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On Blow-up for the Pseudo-Conformally Invariant Nonlinear Schrödinger Equation

By

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Introduction and a main result

We consider the Cauchy problem for the nonlinear Schrödinger equation:

$$(\mathrm{Cp})\qquad \begin{cases} (\mathrm{NLS}) & 2i\dfrac{\partial u}{\partial t} + \Delta u + |u|^{4/n}u = 0\,, \quad (t, x) \in \boldsymbol{R} \times \boldsymbol{R}^n\,, \\ (\mathrm{IV}) & u(0, x) = \phi_0(x)\,, \quad x \in \boldsymbol{R}^n\,. \end{cases}$$

It should be noted that (NLS) has a remarkable property that it is invariant under the pseudo-conformal transformations (see [11], [15] and Preliminary in this paper). Of physical importance is the case $n = 2$, when (NLS) is a model of the self-forcusing of a laser beam. Because of its mathematical interest however, we intend to develop a theory for arbitrary dimensions n.

Here, we list several basic notations which will be used throughout this paper.

$$\partial_t = \partial/\partial t\,, \quad \nabla = (\partial_1, \partial_2, \dots, \partial_n)\,, \quad \partial_j = \partial/\partial x_j\,;$$

$$L^q = L^q(\boldsymbol{R}^n)\,, \quad H^s = H^s(\boldsymbol{R}^n)\,; \quad \|\cdot\|_q = L^q\text{-norm}\,, \quad \|\cdot\| = L^2\text{-norm}\,.$$

$$\Sigma = \{v \in H^1;\ \|v\|^2 + \|\nabla v\|^2 + \|xv\|^2 < +\infty\}\,;$$

$$L^{q,r}(I) = L^r(I; L^q) \quad (I\text{: an interval in } \boldsymbol{R})\,;$$

$$|\cdot|_{q,r,I} = L^{q,r}(I)\text{-norm}\,, \quad |\cdot|_{q,r} = |\cdot|_{q,r,\boldsymbol{R}}\,;$$

$$\langle\cdot, \cdot\rangle = L^2\text{-inner product}\,;$$

$$\mathscr{S} = \mathscr{S}(\boldsymbol{R}^n)\text{: the Schwartz space of rapidly decreasing } C^\infty\text{-functions}\,;$$

$$\mathscr{S}' = \mathscr{S}'(\boldsymbol{R}^n)\text{: the dual of } \mathscr{S}\,.$$

We will consider solving (Cp) in Σ for $t \geqq 0$ (this problem is referred to as $(\mathrm{Cp})^+$), and study the formation of singularity (the Blow-up). The backward problem can be treated in the analogous way, but we restrict our attention to the forward one for simplicity.

It is known that the solutions to (Cp) show the instability phenomena called "Blow-up" for some initial data (see [12], [22], [27] and [29]). That is, there exists $T_m \in \boldsymbol{R}$ such that $\lim_{t \to T_m} \|\nabla u(t)\| = \infty$. Its proof is based on the following equality and the inequality:

$$\|xu(t)\|^2 = \|x\phi_0\|^2 + 2t \operatorname{Im} \langle x \cdot \nabla \phi_0, \phi_0 \rangle + t^2 E(\phi_0), \tag{1}$$

$$\|u(t)\|^2 = \|\phi_0\|^2 \leqq (2/n)\|xu(t)\| \|\nabla u(t)\|, \tag{2}$$

where

$$E(v) = \|\nabla v\|^2 - \frac{2}{p+1}\|v\|_{p+1}^{p+1}, \quad p = 1 + 4/n. \tag{3}$$

The argument goes roughly as follows. Under the suitable assumptions on ϕ_0, (1) implies that $\|xu(t)\| \to 0$ in a finite time. So, one concludes that $\|\nabla u(t)\| \to \infty$ in a finite time by (2) and the standard argument of the nonlinear evolution equation (see Proposition 1.1 in this paper). There arise two questions:

(Q1) Does the time T_0 when $\|xu(t)\|$ vanishes coincide with the maximal existence time T_m (Blow-up time)?

(Q2) How does the Blow-up solution behave near the Blow-up time?

It should be noted that these two questions are not completely independent. If $\lim_{t \to T_m} \|xu(t)\| = 0$, then one easily verifies that $|u(t, x)|^2 \to c\delta(x)$ as $t \to T_m$ in the distribution sense, where c is a constant and δ is the Dirac measure at the origin.

Recently, many mathematicians and physicists have studied the above two questions by numerical analysis for the physical interest mentioned at the beginning of this section and they have given several interesting suggestions (see [19] for example).

We obtain the following theorem concerning (Q1).

Theorem 1. *Let*

$$G(t; a, T) = \exp\,[(1/2)\{-i|x - a|^2/(T - t)\}]$$

and

$$G_0(a, T) = G(0; a, T).$$

We define

$$\mathscr{B}_0 = \{\phi \in \Sigma \backslash \{0\};\ \textit{there exist } a \in \boldsymbol{R}^n,\ T > 0 \textit{ and } \psi \in \mathscr{H}_0 \textit{ such that } \phi = G_0(a, T)\psi.\}, \tag{4}$$

$$\mathscr{B}_1 = \{\phi \in \Sigma \backslash \{0\};\ \textit{there exist } a \in \boldsymbol{R}^n,\ T > 0 \textit{ and } \psi \in \mathscr{H}_1 \textit{ such that } \phi = G_0(a, T)\psi.\}, \tag{5}$$

where

(6) $\mathscr{H}_0 = \{\phi \in \Sigma;\ E(\phi) = 0\}$,

(7) $\mathscr{H}_1 = \{\phi \in \Sigma;\ E(\phi) = 0$ *and* $(Cp)^+$ *with* $u(0) = \phi$ *has a global solution.*$\}$.

Then we have:

(i) *If* $\phi_0 \in \mathscr{B}_0$, *the corresponding* Σ*-solution blows up at* $t = T_m \leqq T$, *where* T *is a positive constant such that* $\phi_0 = G_0(a, T)\psi$ *for some* $a \in \boldsymbol{R}^n$ *and* $\psi \in \mathscr{H}_0$.

(ii) *The* Σ*-solution* $u(t)$ *to* $(Cp)^+$ *blows up at* $t = T_m$ *like the* δ*-function, i.e.,*

$$\lim_{t\to T_m} |u(t, \cdot)|^2 = \|\phi_0\|^2\delta(\cdot - a) \quad in \quad \mathscr{S}' , \tag{8}$$

$$\lim_{t\to T_m} \|(x - a)u(t)\| = 0 , \tag{9}$$

if and only if ϕ_0 *is an element of* $\mathscr{B}_1$ *such that*

$$\phi_0 = G_0(a, T_m)\psi , \quad \psi \in \mathscr{H}_1 , \tag{10}$$

where $\delta(x - a)$ *is the Dirac measure at* $x = a$.

The condition $\phi_0 \in \mathscr{B}_0$, at first sight, seems very restricted, but this is a slight generalization including the space translations of the sufficient conditions for Blow-up listed in [22] (see lemma 1.3 in this paper and also [27]). We note that the set $\mathscr{H}_1$ contains any bound state solution to the problem:

$$\Delta v - v + |v|^{4/n}v = 0 , \tag{11}$$

$$v \in H^1 , \quad v \not\equiv 0 \tag{12}$$

(cf. [4] and [5]).

Our theorem suggests that if the Blow-up occurs at $t = T_m$ for $\phi_0 \notin \mathscr{B}_1$, one has

$$\lim_{t\to T_m} \|(x - a)u(t)\| > 0 \quad \text{for each} \quad a \in \boldsymbol{R}^n , \tag{13}$$

as is suggested by the numerical computation in [19].

We conclude this section by stating the relation between Theorem 1 and the result of Weinstein [28, Theorem 1]. He proved:

Theorem (Weinstein [28]). *Let* R *be the ground state (minimal* L^2*-norm) solution to* (11)–(12). *Suppose that the initial function* ϕ_0 *satisfies*

$$\|\phi_0\| = \|R\| \tag{14}$$

and the corresponding solution $u(t)$ *to* $(Cp)^+$ *blows up at* $t = T_m \in (0, \infty]$, *i.e.* $\lim_{t\to T_m} \|\nabla u(t)\| = \infty$. *Set*

$$\lambda(t) = \|\nabla R\|/\|\nabla u(t)\| .$$

Then, there are functions $y(t) \in \boldsymbol{R}^n$ *and* $\gamma(t) \in \boldsymbol{R}$ *such that*

$$\lim_{t \to T_m} S_{\lambda(t)} u(t, \cdot + y(t)) e^{i\gamma(t)} = R(\cdot) \quad \textit{strongly in } H^1 . \tag{15}$$

Here, $S_\lambda u(t, x) = \lambda^{n/2} u(t, \lambda x)$.

It is easily deduced from (15) that for some $a \in \boldsymbol{R}^n$,

$$\lim_{t \to T_m} |u(t, \cdot)|^2 = \|R\|^2 \delta(\cdot - a) \quad \text{in } \mathscr{S}' . \tag{16}$$

Thus, if we impose the condition $\phi_0 \in \Sigma$ on Weinstein's theorem and suppose $0 < T_m < \infty$, then, by Theorem 1, ϕ_0 should be of the form

$$\phi_0 = G_0(a, T_m)\underline{R} , \tag{17}$$

where $\underline{R}$ is a certain solution to (11)–(12) with $\|\underline{R}\| = \|R\|$, since the ground state is characterized as a solution to the following variational problem (see [4]):

$$\text{minimize} \quad \{\|v\|;\, v \in H^1, E(v) = 0, v \neq 0.\} , \tag{18}$$

and their solutions have an indeterminacy under the group of translations, rotations and (global) gauge transformations. Furthermore, we can give the explicit form of $u(t)$ by using $\underline{R}$:

$$u(t, x) = (T_m - t)^{-n/2} G(t; a, T_m) \underline{R}(y) \exp(is/2) , \tag{19}$$

where $y = (x - a)/(T_m - t)$ and $s = t/T_m(T_m - t)$. This is a direct consequence of the proof of Theorem 1.

Therefore, if the solution belongs to Σ, then Theorem 1 gives a stronger information than Weinstein's result.

§0. Preliminaries

First we formulate the problem (Cp) precisely. We rewrite (Cp) as the integral equation

$$u(t) = U(t)\phi_0 + \frac{i}{2} S(t; F(u)) , \tag{0.1}$$

where F is a nonlinear operator

$$F(u) = |u|^{4/n} u \tag{0.2}$$

and $U(t)$, $S(t;\, \cdot)$ are linear operators given by

(0.3) $$U(t) = \exp\left(\frac{i}{2}t\Delta\right) \quad \text{(free propagator)},$$

(0.4) $$S(t; v) = \int_0^t U(t-\tau)v(\tau)d\tau ,$$

respectively. In what follows the integral in (0.1) is understood to be the Bochner integral in $L^{2+4/n}$. One easily obtains following continuation theorem from [8], [9] and [13].

Theorem 0.1. *For any $\phi_0 \in H^1$, there exist a positive number T_m and a unique solution*

$$u(\cdot),\ \nabla u(\cdot) \in C([0, T_m);\ L^2) \cap L^{2+4/n}_{\mathrm{loc}}(0, T_m;\ L^{2+4/n})$$

to $(Cp)^+$ satisfying (0.1) *with the alternative; either $T_m = +\infty$ or $T_m < +\infty$ and*

(0.5) $$\lim_{t\to T_m} \|\nabla u(t)\| = +\infty .$$

Moreover we have

(0.6) $$\|u(t)\| = \|\phi_0\| ,$$

(0.7) $$E(u(t)) = E(\phi_0), \quad t \in [0, T_m),$$

where E is defined by (3). *Furthermore if $\phi_0 \in \Sigma$, then the above solution is in $C([0, T_m); \Sigma)$ and satisfies*

(0.8) $$\|xu(t)\|^2 = \|x\phi_0\|^2 + 2t \operatorname{Im} \langle x\cdot\nabla\phi_0, \phi_0\rangle + t^2E(\phi_0) .$$

(*For the identity* (0.8), *see, e.g.*, [12], [22] *and* [27].)

Our analysis heavily relies on the invariance properties of the equation (NLS). It is worth while noting that the equation (NLS) is invariant under the same group of space-time-gauge transformations which leave the free Schrödinger equation invariant. Such a group is usually called the Schrödinger group. The Schrödinger group is the largest group of space-time-gauge transformations containing, in addition to the Galilei and the (global) gauge group, the group of dilations and a one parameter group of transformations, called pseudo-conformal (for detailes, see [15] and [11]).

The Schrödinger group consists of the following transformations:

Gauge transformation:

$$J_1 u = [J_1(\theta)u](t, x) = e^{i\theta}u(t, x), \quad \theta \in \boldsymbol{R}/2\pi\boldsymbol{Z},$$

Space translation:

$$J_2 u = [J_2(a)u](t, x) = u(t, x-a), \quad a \in \boldsymbol{R}^n,$$

Time translation:

$$J_3 u = [J_3(b)u](t, x) = u(t - b, x), \quad b \in \boldsymbol{R},$$

Space rotation:

$$J_4 u = [J_4(c)u](t, x) = u(t, A(c)x), \quad A(c) = \exp(c) \in SO(n),$$

$$c = (c_{ij})_{1 \leqq i, j \leqq n} \in Alt(n),$$

Pure Galilei transformation:

$$J_5 u = [J_5(d)u](t, x) = \exp\left(id \cdot x - \frac{i}{2}|d|^2 t\right) u(t, x - dt), \quad d \in \boldsymbol{R}^n,$$

Space-time dilation:

$$J_6 u = [J_6(\lambda)u](t, x) = e^{-(n/2)\lambda} u(e^{-2\lambda} t, e^{-\lambda} x), \quad \lambda \in \boldsymbol{R},$$

Pseudo-conformal transformation:

$$\begin{aligned} J_7 u &= [J_7(\omega)u](t, x) \\ &= (1 - \omega t)^{-n/2} G(t; 0, 1/\omega) u(t/(1 - \omega t), x/(1 - \omega t)), \quad \omega \in \boldsymbol{R}, \end{aligned}$$

where $G(t; a, T) = \exp[(1/2)\{-i|x - a|^2/(T - t)\}]$.

The transformation J_7 will play a very important role to study Blow-up. We conclude this section with the following lemma.

Lemma 0.1. *Put $X_0(I) = C(\bar{I}; \Sigma) \cap L^{2+4/n, 2+4/n}(I)$ for an interval $I \subset \boldsymbol{R}$.*

(i) *Let I be any interval of $\boldsymbol{R}$. Let $u(\cdot) \in X_0(I)$ be a solution to (NLS) satisfying*

$$u(t) = U(t - t_0)u(t_0) + \frac{i}{2} S(t, t_0; F(u)), \quad t, t_0 \in I, \tag{0.9}$$

where $S(t, t_0; \cdot)$ is formally defined by

$$S(t, t_0; \cdot) = S(t; \cdot) - U(t - t_0) S(t_0; \cdot). \tag{0.10}$$

Then $v(\cdot) \equiv [J_\nu u](\cdot) \in X_0(I)$ $(\nu = 1, 2, 4, 5)$ and is also a solution to (NLS) satisfying

$$v(t) = U(t - t_0)v(t_0) + \frac{i}{2} S(t, t_0; F(v)), \quad t, t_0 \in I. \tag{0.11}$$

For J_ν^{-1}, we have similar results.

(ii) *Let I be any interval of $\boldsymbol{R}$. Let $u(\cdot) \in X(I)$ be a solution to (NLS) satisfying (0.9). Then $v(\cdot) \equiv [J_3(b)u](\cdot) \in X(I + b)$ and is also a solution to*

(NLS) satisfying

$$(0.12)\qquad v(t) = U(t-\bar{t}_0)v(\bar{t}_0) + \frac{i}{2}S(t, \bar{t}_0; F(v)), \quad t, \bar{t}_0 \in I + b,$$

where $\bar{t}_0 = t_0 + b$.

For J_3^{-1}, *we have a reverse result.*

(iii) *Let* $I = [0, T]$. *Let* $u(\cdot) \in X_0(I)$ *be a solution to (Cp). Then* J_6 *transforms* $u(t)$ *into a solution* $v(\cdot) \equiv [J_6(\lambda)u](\cdot) \in X_0([0, e^{2\lambda}T])$ *to (Cp) with the initial condition* $v(0) = e^{-(n/2)\lambda}\phi_0(e^{-\lambda}x)$. *For* J_6^{-1} *we have a reverse result.*

(iv) *Let* $I = [0, T]$. *Let* $u \in X_0(I)$ *be a solution to (Cp). Then* J_7 *transforms* $u(t)$ *into a solution* $v(\cdot) \equiv [J_8(\omega)u](\cdot) \in X_0([0, T/(1+\omega T)])$ *to (Cp) with the initial condition* $v(0) = G(0, 0, 1/\omega)\phi_0$. *For* J_7^{-1} *we have a reverse result.*

Proof. Proofs are almost identical with [24, Lemma 2.8 (ii)] and [14, Proposition 2.2 (iii)]. So, we omit it.

§1. Proof of Theorem 1

In this section, we will prove Theorem 1, and give two corollaries which are immediate consequences of Theorem 1.

First we give the Blow-up theorem due to M. Tsutsumi [22] (see also [27], [12] and [29]) adapted to $(\mathrm{Cp})^+$.

Proposition 1.1. *Let* $\phi_0 \in \Sigma$ *and either*

$$(1.1)\qquad E(\phi_0) < 0,$$

$$(1.2)\qquad E(\phi_0) = 0 \quad \text{and} \quad \mathrm{Im}\,\langle x\cdot\nabla\phi_0, \phi_0\rangle < 0$$

or

$$(1.3)\qquad E(\phi_0) > 0 \qquad \text{and} \qquad \mathrm{Im}\,\langle X\cdot\nabla\phi_0, \phi_0\rangle \leqq -\sqrt{E(\phi_0)}\,\|x\phi_0\|.$$

Then there exists $0 < T_m < +\infty$ *such that* $\lim_{t\to T_m}\|\nabla u(t)\| = +\infty$, *where* $u \in C([0, T_m); \Sigma)$ *is the corresponding solution to* $(Cp)^+$.

Proof. Suppose that the solution $u(t)$ exists globally in time under the hypothesis (1.1), (1.2) or (1.3). Then the identity (0.8) implies that there exists $0 < T_0 < +\infty$ such that $\lim_{t\to T_0}\|xu(t)\| = 0$. By the Weyl-Heisenberg inequality;

$$(1.4)\qquad \|u(t)\|^2 \leqq (2/n)\|xu(t)\|\,\|\nabla u(t)\|$$

and the conservation law (0.6), one has $\lim_{t\to T_0}\|\nabla u(t)\| = +\infty$, which violates the continuation theorem (Theorem 0.1).

Remark 1.1. We note that by Lemma 0.1 (i) one can easily generalize the conditions (1.1), (1.2) and (1.3) as follows: or;

(1.1)' $E(\phi_0) < 0$,

(1.2)' $E(\phi_0) = 0$ and $\exists a \in \boldsymbol{R}^n$ such that $\operatorname{Im} \langle (x - a) \cdot \nabla \phi_0, \phi_0 \rangle < 0$,

(1.3)' $E(\phi_0) > 0$ and $\exists a \in \boldsymbol{R}^n$ such that

$$\operatorname{Im} \langle (x - a) \cdot \nabla \phi_0, \phi_0 \rangle \leqq - \|(x - a) \cdot \phi_0\| \sqrt{E(\phi_0)} .$$

Before we state the proof of Theorem 1, we introduce a transformation $J(a, T)$.

Definition 1.1. We define the mapping $J(a, T)$ as follows: for any $a \in \boldsymbol{R}^n$ and $T > 0$,

$$\begin{aligned} (1.5) \quad [J(a, T)u](t, x) &= [J_2(a) \circ J_6(\log T) \circ J_7(T)u](t, x) \\ &= J_2(a)\{(T - t)^{-n/2} G(t; 0, T) u(t/T(T - t), x/(T - t))\} \\ &= (T - t)^{-n/2} G(t; a, T) u(t/T(T - t), (x - a)/(T - t)) \end{aligned}$$

for an appropriate function $u(t, x)$. We also define the mapping $j(a, T)$ as follows: for an appropriate function $\phi(x)$,

$$(1.6) \qquad [j(a, T)\phi](x) = T^{-n/2} G(0; a, T) \phi((x - a)/T) .$$

For the transformations $J(a, T)$ and $j(a, T)$, we immediately have by Lemma 0.1,

Lemma 1.1. (i) *Let $\phi(\cdot) \in L^2$ and $v(t, \cdot) \in L^2$, then*

$$(1.7) \qquad \|j(a, T)\phi\| = \|\phi\| ,$$

$$(1.8) \qquad \|[J(a, T)v](t)\| = \|v(t)\| .$$

(ii) *Let T_0 be any positive constant. Let $u(t)$ be a solution to $(Cp)^+$ with $\phi_0 = u_0$ such that $u \in C([0, T_0]; \Sigma)$. Then $J(a, T)^{-1}$ transforms $u(t)$ into a solution $v(t) \equiv [J(a, T)^{-1}u](t)$ to $(Cp)^+$ with $\phi_0 = j(a, T)^{-1}u_0$ such that $v \in C([0, T_0/T(T - T_0)]; \Sigma)$.*

For $J(a, T)$, we have a reverse result.

We prepare several lemmas.

Lemma 1.2. *Let $u(\cdot) \in C([0, T); \Sigma)$ be a solution to $(Cp)^+$ with $\phi_0 = u_0$. Put $v(\cdot) = [J(a, T)^{-1}u](\cdot)$. Then $v(\cdot)$ is also a solution to $(Cp)^+$ with $\phi_0 = j(a, T)^{-1}u_0 \equiv v_0$ such that $v(\cdot) \in C([0, +\infty); \Sigma)$. Furthermore we have*

(1.9) $$E(v_0) = \|(x-a)u_0\|^2 + 2T\,\mathrm{Im}\,\langle (x-a)\cdot \nabla u_0, u_0\rangle + T^2E(u_0)$$

(1.10) $$\begin{aligned}\|(x-a)u(t)\|^2 &= (T/(1+Ts))^2\|yv(s); dy\|^2\\ &= (T/(1+Ts))^2\{\|yv_0; dy\|^2 + 2s\,\mathrm{Im}\,\langle y\cdot\nabla_y v_0, v_0; dy\rangle\\ &\qquad + s^2E(v_0)\}\\ &> 0\end{aligned}$$

on $[0, T)$, *where we have used the notations*;

(1.11) $$s = t/T(T-t)\,,$$

(1.12) $$y = (x-a)/(T-t)\,.$$

Proof. It follows from Lemma 1.1 (ii) that v is a solution to $(\mathrm{Cp})^+$ with $\phi_0 = v_0$ such that $v(\cdot)\in C([0, +\infty); \Sigma)$ and satisfies (0.8) by Theorem 0.1, so that a direct calculation yields (1.10). (1.9) is also easily obtained from a direct calculation and the definition of E.

Lemma 1.3. *The condition* $\phi_0 \in \mathscr{B}_0$ *is equivalent to* (1.1)$'$ $\sim$ (1.3)$'$.

Proof. This is an easy consequence of the identity (1.9).

Lemma 1.4. *Let* $v_0 \in \mathscr{H}_1$ *(defined by* (7)*) and* $T > 0$ *be any fixed positive number. Put* $\phi_0 = j(a, T)v_0 \equiv u_0$. *Then the corresponding solution* $u(t)$ *to* $(Cp)^+$ *blows up at* $t = T$ *and satisfies*

(1.13) $$\lim_{t\to T}\|(x-a)u(t)\| = 0\,,$$

(1.14) $$\lim_{t\to T}|u(t,\cdot)|^2 = \|\phi_0\|^2\delta(\cdot - a) \quad \text{in } \mathscr{S}'\,.$$

Proof. We solve $(\mathrm{Cp})^+$ with the initial datum v_0. Then there exists a unique global solution in time $v(t)$ such that $v(\cdot)\in C([0, +\infty); \Sigma)$ by the definition of $\mathscr{H}_1$. Putting $w(t) = [J(a, T)v](t)$, one finds that $w(t)$ is a solution to $(\mathrm{Cp})^+$ with $\phi_0 = w(0) = u_0$ such that $w(\cdot)\in C([0, T); \Sigma)$ by Lemma 1.1. Hence the uniqueness of solutions implies that $u = w$. By the identity (1.10), we have (1.13) since $E(v_0) = 0$.

(1.14) means that

(1.15) $$\lim_{t\to T}\int_{\boldsymbol{R}^n}\rho(x)|u(t,x)|^2dx = \|u_0\|^2\rho(a)\,, \quad \text{for any } \rho\in\mathscr{S}\,.$$

This can be easily obtained by (0.6), (1.13) and the Chebychev inequality:

(1.16) $$\int_{|x-a|>\varepsilon}|\phi(x)|^2dx \le \frac{1}{\varepsilon^2}\|(x-a)\phi\|^2\,.$$

We are now in a position to prove Theorem 1.

Proof of Theorem 1. We first prove (i). Let $\phi_0 = G_0(a, T)\psi \in \mathscr{B}_0$ for some $a \in \boldsymbol{R}^n$, $T > 0$ and $\psi \in \mathscr{H}_0$. Let $u(t)$ be the corresponding solution to $(\mathrm{Cp})^+$. It is obvious from Proposition 1.1 and Lemma 1.3 that $u(t)$ cannot exist globally in time. Thus, we have only to show that the maximal existence time T_m is not larger than T. Suppose the contrary, i.e., $T_m > T$. So, we have

$$\sup_{t \in [0,T]} \|\nabla u(t)\| < +\infty . \tag{1.17}$$

Putting $v(t) = [J(a, T)^{-1}u](t)$ and $v_0 = j(a, T)^{-1}\phi_0$, one finds that $v(\cdot) \in C([0, +\infty); \Sigma)$, $v(0) = v_0$ and $E(v_0) = 0$ by Lemma 1.1 together with the fact that $\psi(x) = T^{-n/2}v_0((x-a)/T)$. Since $E(v_0) = 0$, we have, by (1.10),

$$\lim_{t \to T} \|(x-a)u(t)\| = 0 , \tag{1.18}$$

so that one concludes that

$$\lim_{t \to T} \|u(t)\| = 0 , \tag{1.19}$$

by (1.17) and the modified Weyl-Heisenberg inequality:

$$\|u(t)\|^2 \leqq (2/n)\|(x-a)u(t)\|\,\|\nabla u(t)\| . \tag{1.4$'$}$$

The fact (1.19) is contrary to the L^2-conservation law (0.6).

We next prove (ii). Let $\phi_0 \in \mathscr{B}_1$. Then there exist $\psi \in \mathscr{H}_1$, $a \in \boldsymbol{R}^n$ and $T_m > 0$ such that

$$\phi_0 = G_0(a, T_m)\psi . \tag{1.20}$$

Define $\phi \in \Sigma$ so that $\psi(x) = T^{-n/2}\phi((x-a)/T)$. Lemma 0.1 (i) and the uniqueness of the solution imply $\phi \in \mathscr{H}_1$, since $E(\phi) = 0$. Hence one has

$$\phi_0 = j(a, T_m)\phi , \tag{1.21}$$

so that the corresponding solution to $(\mathrm{Cp})^+$ satisfies (8) and (9) by lemma 1.4.

Conversely, suppose that the solution to $(\mathrm{Cp})^+$ blows up at $t = T_m$ and satisfies (8) and (9). Put $v(t) = [J(a, T_m)^{-1}u](t)$. Then $v(\cdot) \in C([0, \infty); \Sigma)$ and

$$E(v) = E(j(a, T_m)^{-1}\phi_0) = \lim_{t \to T_m} \|(x-a)u(t)\| = 0 \tag{1.22}$$

by (9) and (1.9) with $T = T_m$. Hence $\phi_0 \in \mathscr{B}_1$.

As is suggested by the numerical computation in [19], we have

Corollary 1.1. *Suppose that $\phi_0 \notin \mathscr{B}_1$ and ϕ_0 leads to the Blow-up solution $u(t)$ to $(Cp)^+$. Then we have*

$$\lim_{t \to T_m} \|(x-a)u(t)\| > 0 \tag{1.23}$$

for each $a \in \boldsymbol{R}^n$, where T_m is the maximal existence time.

This is a direct consequence of Theorem 1. Theorem 1 also suggests that it is important to study properties of the subsets $\mathscr{H}_0$ and $\mathscr{H}_1$. For $\mathscr{H}_1$ we have the following instability result.

Corollary 1.2. *For any $\psi \in \mathscr{H}_1$ and $\varepsilon > 0$, there exists $\phi \in \mathscr{B}_1$ such that*

$$\|\psi - \phi\|_\Sigma < \varepsilon . \tag{1.24}$$

Proof. Taking $\phi = G_0(0, T)\psi$ for sufficiently large $T > 0$, we have (1.24) by the Lebesgue dominated convergence theorem.

This corollary says that any bound state solution to the problem (11)–(12) are unstable as is stated in [2], [6], [17] and [27], since such solutions belong to $\mathscr{H}_1$.

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