

## One Parameter Semi-Groups of Operators of Schatten Class $C_p$

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### I. Introduction

Let  $H$  be a Hilbert space. A one parameter family  $T(t)$ ,  $0 \leq t < \infty$ , is called a semi-group of operators if:

- (i)  $T(0) = I$ , the identity operator of  $H$
- (ii)  $T(s + t) = T(s)T(t)$  for every  $t, s \geq 0$ .

The semi group  $T(t)$  is called  $C_0$ -semi group if  $\lim_{t \rightarrow 0} T(t)x = x$  for all  $x \in H$ . The infinitesimal generator of the semigroup  $T(t)$  is by definition the linear operator  $A$  defined by

$$D(A) = \left\{ x \in H : \lim_{t \rightarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t \rightarrow 0} \frac{T(t)x - x}{t} \quad \text{for every } x \in D(A).$$

It is well known that if  $T(t)$  is a  $C_0$ -semigroup, then  $A$  is a densely defined closed operator, [1]. The compactness of  $T(t)$ ,  $0 < t < \infty$ , was discussed in Pazy [2].

The object of this paper is to discuss when  $T(t)$ ,  $0 < t < \infty$  is in the Schatten class  $C_p$ ,  $0 < p < \infty$ . For semigroups on Hilbert spaces, the problem of being in  $C_p$  is more interesting than of being compact. This is due to the fact that for a  $C_0$ -semigroup  $T(t)$ ,  $0 < t < \infty$ ,  $\{\|T(t)\|, 0 < t \leq a\}$  is bounded in  $H$  for every finite  $a$ . But if  $T(t) \in C_p$ ,  $0 < t < \infty$ , then  $\|T(t)\|_p$  need not be bounded in any interval  $(0, a)$  for any finite  $a$ . For the basic theory of semigroups we refer to Hille and Phillips [1].

### II. When $T(t) \in C_p$

For a Hilbert space  $H$ , let  $L(H)$  be the space of all bounded linear operator on  $H$ . For  $1 \leq p < \infty$ , set:

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$$C_p = \left\{ T \in L(H) : \sup \sum_{n=1}^{\infty} |\langle Te_n, f_n \rangle|^p < \infty \right\},$$

where the supremum is taken over all orthonormal bases  $(e_n)$  and  $(f_n)$  of  $H$ . For  $T \in C_p$ , one defines  $\|T\|_p = \sup \left( \sum |\langle Te_n, f_n \rangle|^p \right)^{1/p}$ . This defines a norm on  $C_p$ . With this norm,  $C_p$  is a two sided Banach ideal in  $L(H)$ . For more on  $C_p$ , we refer to Pietsch [4].

The following lemma (whose proof is known in the literature) will be used oftenly throughout the paper. We give a proof of the lemma for completeness.

**Lemma 2.1.** *Let  $T_n \in C_p$  such that  $\sup_n \|T_n\|_p < \infty$ . If  $T_n \rightarrow T$  ( $n \rightarrow \infty$ ) in the operator norm, then  $T \in C_p$ .*

*Proof.* Since  $T_n \in C_p$ , each  $T_n$  is compact. Hence

$$T_n = \sum_{k=1}^{\infty} \sigma_{nk} e_{nk} \otimes f_{nk},$$

where  $\sum_{k=1}^{\infty} |\sigma_{nk}|^p \leq \lambda < \infty$  for all  $n$ ,  $(e_{nk})$ ,  $(f_{nk})$  are orthonormal sequences for each  $n$ . Since  $\|T_n - T\| \rightarrow 0$  ( $n \rightarrow \infty$ ), it follows that  $T$  is compact. Let  $T = \sum_{k=1}^{\infty} \sigma_k e_k \otimes f_k$ . Using Theorem 1.20 of [6], we get  $\sigma_{nk} \rightarrow \sigma_k$  ( $n \rightarrow \infty$ ) for all  $k$ . Since

$$\sum_{k=1}^r |\sigma_k|^p = \sum_{k=1}^r \lim_n |\sigma_{nk}|^p = \lim_n \sum_{k=1}^r |\sigma_{nk}|^p \leq \lambda$$

is true for every  $r$ , it follows that  $\|T\|_p = \left( \sum_{k=1}^{\infty} |\sigma_k|^p \right)^{1/p} \leq \lambda$ . This ends the proof.

**Lemma 2.2.** *Let  $(T(t))$  be a  $C_0$ -semigroup in  $L(H)$ . If for some  $t_0 > 0$ ,  $T(t_0) \in C_p$ , then  $T(t) \in C_p$  for all  $t > t_0$ . Further there exists  $M$  and  $a$  in  $(0, \infty)$  such that  $\|T(t)\|_p \leq \|T(t_0)\|_p M e^{a(t-t_0)}$ .*

*Proof.* From the semigroup property, we have  $T(t) = T(t_0)T(t-t_0)$ . Since  $C_p$  is a two sided ideal, it follows that  $T(t) \in C_p$ . Further the Banach ideal property of  $C_p$  gives  $\|T(t)\|_p \leq \|T(t_0)\|_p \|T(t-t_0)\|$ . Since  $(T(t))$  is a  $C_0$ -semigroup, then there exists an  $M$  and  $a$  in  $(0, \infty)$  such that  $\|T(s)\| \leq M e^{as}$ , [3]. This gives the result.

**Definition 2.3.** Let  $(T(t))$  be a  $C_0$ -semigroup in  $L(H)$ . We say  $(T(t))$  is of type  $p$  if:

- (i)  $T(t) \in C_p$  for all  $t > 0$
- (ii) There exists an  $\varepsilon > 0$  and an  $\alpha > 0$  such that  $\|T(t)\|_p \leq \alpha$  for all  $t \in (0, \varepsilon)$ .

Let  $(T(t))$  be a  $C_0$ -semigroup of operators with generator  $A$ . Let  $\lambda \in \rho(A)$  such that  $\operatorname{Re}(\lambda) > a$ , where  $\|T(t)\| \leq M e^{at}$ . We define a family of operators

$(R_t(\lambda, A))$ , where  $R_t(\lambda, A)x = \int_t^\infty e^{-\lambda s} T(s)x ds$ . We say  $(R_t(\lambda, A))$  is of type  $p$  if

- (i)  $R_t(\lambda, A) \in C_p$  for all  $t$  and all  $\lambda \in \rho(A)$ ,  $\operatorname{Re}(\lambda) > a$ .
- (ii) There exists  $\beta > 0$  such that  $\|\lambda R_t(\lambda, A)\|_p \leq \beta$  for all  $t \in (0, \infty)$  and  $\lambda \in \rho(A)$ ,  $\operatorname{Re}(\lambda) > a_1 > a$ .

Now we prove.

**Theorem 2.4.** Let  $(T(t))$  be a  $C_0$ -semigroup with generator  $A$ . Then the following are equivalent:

- (i)  $(T(t))$  is of type  $p$
- (ii)  $(R_t(\lambda, A))$  is of type  $p$  and  $(T(t))$  is uniformly continuous on  $(0, \infty)$ .

*Proof.* (i)  $\rightarrow$  (ii). Since  $T(t) \in C_p$ , it follows that  $T(t)$  is compact for all  $t \in (0, \infty)$ . Hence  $T(t)$  is uniformly continuous on  $(0, \infty)$ , [2]. Consequently

$$R(\lambda, A) = \int_0^\infty e^{-\lambda s} T(s) ds = \lim_{t \rightarrow 0} R_t(\lambda, A), \text{ where the limit is the uniform limit.}$$

Now,

$$\begin{aligned} R_t(\lambda, A) &= \int_t^\infty e^{-\lambda s} T(s) dx \\ &= T(t) \int_t^\infty e^{-\lambda s} T(s-t) ds. \end{aligned}$$

Since  $T(t) \in C_p$ , it follows that  $R_t(\lambda, A) \in C_p$ . Further

$$\begin{aligned} \|R_t(\lambda, A)\|_p &\leq \|T(t)\|_p \int_0^\infty e^{-\lambda s} M e^{a(s-t)} ds \\ &\leq b \|T(t)\|_p, \end{aligned}$$

where  $b$  is a constant that is independent of  $t$  and  $\lambda$ . Hence  $R_t(\lambda, A) \in C_p$  for all  $\lambda$  and  $t > 0$ .

Now, consider:

$$\begin{aligned} \|\lambda R_t(\lambda, A)\|_p &= |\lambda| \left\| \int_t^\infty e^{-\lambda s} T(s) ds \right\|_p \\ &\leq |\lambda| \|T(t)\|_p \left| \int_t^\infty e^{-\lambda s} M e^{a(s-t)} ds \right| \\ &\leq \|T(t)\|_p \xi \frac{|\lambda|}{|a - \lambda|}. \end{aligned}$$

Consequently, if  $t \in (0, \delta)$ ,  $\delta \leq \varepsilon$ , we get  $\|\lambda R_t(\lambda, A)\|_p \leq \beta$ .

Conversely. (ii)  $\rightarrow$  (i). Since  $(T(t))$  is uniformly continuous, it follows that

$R_t(\lambda, A) \rightarrow R(\lambda, A) = \int_0^\infty e^{-\lambda s} T(s) ds$  uniformly. By the assumption,  $\|R_t(\lambda, A)\|_p \leq \beta$ . It follows from convergence theorems in  $C_p$ , [5] that  $R(\lambda, A) \in C_p$ . Further

$$\|\lambda R(\lambda, A)\|_p \leq \varliminf_t \|\lambda R_t(\lambda, A)\|_p \leq \beta.$$

Further; it follows from [3; the proof of Theorem 3.3] that

$$\lambda R(\lambda, A)T(t) \rightarrow T(t) \text{ uniformly,}$$

and

$$\begin{aligned} \|\lambda R(\lambda, A)T(t)\|_p &\leq \|T(t)\| \|\lambda R(\lambda, A)\|_p \\ &\leq \beta \|T(t)\|. \end{aligned}$$

For  $t \in (0, \varepsilon]$ ,  $\|T(t)\| \leq \eta$  for some  $\eta$ . Thus  $\lambda R(\lambda, A)T(t)$  is uniformly bounded in  $C_p$ . Consequently, [5],  $T(t) \in C_p$  for all  $t \in (0, \varepsilon]$ . It follows from the semigroup property that  $T(t) \in C_p$  for all  $t > 0$ . Further:

$$\begin{aligned} \|T(t)\|_p &\leq \varliminf_\lambda \|\lambda R(\lambda, A)T(t)\|_p \\ &\leq \beta \|T(t)\| \\ &\leq \beta \eta \end{aligned}$$

for  $t \in (0, \varepsilon]$ . This ends the proof.

*Remarks.* (i) If  $(T(t))$  is of type  $p$ , then the resolvent operator  $R(\lambda, A) \in C_p$ . To see that:

$$R_t(\lambda, A) \in C_p \text{ and } \|R_t(\lambda, A)\|_p \leq \beta.$$

Further  $R_t(\lambda, A) \rightarrow R(\lambda, A)$  ( $t \rightarrow 0$ ) uniformly. Hence, [5],  $R(\lambda, A) \in C_p$ .

(ii) There exists a  $C_0$ -semigroup of operators  $(T(t))$  such that  $T(t) \in C_p$  for all  $t \in (0, \infty)$ , but  $\|T(t)\|_p \rightarrow \infty$  as  $t \rightarrow 0$  as the following example shows:

*Example 2.5.* Let  $A$  be a positive compact operator which is not of finite rank and  $\|A\| \leq 1$ . So  $A = \sum_{n=1}^\infty \lambda_n e_n \otimes e_n$ , for some  $0 < \lambda_n < 1$  and decreasing, and  $(e_n)$  is some orthonormal basis. Define a one parameter family of operators as follows:

$$T(t) = \sum_{n=1}^\infty \lambda_n^t e_n \otimes e_n.$$

It is easily seen that  $(T(t))$  is a  $C_0$ -semigroup of operators on  $H$ . Choose  $(\lambda_n) \in \bigcap_{p>0} l^p$ , where  $l^p$  is the space of  $p$ -summable sequences. Then  $T(t) \in C_p$  for all

$p$  and all  $t$ . Now,  $\|T(t)\|_p = (\sum_{n=1}^{\infty} \lambda_n^{tp})^{1/p}$ . Further  $\|T(t)\|_p \leq \|T(s)\|_p$  for  $t > s$ . The Monotone Convergence Theorem implies that  $\|T(t)\|_p \rightarrow \infty$  as  $t \rightarrow 0$ .

Another main result of this section:

- Theorem 2.6.** *Let  $T(t)$  be  $C_0$ -semigroup of operators in  $L(H)$  with generator  $A$ . If  $w \in (0, \infty)$  such that  $\|T(t)\| \leq e^{-wt}$ , then the following are equivalent.*
- (i)  $T(t) \in C_p$  for  $t \in (0, \infty)$  and  $\|T(1/n)\|_p \leq \gamma$  for all  $n \geq n_0$ , for some  $n_0$ .
  - (ii)  $R(\lambda, A) \in C_p$  and  $\|R(\lambda, A)\|_p \leq \gamma/(\lambda + w)$  for some  $\gamma > 0$  and all  $\lambda > 0$ .

*Proof.* (i)  $\rightarrow$  (ii). Set  $R_n(\lambda, A)x = \int_{1/n}^{\infty} e^{-\lambda s} T(s)x ds$ . Since  $T(t) \in C_p$  and  $\|T(t)\|_p \leq \gamma$ , for all  $t$  in some neighborhood of zero, then

$$R_n(\lambda, A)x = T\left(\frac{1}{n}\right) \int_{1/n}^{\infty} e^{-\lambda s} T\left(s - \frac{1}{n}\right) x ds$$

is an element of  $C_p$  and

$$\begin{aligned} \|R_n(\lambda, A)\|_p &\leq \left\| T\left(\frac{1}{n}\right) \right\|_p \frac{1}{\lambda + w} \\ &\leq \frac{\gamma}{\lambda + w} \end{aligned}$$

for large values of  $n$ . But  $R_n(\lambda, A)x \rightarrow R(\lambda, A)x$  for all  $x \in H$ . Consequently Lemma 2.1, implies that  $R(\lambda, A) \in C_p$  and  $\|R(\lambda, A)\|_p \leq \gamma/(\lambda + w)$ .

(ii)  $\rightarrow$  (i) By the expansion formula of  $T(t)$ , [1, p. 352] we have  $T(t)x = \lim_{\lambda \rightarrow \infty} e^{-\lambda t} \sum_{n=0}^{\infty} (\lambda^n t^n)/(n!) [\lambda R(\lambda, A)]^n x$ , for  $\lambda > -w$ , where  $w$  is as given in the assumption. Then

$$\|T(t)\|_p \leq \lim_{\lambda \rightarrow \infty} e^{-\lambda t} \sum_0^{\infty} \frac{\lambda^n t^n}{n!} \lambda^n \|R(\lambda, A)\|^{n-1} \|R(\lambda, A)\|_p.$$

But  $\|R(\lambda, A)\| \leq \int_0^{\infty} e^{-\lambda s} \|T(s)\| ds \leq \frac{1}{\lambda + w}$ . Hence

$$\begin{aligned} \|T(t)\|_p &\leq \lim_{\lambda \rightarrow \infty} e^{-\lambda t} \sum_0^{\infty} \frac{\lambda^n t^n}{n!} \frac{\lambda^n}{(\lambda + w)^{n-1}} \frac{\gamma}{(\lambda + w)} \\ &\leq \lim_{\lambda \rightarrow \infty} e^{-\lambda t} \gamma \sup_n \frac{\lambda^n}{(\lambda + w)^n} \sum_0^{\infty} \frac{\lambda^n t^n}{n!} \\ &\leq \gamma \limsup_{\lambda \rightarrow \infty} \frac{\lambda^n}{(\lambda + w)^n}. \end{aligned}$$

Since  $\lambda > 0$  and  $w > 0$ , then  $\sup \lambda^n/(\lambda + w)^n = \lambda/(\lambda + w)$ . Hence  $\|T(t)\|_p \leq \gamma$ . Consequently  $T(t) \in C_p$ . This ends the proof.

### III. Further Results

In this section we prove:

**Theorem 3.1.** *Let  $(T(t))$  be a  $C_0$ -semigroup. If  $T(t)$  is self adjoint in  $C_p$  for all  $t > 0$ , and some  $p_0 > 0$ , then  $T(t) \in C_{p_0}$  for all  $t > 0$  and all  $p > 0$ .*

*Proof.* Since  $T(t)$  is self adjoint, then for each  $t > 0$  there exist a positive decreasing sequence  $(\lambda_{n,t})$  and an orthonormal sequence  $(e_{n,t})$  such that

$$T(t) = \sum_{n=1}^{\infty} \lambda_{n,t} e_{n,t} \otimes e_{n,t}.$$

With no loss of generality, we assume  $T(t) \in C_1$  for all  $t > 0$ . For simplicity, we show that  $T(1) \in C_p$  for all  $p$ .

If  $p > 1$ , then  $C_1 \subseteq C_p$  and there is nothing to prove. Assume  $p < 1$ . Thus  $p = 1/\varepsilon$  for some  $\varepsilon > 1$ . Choose a positive integer  $n$  such that  $n > \varepsilon$ . Assume

$$T\left(\frac{1}{n}\right) = \sum_{k=1}^{\infty} \xi_k e_k \otimes e_k,$$

$(e_k)$  is orthonormal basis and  $(\xi_k)$  is a positive decreasing sequence in  $l^1$ . Now:

$$T(1) = T^n\left(\frac{1}{n}\right) = \sum_{k=1}^{\infty} \xi_k^n e_k \otimes e_k.$$

Since  $(\xi_k) \in l^1$ , it follows that  $(\xi_k^n) \in l^{1/n} \subseteq l^p$ . Thus  $T(1) \in l^p$ . In a similar way we can show  $T(t) \in l^p$  for  $t > 0$ . This ends the proof.

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