \cosh \rho + \sum_{k=1}^{\ell} a_k(ij)^{\alpha_k} \sinh \rho$. Multiplying (3.4) by A, using (3.5) and standard identities for hyperbolic functions, we get:

$$(3.6)\quad \begin{aligned} Au(y) &= g \sinh \rho y - \int_0^1 f(t)\Big[\cosh \rho(1-t) \sinh \rho y \\ &\qquad + \frac{1}{\rho}\Big(\sum_{k=1}^{\ell} a_k(ij)^{\alpha_k}\Big) \sinh \rho(1-t) \sinh \rho y\Big] dt \\ &\quad + \int_0^y f(t)\Big[\sinh \rho(y-t) \cosh \rho \\ &\qquad + \frac{1}{\rho}\Big(\sum_{k=1}^{\ell} a_k(ij)^{\alpha_k}\Big) \sinh \rho(y-t) \sinh \rho\Big] dt \\ &= g \sinh \rho y - \int_0^1 f(t)\Big[\cosh \rho(1-t) \sinh \rho y \\ &\qquad + \frac{1}{\rho}\Big(\sum_{k=1}^{\ell} a_k(ij)^{\alpha_k}\Big) \sinh \rho(1-t)\ \sinh \rho y\Big] dt \\ &\quad - \int_0^y f(t)\Big[\sinh \rho t \cosh \rho(y-1) \\ &\qquad + \frac{1}{\rho}\Big(\sum_{k=1}^{\ell} a_k(ij)^{\alpha_k}\Big) \sinh \rho t \sinh \rho(1-y)\Big] dt\,. \end{aligned}$$

Notice that by our assumptions $|A| \geqslant ce^{\rho}(\rho + j^{\alpha_k})$, for all $1 \leqslant k \leqslant \ell$. Then from (3.6) we easily estimate (see [2, p. 876] for a similar argument)

$$(3.7)\qquad |u(y)| \leqslant c\left(\frac{|g|}{\rho + j^{\alpha_k}} + \frac{1}{\rho}\left(\int_0^1 |f(t)|^2 dt\right)^{1/2}\right).$$

This implies (restoring the subscripts)

$$(3.8)\qquad \int_0^1 |u_j(y)|^2 dy \leqslant c\left(\frac{|g_j|^2}{\rho^2} + \frac{1}{\rho^2}\int_0^1 |f_j|^2 dt\right).$$

In the case $\rho = 0$ we easily get from (3.3)

$$(3.9)\qquad \int_0^1 |u_0|^2 dy \leqslant c\left(|g_0|^2 + \int_0^1 |f_0(t)|^2 dt\right).$$

Differentiating (3.6) and going through the same steps as in derivation of (3.7), we get

$$(3.10)\qquad |u_j'|^2 \leqslant c\left(|g_j|^2 + \int_0^1 |f_j|^2 dt\right).$$

From (3.8–10) and from (3.7) with $y = 1$ we conclude the estimate (3.2) with $m = 0$. Higher estimates are easily proved by induction, differentiating (3.1) in x.

Turning to the general case, we write the boundary condition at $y = 1$ in the form

$$u_y + \sum_{k=1}^{\ell} a_k D^{\alpha_k} u = g + \sum_{k=1}^{\ell} \rho_k(x) D^{\alpha_k} u \,,$$

and apply the estimate (3.2) for the constant coefficient case.

Since

$$\|\rho_k(x) D^{\alpha_k} u\|_m \leqslant c |\rho_k|_m \|D^{\alpha_k} u\|_m \,,$$

with $|\rho_k|_m$ small, the proof follows.

Remark 1. Notice that the above argument establishes existence of solution $u(x, y) \in G^{m+1}$ to the problem (3.1) in the case $\rho_k(x) \equiv 0$ for all k, provided $f \in H^{m+k_0}(V)$, $g \in H^m(V_t)$, $m \geqslant 0$. For the general case existence follows in the same way as in the theorem 3.1 below, under the additional condition $m \geqslant k_0 + [(n+1)/2] + 1$.

Remark 2. All conditions on a_k which correspond to $|\alpha_k|$ odd can be dropped if none of the corresponding D^{α_k} is a dominating derivative in the set S.

Remark 3. A partition of unity argument (which allows one to remove the smallness conditions on ρ_k) produces the estimate (3.2) with an extra term $\|u\|_0$ on the right. We do not know how to remove this term (notice, there is no apparent maximum principle).

Sharper estimates can be obtained in the following special case.

Lemma 3.2. *Consider the problem* (3.1) *with* $\ell = 1$. *Assume that* $(-1)^{|\alpha_1|/2}$ $a_1 > 0$ *in case* $|\alpha_1|$ *is even and* $a_1 \neq 0$ *in case* $|\alpha_1|$ *is odd. Then for* $|\rho_1|_m$ *sufficiently small we have* (*integer* $m \geqslant 0$)

$$\|u\|_{m+1} + \overline{\|D^{\alpha_1} u\|}_m \leqslant c(\|f\|_m + \|g\|_m) \,.$$

Proof. It is sufficient to consider the case $\rho_1 \equiv 0$, from which the general case will follow as before. Follow the proof of lemma 3.1. From (3.10) it follows (by taking derivatives of (3.1) in x and setting $y = 1$)

$$\overline{\|u_y\|}_m \leqslant c(\|f\|_m + \|g\|_m) \,,$$

and hence

$$\overline{\|D^{\alpha_1} u\|}_m \leqslant \overline{\|u_y\|}_m + \|g\|_m \leqslant c(\|f\|_m + \|g\|_m) \,.$$

Combining this with (5.8–9), we conclude the proof.

Theorem 3.1. *Consider the problem*

$$(3.11)\qquad \begin{aligned} u_y &= \rho(x, D^{\alpha_1}u, \ldots, D^{\alpha_\ell}u) && y = 1\,, \\ \Delta u &= \varepsilon f(x, y) && 0 < y < 1\,, \\ u &= 0 && y = 0\,. \end{aligned}$$

Assume that $\rho(x, 0, \ldots, 0) \equiv 0$. *Denote* $r_k = -(\partial\rho/\partial D^{\alpha_k}u)(x, 0, \ldots, 0)$, *and assume that* $r_k(x)$ *satisfy the same conditions as in lemma* 3.1. *Let* $k_0 = \max_{1 \leqslant k \leqslant \ell} |\alpha_k| \geqslant 1$, $m_0 = k_0 + [(n+1)/2] + 1$, $f \in C^{m_0+k_0-1}$, $\rho \in C^{m_0}$ *for* $(x, y) \in V$ *and in small balls around the origin for other variables. Then for* ε *and* $\max_k |\rho_k|_{m_0}$ *sufficiently small the problem* (3.11) *has a solution* $u \in C^2(V) \cap C^{k_0}(V_t)$.

Proof. Define a map $T: u \in G^{m_0} \to v \in G^{m_0}$ by solving (see lemma 2.3)

$$\begin{aligned} v_y + \sum_{k=1}^{\ell} a_k D^{\alpha_k} v = \rho(D^{\alpha_1}u, \ldots, D^{\alpha_\ell}u) + \sum_{k=1}^{\ell} a_k D^{\alpha_k} u &\qquad y = 1\,, \\ \Delta v = \varepsilon f(x, y) &\qquad 0 < y < 1\,, \\ v = 0 &\qquad y = 0\,. \end{aligned}$$

Using lemma 3.1 it is easy to see that T is a contraction on sufficiently small balls around the origin in G^{m_0}.

Remarks. 1. It is easy to see that a similar perturbation result will hold for $f = f(x, y, u, Du, D^2u)$, provided $\max_{1 \leqslant k \leqslant \ell} |\alpha_k| \leqslant 1$.

2. Clearly the smoothness of solution increases with that of ρ and f. In particular if $\rho, f \in C^\infty$ so does u.

Example. Let $u = u(x, y, z)$. The non-coercive problem

$$\begin{aligned} u_y &= u_{xx}^2 - u_{xxxx} + u_{xzz} && y = 1\,, \\ \Delta u &= \varepsilon \sin x \sin z && 0 < y < 1\,, \\ u &= 0 && y = 0 \end{aligned}$$

verifies conditions of the theorem 3.1, and hence it has a C^∞ solution, 2π periodic in x and z, provided ε is small enough.

Theorem 3.2. *Consider the problem*

$$(3.12)\qquad \begin{aligned} u_y &= \rho(x, D^{\alpha_1}u, \ldots, D^{\alpha_\ell}u) && y = 1\,, \\ \Delta u &= \varepsilon f(x, y, u, Du) && 0 < y < 1\,, \\ u &= 0 && y = 0\,. \end{aligned}$$

Assume that $\rho(x, 0, \ldots, 0) \equiv 0$. *Assume that one of the derivatives in the set* S, *say* D^{α_1}, *dominates all others. With* $r_k(x)$ *as defined in the theorem* 3.1, *we assume that* $r_1(x)$ *satisfies the conditions of lemma* 3.2, *and* $r_k \equiv 0$ *for* $k = 2, \ldots, \ell$. *With* k_0 *and* m_0 *as above assume that* $f, \rho \in C^{m_0}$ *for* $(x, y) \in V$ *and in small balls around the origin for other variables. Then for* ε *and* $|\rho_1|_{m_0}$ *sufficiently small the problem* (3.12) *has a solution* $u \in C^2(V) \cap C^{k_0}(V_t)$.

The proof is similar to that of the theorem 3.1.

4. Small divisors in dimensions two and three

We show first that the situation is rather simple for the two-dimensional case, i.e. $u = u(x, y)$. Namely, except for $F = (\coth j)/j$ where zero divisors appear, for other F the divisors are bounded away from zero.

Lemma 4.1. *Consider the problem* (u, f *and* g *are* 2π *periodic in* x)

$$\begin{aligned} u_y + F u_{xx} &= g(x) & y &= 1\,, \\ \Delta u &= f(x, y) & 0 < y &< 1\,, \\ u &= 0 & y &= 0\,. \end{aligned} \tag{4.1}$$

Assume that $F \neq (\coth j)/j$, $j = 1, 2, \ldots$, *and* $F \neq 0$. *Then*

$$\|u\|_{m+2} + \overline{\|u\|}_{m+2} \leqslant c(\|f\|_{m+1} + \|g\|_m)\,. \tag{4.2}$$

Proof. Look for solution in the form $u(x, y) = \sum_{j=-\infty}^{\infty} u_j(y) e^{ijx}$, and follow the proof of lemma 3.1. This time $A = j \cosh j - F j^2 \sinh j$. Notice that if $F \neq (\coth j)/j$ and $F \neq 0$, then $|A| \geqslant c_0 j^2 e^j$ for some $c_0 > 0$. The rest of the proof is similar to that of lemma 3.1.

Theorem 4.1. *Consider the problem*

$$\begin{aligned} u_y &= \rho(u, u_x, u_{xx}) & y &= 1\,, \\ u_{xx} + u_{yy} &= f(x, y, u, u_x, u_y) & 0 < y &< 1\,, \\ u &= 0 & y &= 0\,. \end{aligned} \tag{4.3}$$

Assume that f *is* 2π *periodic in* x, *and the following*

(i) $\rho(0, 0, 0) = \rho_u(0, 0, 0) = \rho_{u_x}(0, 0, 0) = 0$;

$$F_0 = -\rho_{u_{xx}}(0, 0, 0) \neq \frac{1}{j} \coth j, \qquad j = 1, 2, \ldots, F_0 \neq 0\,.$$

(ii) $\rho \in C^3$, $f, f_u, f_{u_x}, f_{u_y} \in C^3$ *in all arguments* (*for* $0 \leqslant x \leqslant 2\pi$, $0 \leqslant y \leqslant 1$ *and in small balls around the origin for other variables*).

Then for $\|f(x, y, 0, 0, 0)\|_3$, $\|f_u(x, y, 0, 0, 0)\|_3$, $\|f_{u_x}(x, y, 0, 0, 0)\|_3$ *and* $\|f_{u_y}(x, y, 0, 0, 0)\|$ *sufficiently small the problem* (4.3) *has a* C^2 *solution,* 2π *periodic in* x.

Proof. Let $G^m(m = \text{integer} \geqslant 1)$ be a subset of $H^m(V)$ consisting of functions $u \in H^m(V)$ such that in addition $u \in H^m(V_t)$. By lemma 2.3 G^m with the norm $|||u|||_m = \|u\|_m + \overline{\|u\|}_m$ is a Banach space (notice that by lemma 2.1 this norm is equivalent to $\|u\|_m + \overline{\|u_x\|}_{m-1}$ for $n = 1$). Define a map T of G^m into itself by solving ($v = Tu$)

$$v_y + F_0 v_{xx} = \rho(u, u_x, u_{xx}) + F_0 u_{xx} \qquad y = 1\,,$$

$$\Delta v = f(x, y, u, u_x, u_y) \qquad 0 < y < 1\,,$$

$$v = 0 \qquad y = 0\,.$$

Using lemma 4.1 it is straightforward to show that the map T takes a sufficiently small ball around the origin in G^4 into itself and is a contraction (see [2] for a similar argument).

Next we turn to the three-dimensional case, i.e., $u = u(x, y, z)$, where the situation is more involved. Notice first that the set of F corresponding to the set $\mathscr{F} = \{F_{jk}\}$ of zero divisors $F_{j,k} = (\sqrt{j^2 + k^2}/j^2)\coth\sqrt{j^2 + k^2}$ is everywhere dense on the positive real axis, as we showed in [2].

We restrict now to the rational $F = p/q$, and see that the situation changes depending on whether $F > 1/2$ or $F \leqslant 1/2$. For $F = p/q > 1/2$ and $F \notin \mathscr{F}$ we show that the denominators are bounded away from zero, which allows us to derive a priori estimates. For $F = p/q \leqslant 1/2$ it is possible that $F \notin \mathscr{F}$, but the denominators can get arbitrarily small. We also show that for each rational F condition $F \in \mathscr{F}$ can be decided by a finite number of computations.

Lemma 4.2. *Let* $F = p/q > 1/2$ *be an irreducible fraction. Then there exists a constant* $c_0 > 0$, *such that*

$$d \equiv |\sqrt{(j^2 + k^2)} - Fj^2| \geqslant c_0\,,$$

for all integers j, k, *possibly with the exception of finitely many pairs* (j, k).

Proof. Without loss of generality we restrict to positive j and k. We consider three cases.

(i) $k \geqslant Fj^2$. Then

$$d \geqslant \sqrt{(j^2 + F^2j^4)} - Fj^2 = \frac{j^2}{\sqrt{(j^2 + F^2j^4)} + Fj^2} \geqslant \frac{1}{\sqrt{(1 + F^2)} + F}$$

(ii) $k = Fj^2 - \ell$, $\ell \geqslant 2$. Then one easily gets:

$$d = \frac{k(2\ell - 1/F) + \ell(\ell - 1/F)}{\sqrt{(1/F(k+\ell) + k^2)} + k + \ell} \geqslant \frac{2k + \ell(\ell - 1/F)}{\sqrt{(2(k+\ell) + k^2)} + k + \ell} \geqslant \frac{c_1'(k+\ell)}{c_1''(k+\ell)}$$
$$\equiv c_1 > 0 .$$

(iii) It is easy to see that it remains to consider the case $k = Fj^2 - \ell$ with $1/q < \ell < 2 - 1/q$, where $\ell = m/q$ (for some m = integer $\geqslant 0$) may be a reducible fraction. Then

$$d = \frac{|F^2j^4 - j^2 - (Fj^2 - \ell)^2|}{\sqrt{(j^2 + k^2)} + Fj^2} = \frac{|-j^2 + 2F\ell j^2 - \ell^2|}{\sqrt{(j^2 + k^2)} + Fj^2}$$

Notice that $2F\ell = 2(p/q)(m/q) \neq 1$, for otherwise we would have $2m = q^2/p$ with p and q being mutually prime a contradiction. Denote $|2F\ell - 1| = \bar{c}_1 > 0$. Then for $j \geqslant j_0$ – large,

$$d \geqslant \frac{c_1 j^2 - \ell^2}{\sqrt{j^2 + (Fj^2 - \ell)^2} + Fj^2} \geqslant c_0 > 0.$$

Remark 4.1. If $F = 1$, then we easily estimate $d > 3/8$, with the exception of $j = k = 0$ and $j = \pm 1$, $k = 0$.

Lemma 4.3. *Let $F > 1/2$ be rational, and $F \neq F_{j,k}$. Then there exists a constant $c_3 > 0$, such that for any pair of integers j, k we have*

$$|\Delta| \equiv |\sqrt{j^2 + k^2} \cosh \sqrt{j^2 + k^2} - Fj^2 \sinh \sqrt{j^2 + k^2}| \geqslant c_3 e^{\sqrt{j^2+k^2}} .$$

Proof. Write

$$|\Delta| = \frac{e^{\sqrt{j^2+k^2}}}{2} |\sqrt{j^2 + k^2} - Fj^2 + e^{-2\sqrt{j^2+k^2}}(\sqrt{j^2 + k^2} + Fj^2)|$$
$$\equiv \frac{e^{\sqrt{j^2+k^2}}}{2} |d + d_1| .$$

By Lemma 4.2, $|d| \geqslant c_0$ for $|j| \geqslant j_0$. Also $|d| \geqslant c_0$ for $|j| < j_0$ and $|k| \geqslant k_0$, k_0 – large. By increasing j_0 and k_0, if necessary, we can also assume that $|d_1| \leqslant c_0/2$ for $|j| \geqslant j_0$ or $|k| \geqslant k_0$. Let now $\bar{c}_2 = \min_{|j|<j_0, |k|<k_0} |d + d_1|$. Notice that $\bar{c}_2 > 0$, since $F \neq F_{j,k}$. The lemma now follows with $c_3 = \min(c_0/4, \bar{c}_2/2)$.

Lemma 4.4. *For each rational $F > 1/2$, condition $F \in \mathscr{F}$ can be decided by a finite number of computations. (Recall that the set $\mathscr{F}$ is everywhere dense).*

Proof. Condition $F \in \mathscr{F}$ implies that for some j and k

$$e^{-2\sqrt{j^2+k^2}} = \frac{Fj^2 - \sqrt{j^2 + k^2}}{Fj^2 + \sqrt{j^2 + k^2}} . \tag{4.4}$$

By Lemma 4.2,

$$\frac{|Fj^2 - \sqrt{j^2 + k^2}|}{Fj^2 + \sqrt{j^2 + k^2}} \geqslant \frac{c_0}{F(j^2 + k^2) + \sqrt{j^2 + k^2}},$$

and the left hand side of (4.4) is less than that quantity for large j, k, e.g. for

$$\sqrt{j^2 + k^2} > \ln \sqrt{\frac{F+1}{c_0}(j^2 + k^2)}, \tag{4.5}$$

which concludes the proof.

Remark 4.2. $1 \neq \mathscr{F}$. Indeed, by Remark 4.1 we can take $c_0 = 3/8$. Then in view of (4.5) we have only to check $j = \pm 1$, $k = 0$ and $j = 0$, $k = \pm 1$, which is easily done.

Lemma 4.5. *Condition $F > 1/2$ in lemmas* 4.2, 4.3 *cannot be removed.*

Proof. Namely, we show that $1/2 \notin \mathscr{F}$, but $d = \sqrt{j^2 + k^2} - j^2/2$ gets arbitrarily small for large j, k. Indeed, take $k = j^2/2 - 1$, j – even. Then $d = 1/(\sqrt{j^2 + k^2} + j^2/2)$. Condition $1/2 \notin \mathscr{F}$ is equivalent to checking impossibility for any j and k of

$$e^{-2\sqrt{j^2+k^2}} = \frac{j^2/2 - \sqrt{j^2 + k^2}}{j^2/2 + \sqrt{j^2 + k^2}}. \tag{4.6}$$

For this, one first notices that the left hand side of (4.6) is less than the absolute value of the right hand side if $j^2 + k^2 \geqslant 8$, and then one eliminates all remaining possibilities.

We can now obtain the following a priori estimates.

Theorem 4.2. *Consider the problem* ($u = u(x, y, z)$)

$$\begin{aligned} u_y + Fu_{xx} &= g(x, z) && y = 1, \\ \Delta u &= f(x, y, z) && 0 < y < 1, \\ u &= 0 && y = 0. \end{aligned} \tag{4.7}$$

Assume that $F > 1/2$ is rational, $F \notin \mathscr{F}$; f and g are 2π periodic in x and z. Then we have the following estimate

$$\|u\|_m + \overline{\|u\|}_m \leqslant c(\|f\|_{m+1} + \|g\|_m). \tag{4.8}$$

Proof. Look for solution in the form $u(x, y, z) = \sum_{j,k=-\infty}^{\infty} u_{jk}(y)e^{ijx+ikz}$. This time $|A| = |\sqrt{j^2 + k^2} \cosh \sqrt{j^2 + k^2} - Fj^2 \sinh \sqrt{j^2 + k^2}| \geqslant c_0 e^{\sqrt{j^2+k^2}}$ by lemma 3.3. Then proceed as in lemma 5.1.

Corollary. *If $f \in H^5(V)$, $g \in H^4(V_t)$ then the problem* (4.7) *has a unique $C^2(V)$ solution.*

Acknowledgements. It is a pleasure to thank L. Nirenberg for posing the problem, and H. Levine and G. Lieberman for their interest in my work and useful discussions.

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(Ricevita la 7-an de junio, 1988)