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Theorems of Sibuya-Malgrange Type for Gevrey Functions of Several Variables

By

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In the study of Riemann-Hilbert-Birkhoff problem Sibuya [11] obtained a decomposition theorem in asymptotic analysis of one variable analogous to Cartan's decomposition theorem in complex analysis, and he solved the problem locally by using the theorem. Related with Borel-Ritt theorem, the result of Sibuya is reformulated to the isomorphism theorem by Malgrange [6]. We shall call them the theorems of Sibuya-Malgrange.

In this paper we prove the theorems of Sibuya-Malgrange type for Gevrey functions of n variables. It was Ramis [7], [8], [9], [10] who proved the theorems of Sibuya-Malgrange type for Gevrey functions of one variable. In the first section of this paper we extend the definition of Gevrey functions to the case of n variables, using the concept of strongly asymptotic expansions. This concept is due to Majima [4], who extended the theorems of Sibuya-Malgrange to the case of several variables. The second and the third sections are devoted to establishing the theorem of Borel-Ritt type and the theorems of Sibuya type for Gevrey functions. We first proved the non-abelian version of the theorem of Sibuya type by using the combinatorial anlaysis (Haraoka [2]). However it can be obtained simply from the abelian version of the theorem and the non-abelian theorem without Gevrey condition. B. Malgrange suggested to the author the way of the reduction, which is adopted in this paper. After introducing some sheaves, the theorems of Malgrange type are stated in the fourth section.

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§0. Notations

The following is a list of notations used throughout this paper.

1) $N=\{0, 1, 2, \dots\}$: the set of non-negative integers.
2) For $n, n' \in N$ such that $n < n'$, we put

$$[n, n'] = \{i \in N; n \leqq i \leqq n'\}.$$

3) $\boldsymbol{R}_+$: the set of all positive real numbers.

4) For $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)\in(\boldsymbol{R}_+)^n$ or $\boldsymbol{N}^n$, $a\geqq b$ means $a_i\geqq b_i$ for all $i=1,\dots,n$, $a\ngeqq b$ means $a_i<b_i$ for some i, and we put

$$a^b=a_1^{b_1}\cdots a_n^{b_n}.$$

5) For $\alpha=(\alpha_1,\dots,\alpha_n)\in\boldsymbol{N}^n$ and for $s=(s_1,\dots,s_n)\in(\boldsymbol{R}_+)^n$,

$$|\alpha|=\alpha_1+\cdots+\alpha_n,$$

$$\alpha!=\alpha_1!\dots a_n!,$$

$$(\alpha!)^s=(\alpha_1!)^{s_1}\dots(\alpha_n!)^{s_n},\qquad\text{and}$$

$$(\alpha!)^{s-1}=(\alpha_1!)^{s_1-1}\dots(\alpha_n!)^{s_n-1}.$$

6) $\boldsymbol{C}$: the set of all complex numbers.

7) $D(r)=\{x\in\boldsymbol{C};|x|<r\}$.

8) $V(\tau_-,\tau_+;r)=\{x\in\boldsymbol{C};0<|x|<r,\tau_-<\arg x<\tau_+\}$: open sector.

9) $V[\tau_-,\tau_+;r]=\{x\in\boldsymbol{C};0<|x|<r,\tau_-\leqq\arg x\leqq\tau_+\}$: closed sector.

10) For a domain D of $\boldsymbol{C}^n$, $\mathcal{O}(D)$ represents the set of all holomorphic functions in D.

11) For $D(r_i)$, $i=1,\dots,n$, we put

$$D(r_1)\times\cdots\times\check{D}(r_i)\times\cdots\times D(r_n)=D(r_1)\times\cdots\times D(r_{i-1})\times D(r_{i+1})\times\cdots\times D(r_n).$$

12) $\mathcal{O}(D(r_1)\times\cdots\times\check{D}(r_i)\times\cdots\times D(r_n))[[x_i]]$: $\boldsymbol{C}$-algebra of formal power series of one variable x_i with coefficients in $\mathcal{O}(D(r_1)\times\cdots\times\check{D}(r_i)\times\cdots\times D(r_n))$.

13) For $r=(r_1,\dots,r_n)\in(\boldsymbol{R}_+)^n$, we put

$$\hat{\mathcal{O}}'_n(r)=\bigcap_{i=1}^n\mathcal{O}(D(r_1)\times\cdots\times\check{D}(r_i)\times\cdots\times D(r_n))[[x_i]]$$

14) $\hat{\mathcal{O}}'_n=\text{dir.}\lim_{r\to0}\hat{\mathcal{O}}'_n(r)$

15) $\hat{\mathcal{O}}_n$: $\boldsymbol{C}$-algebra of formal power series in $(x_1,\dots,x_n)$ with coefficients in $\boldsymbol{C}$.

16) $\mathcal{O}_n$: $\boldsymbol{C}$-algebra of convergent power series in $(x_1,\dots,x_n)$ with coefficients in $\boldsymbol{C}$.

17) For a subset J of $[1,n]$, we put

$$J^c=\{i\in[1,n];i\notin J\},$$

$$\#J:\text{ cardinality of }J,$$

$$\boldsymbol{N}^J=\{(\alpha_j)_{j\in J};\alpha_j\in\boldsymbol{N}(j\in J)\},$$

$$x_J=(x_j)_{j\in J},\quad\alpha_J=(\alpha_j)_{j\in J},$$

$$x_J^{\alpha_J}=\prod_{j\in J}x_j^{\alpha_j}.$$

18) For subsets J and J' of $[1, n]$ such that $J \cap J' = \phi$, we put

$$\alpha_{J \cup J'} = (\alpha_j)_{j \in J \cup J'} .$$

19) I_m: m-by-m unit matrix.
20) O_m (resp. $O_{m,m'}$): m-by-m (resp. m-by-m') zero matrix.
21) For an m-by-m' matrix $A = (a_{i,j})$, we put

$$|A| = \max\{|a_{i,j}|; i = 1, \dots, m, j = 1, \dots, m'\} .$$

§1. Definitions and propositions

Let $x = (x_1, \dots, x_n)$ be coordinates of $\boldsymbol{C}^n$, and $s = (s_1, \dots, s_n)$ be in $(\boldsymbol{R}_+)^n$ with $s_i > 1$ for $i = 1, \dots, n$. In this section we define the concept of formal series of Gevrey order s, functions of Gevrey order s and s-Gevrey strongly asymptotically developable functions, and establish several propositions about the relations of these concepts.

For a formal series $\hat{f} = \sum_{\alpha \in \boldsymbol{N}^n} f_\alpha x^\alpha$ of n variables we can show, by using the asymptotic behaviour of Γ function, that the following four conditions are equivalent;

(i) there exist a constant C and a constant vector $A = (A_1, \dots, A_n)$ such that

$$|f_\alpha| \leqq C(\alpha!)^{s-1} A^\alpha \tag{1.1}$$

for any $\alpha \in \boldsymbol{N}^n$,

(ii) there exist a constant C' and a constant vector $A' = (A'_1, \dots, A'_n)$ such that

$$|f_\alpha| \leqq C' \Gamma\left(1 + \frac{\alpha_1}{k_1}\right) \dots \Gamma\left(1 + \frac{\alpha_n}{k_n}\right) A'^\alpha$$

for any $\alpha \in \boldsymbol{N}^n$, where we put $k_i = (s_i - 1)^{-1}$ for $i = 1, \dots, n$,

(iii) the formal series

$$\sum_{\alpha \in \boldsymbol{N}^n} \frac{f_\alpha}{(\alpha!)^{s-1}} x^\alpha$$

is convergent,

(iv) the formal series

$$\sum_{\alpha \in \boldsymbol{N}^n} \frac{f_\alpha}{\Gamma\left(1 + \frac{\alpha_1}{k_1}\right) \dots \Gamma\left(1 + \frac{\alpha_n}{k_n}\right)} x^\alpha$$

is convergent, where $k_i (i = 1, \dots, n)$ is as above.

Definition 1. A formal series $\hat{f}=\sum_{\alpha\in N^n} f_\alpha x^\alpha$ which satisfies one (and hence all) of the above four conditions is called a *formal series of Gevrey order s*. We denote by $\hat{\mathcal{O}}_{n,s}$ the set of all formal series of Gevrey order s of n variables.

Next we define a function of Gevrey order s in an open polysector $V = V_1 \times \cdots \times V_n$ at the origin in C^n.

Definition 2. A holomorphic function $f(x)$ in V is called a *function of Gevrey order s in V*, if, for any closed subpolysector W of V, there exist a constant C_W and a constant vector $A_W = (A_{W,1}, \dots, A_{W,n})$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha f(x)\right| \leqq C_W(\alpha!)^s A_W^\alpha$$

for any $x \in W$ and for any $\alpha \in N^n$. We denote by $\mathcal{O}_s(V)$ the set of all functions of Gevrey order s in V.

Proposition 1.

(1) *If* $f, g \in \mathcal{O}_s(V)$, *then* $f+g, fg \in \mathcal{O}_s(V)$.

(2) *If* $f \in \mathcal{O}_s(V)$, $f(x) \neq 0$ *in* V *and* $\liminf_{\substack{x\to 0\\ x\in V}} |f(x)| \neq 0$, *then* $f^{-1} \in \mathcal{O}_s(V)$.

(1) is a direct result from the definition. As to the proof of (2), see Komatsu [3], Lemma 5. Proposition 1 implies that $\mathcal{O}_s(V)$ makes a C-algebra and the subset of $\mathcal{O}_s(V)$ which consists of the elements satisfying the assumptions of (2) makes a multiplicative group.

For functions depending on several parameters, we provide the following definition. Suppose that $f(x, t)$ is a function holomorphic in x in V with parameters $t = (t_1, \dots, t_m)$. We say that $f(x, t)$ is *of Gevrey order s in V uniformly in t*, if, for any closed subpolysector W of V, there exist a constant C_W and a constant vector $A_W = (A_{W,1}, \dots, A_{W,n})$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha f(x, t)\right| \leqq C_W(\alpha!)^s A_W^\alpha$$

for any t, for any $x \in W$ and for any $\alpha \in N^n$.

Before the definition of s-Gevrey strongly asymptotically developable functions, we introduce the concept of strongly asymptotically developable functions, which is an extension of asymptotically developable functions to the case of several variables and is defined by Majima [4].

Let $V = V_1 \times \cdots \times V_n$ be an open polysector at the origin in C^n.

Definition A. A holomorphic function $f(x)$ in V is called *strongly asymptotically developable as x tends to* 0 *in V*, if there exists a family of functions

$$\{f_{\alpha_J}(x_{J^c})\}_{J\subset[1,n],\, J\neq\phi,\, \alpha_J\in N^J} \tag{A.1}$$

having the following properties;

(A.2) $f_{\alpha_J}(x_{J^c})$ is holomorphic in $V_{J^c} = \prod_{j \in J^c} V_j$ for any non-empty proper subset J of $[1, n]$ and for any $\alpha \in \boldsymbol{N}^J$, and $f_{\alpha_{[1,n]}}(x_{[1,n]^c})$ is a constant for any $\alpha_{[1,n]} \in \boldsymbol{N}^n$,

(A.3) for any $N \in \boldsymbol{N}^n$ and for any closed subpolysector W of V, there exists a constant $K_{N,W}$ such that

$$|f(x) - \mathrm{App}_N(x, f)| \leqq K_{N,W}|x|^N$$

for any $x \in W$, where $\mathrm{App}_N(x, f)$ is defined by

$$\text{(A.4)} \qquad \mathrm{App}_N(x, f) = \sum_{\phi \neq J \subset [1,n]} (-1)^{\# J+1} \sum_{j \in J} \sum_{\alpha_j=0}^{N_j-1} f_{\alpha_J}(x_{J^c}) x_J^{\alpha_J}.$$

We shall give several remarks and provide terminology concerning the above definition. If f is strongly asymptotically developable, the family (A.1) of functions satisfying (A.2) and (A.3) is uniquely determined; it is called the *total family of coefficients of strongly asymptotic expansion of f* and is denoted by $TA(f)$. In this case $f_{\alpha_J}(x_{J^c})$, which is denoted by $TA_{\alpha_J}(f)$, is as a function of $\# J^c$ variables strongly asymptotically developable as x_{J^c} tends to 0 in V_{J^c} with a total family of coefficients of strongly asymptotic expansion

$$\{f_{\alpha_{I \cup J}}(x_{(I \cup J)^c})\}_{I \subset J^c, I \neq \phi, \alpha_{I \cup J} \in \boldsymbol{N}^{I \cup J}}.$$

Define a formal series $FA_J(f)$ by

$$FA_J(f) = \sum_{\alpha_J \in \boldsymbol{N}^J} TA_{\alpha_J}(f) x_J^{\alpha_J}.$$

For $J = [1, n]$, $FA_{[1,n]}(f)$ reduces to a formal series of n variables with constant coefficients; we use $FA(f)$ instead of $FA_{[1,n]}(f)$ and call it the *formal series of strongly asymptotic expansion of f*.

Suppose that f is strongly asymptotically developable in V with such the coefficients of strongly asymptotic expansion that

$$FA(f) = \sum_{\alpha \in \boldsymbol{N}^n} f_\alpha x^\alpha \in \hat{\mathcal{O}}'_n(r),$$

$$TA_{\alpha_J}(f) = \sum_{\alpha_{J^c} \in \boldsymbol{N}^{J^c}} f_{\alpha_{J \cup J^c}} x_{J^c}^{\alpha_{J^c}}$$

for any non-empty subset J of $[1, n]$ and for any $\alpha_J \in \boldsymbol{N}^J$, where $r \in (\boldsymbol{R}_+)^n$ is the multi-radius of V. Then the total family of coefficients $TA(f)$ is determined only by $\hat{f} = FA(f)$. In this case we may say that f is strongly asymptotically developable to $\hat{f}$ in V.

We denote by $A(V)$ the set of all functions holomorphic and strongly asymptotically developable in V, and by $A'(V)$ the set of all functions holomorphic and strongly asymptotically developable to some elements of $\hat{\mathcal{O}}'_n(r)$ in V.

Definition 3. A holomorphic function $f(x)$ in V is called *s-Gevrey strongly asymptotically developable as x tends to* 0 *in V*, if it is strongly asymptotically developable in V and if, for any closed subpolysector W of V, there exist a constant C_W and a constant vector $A_W=(A_{W,1},\dots,A_{W,n})$ such that

$$|f(x)-\mathrm{App}_N(x,f)|\leqq C_W(N!)^{s-1}A_W^N|x|^N \tag{1.3}$$

for any $x\in W$ and for any $N\in \boldsymbol{N}^n$.

We denote by $A_s(V)$ the set of all functions holomorphic and s-Gevrey strongly asymptotically developable in V, and we put $A_s'(V)=A_s(V)\cap A'(V)$.

If $f(x)$ is s-Gevrey strongly asymptotically developable in V, $TA_{\alpha_J}(f)$ is s_{J^c}-Gevrey strongly asymptotically developable in V_{J^c} for any non-empty subset J of $[1,n]$ and for any $\alpha\in\boldsymbol{N}^J$. More precisely we have, for any non-empty subset J of $[1,n]$, for any $\alpha_J\in\boldsymbol{N}^J$ and for any $N_{J^c}\in\boldsymbol{N}^{J^c}$,

$$\begin{aligned}&|TA_{\alpha_J}(f)(x_{J^c})-\mathrm{App}_{N_{J^c}}(x_{J^c},TA_{\alpha_J}(f))|\\&\qquad\leqq C_W(\alpha!)^{s_J-1}A_{W_J}^{\alpha_J}(N_{J^c}!)^{s_{J^c}-1}A_{W_{J^c}}^{N_{J^c}}|x_{J^c}|^{N_{J^c}}\end{aligned}\tag{1.4}$$

for any $x_{J^c}\in W_{J^c}$, where C_W etc. are as in Definition 3.

We shall establish several fundamental relations among the concepts defined in this section.

Proposition 2. *If f is s-Gevrey strongly asymptotically developable in V, its formal series $FA(f)$ is a formal series of Gevrey order s.*

Proof. This is an immediate consequence of the formula (1, 4); in fact, put $J=[1,n]$ in (1, 4), then we have the proposition.

Proposition 3. *For a holomorphic function f in V, the following two conditions are equivalent;*

(i) *f is a function of Gevrey order s and strongly asymptotically developable in V,*

(ii) *f is s-Gevrey strongly asymptotically developable in V.*

Proof.

(i) $\Rightarrow$ (ii) Suppose that f is strongly asymptotically developable in V and put $TA_{\alpha_J}(f)=f_{\alpha_J}$. Similarly to the case of asymptotically developable functions of one variable, we obtain the following formulas;

$$\frac{1}{\alpha_J!}\lim_{x_J\to 0,\,x_J\in V_J}\left(\frac{\partial}{\partial x_J}\right)^{\alpha_J}f(x)=f_{\alpha_J}(x_{J^c}),$$

$$\frac{1}{\alpha_I!}\lim_{x_I\to 0,\,x_I\in V_I}\left(\frac{\partial}{\partial x_I}\right)^{\alpha_I}f_{\alpha_J}(x_{J^c})=f_{\alpha_{I\cup J}}(x_{(I\cup J)^c})$$

where $I\subset J^c$. By using these formulas we can show that

$$f(x) - \mathrm{App}_N(x, f) = \prod_{i=1}^n \left(\int_0^{x_i} dt_{i,1} \int_0^{t_{i,1}} dt_{i,2} \cdots \int_0^{t_{i,N_i-1}} dt_{i,N_i} \right)$$

$$\times \left(\frac{\partial}{\partial t_{i,N_1}} \right)^{N_1} \cdots \left(\frac{\partial}{\partial t_{n,N_n}} \right)^{N_n} f(t_{1,N_1}, \ldots, t_{n,N_n}),$$

where $\int_0^{t_{i,j}} dt_{i,j+1}$ represents an integral on a segment from 0 to $t_{i,j}$ in V_i and $\prod_{i=1}^n \left(\int_{a_i}^{b_i} dt_i \cdots \int_{a_i'}^{b_i'} dt_i' \right)$ is used instead of $\int_{a_1}^{b_1} dt_1 \cdots \int_{a_1'}^{b_1'} dt_1' \cdots\cdots \int_{a_n}^{b_n} dt_n \cdots \int_{a_n'}^{b_n'} dt_n'$.

Let x be contained in a closed subpolysector W of V. Since f is of Gevrey order s in V, there are constants $C_W, A_{W,1}, \ldots, A_{W,n}$ such that

$$\left| \left(\frac{\partial}{\partial t_{1,N_1}} \right)^{N_1} \cdots \left(\frac{\partial}{\partial t_{n,N_n}} \right)^{N_n} f(t_{1,N_1}, \ldots, t_{n,N_n}) \right| \leqq C_W (N!)^s A_W^N .$$

Therefore, for any $x \in W$ and for any $N \in \boldsymbol{N}^n$, we have

$$\begin{aligned} |f(x) - \mathrm{App}_N(x, f)| &\leqq \prod_{i=1}^n \left(\int_0^{x_i} |dt_{i,1}| \cdots \int_0^{t_{i,N_{i-1}}} |dt_{i,N_i}| \right) C_W (N!)^s A_W^N \\ &= C_W (N!)^s A_W^N \frac{|x|^N}{N!} \\ &= C_W (N!)^{s-1} A_W^N |x|^N . \end{aligned}$$

This proves (ii).

(ii) $\Rightarrow$ (i) Suppose f to be s-Gevrey strongly asymptotically developable in V. We need to show that f is of Gevrey order s in V. For any closed subpolysector $W = W_1 \times \cdots \times W_n$ of V, take, for each $i = 1, \ldots, n$, a closed sector W_i' such that $W_i \Subset W_i' \subset V_i$; that is W_i is contained in the interior of W_i'. Further take a positive number σ_i so small that

$$\gamma_i(x_i) = \{x_i + \sigma_i |x_i| \exp(\sqrt{-1}\theta); \theta \in [0, 2\pi]\}$$

is contained in W_i' for any $x_i \in W_i$.

By using the Cauchy's integrel formula, we have

$$\begin{aligned} \left(\frac{\partial}{\partial x} \right)^\alpha f(x) &= \left(\frac{\partial}{\partial x} \right)^\alpha (f(x) - \mathrm{App}_\alpha(x, f)) \\ &= \frac{\alpha!}{(2\pi\sqrt{-1})^n} \prod_{i=1}^n \left(\int_{\gamma_i(x_i)} dt_i \right) \frac{f(t) - \mathrm{App}_\alpha(t, f)}{(t - x)^{\alpha+1}} \end{aligned}$$

for any $\alpha \in \boldsymbol{N}^n$. Therefore we obtain

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha f(x)\right| \leqq \alpha! \prod_{i=1}^n \sigma_i \frac{C_{W'}(\alpha!)^{s-1}A_{W'}^\alpha(1+\sigma)^\alpha}{\sigma^{\alpha+1}}$$

$$= C_{W'}(\alpha!)^s \frac{(1+\sigma)^\alpha}{\sigma^\alpha} A_{W'}^\alpha,$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $1 + \sigma = (1 + \sigma_1, \ldots, 1 + \sigma_n)$. Thus f is of Gevrey order s in V. This completes the proof.

We can show the following proposition in the manner analogous to the proof of Proposition 3.

Proposition 4. *Suppose f to be a function of Gevrey order s in V. If g is a function holomorphic and bounded in V, then fg is also of Gevrey order s in V.*

§2. Theorem of Borel-Ritt type

For an s-Gevrey strongly asymptotically developable function, its formal series of strongly asymptotic expansion is always of Gevrey order s. Conversely it will be shown that, for any formal series of Gevrey order s, there exists an s-Gevrey strongly asymptotically developable function whose formal series coincides with the given one.

Let $V = \prod_{i=1}^n V_i = \prod_{i=1}^n V_i(\tau_{-i}, \tau_{+i}; u_i)$ be an open polysector at the origin in $\boldsymbol{C}^n$ and put $u = (u_1, \ldots, u_n) \in (\boldsymbol{R}_+)^n$.

Theorem 1 (Borel-Ritt-Gevrey).

(1) *For any $\hat{f} \in \hat{\mathcal{O}}_{n,s}$ and for any V such that $\tau_{+i} - \tau_{-i} < (s_i - 1)\pi$ for $i = 1, \ldots, n$, there exists a function f holomorphic and s-Gevrey strongly asymptotically developable in V such that $FA(f) = \hat{f}$.*

(2) *Suppose further that $\hat{f} \in \hat{\mathcal{O}}'_{n,s}(u) = \hat{\mathcal{O}}_{n,s} \cap \hat{\mathcal{O}}'_n(u)$. Then there exists a function f holomorphic and s-Gevrey strongly asymptotically developable to $\hat{f}$ in V.*

Proof. By an appropriate rotation of the coordinate axes, we may assume that

$$V_i = V_i(-\tau_i, \tau_i; u_i), \quad 0 < \tau_i < \frac{(s_i - 1)}{2}\pi,$$

for each $i = 1, \ldots, n$.

(1) In the case of $n = 1$ the theorem has been obtained by Ramis [9]. However we shall sketch the proof of the case of $n = 1$ which will be used in the proof of a general case.

Put $\hat{f} = \sum_{\alpha=0}^{\infty} f_\alpha x^\alpha$. Since $\hat{f} \in \hat{\mathcal{O}}_{1,s}$, the radius R of convergence of a series

$$\phi(x) = \sum_{\alpha=0}^{\infty} \frac{f_\alpha}{\Gamma\left(1 + \frac{\alpha}{k}\right)} x^\alpha$$

is positive, where we put $k = (s-1)^{-1}$. Take a positive number r such that $0 < r < R$ and fix it. Then a function possessing the desired properties is given by

$$f(x) = \frac{k}{x^k} \int_0^r \phi(t) \exp\left(-\frac{t^k}{x^k}\right) t^{k-1} dt .$$

Now let us show that $f(x)$ is s-Gevrey asymptotically developable to $\hat{f}$ in V. Take any $N \in \boldsymbol{N}$ and any closed sector W of V and fix them. Let x be in W. By using the integral representation of Γ function and by a change of a variable $t^k/x^k = w$, we have

$$\text{(2.1)} \quad f(x) - \sum_{\alpha=0}^{N-1} f_\alpha x^\alpha = \sum_{\alpha=0}^{N-1} \hat{f}_\alpha x^\alpha \int_{r^k/x^k}^{\infty e^{\sqrt{-1}\theta}} w^{\alpha/k} \exp(-w) dw$$

$$+ x^N \int_0^{r^k/x^k} \sum_{\alpha=N}^{\infty} \hat{f}_\alpha (x w^{1/k})^{\alpha-N} w^{N/k} \exp(-w) dw ,$$

where $\theta = -k \arg x$ and $\hat{f}_\alpha = \dfrac{f_\alpha}{\Gamma\left(1 + \frac{\alpha}{k}\right)}$. The first and the second terms of the second member of (2.1) are estimated as follows;

$$\left| \sum_{\alpha=0}^{N-1} \hat{f}_\alpha x^\alpha \int_{r^k/x^k}^{\infty e^{\sqrt{-1}\theta}} w^{\alpha/k} \exp(-w) dw \right| \leqq C_1 (N!)^{s-1} A^N |x|^N ,$$

$$\left| \int_0^{r^k/x^k} \sum_{\alpha=N}^{\infty} \hat{f}_\alpha (x w^{1/k})^{\alpha-N} w^{N/k} \exp(-w) dw \right| \leqq C_2 (N!)^{s-1} A^N ,$$

where C_1, C_2 and A are constants which depend only on W. Put $C = C_1 + C_2$, then we have immediately

$$|f(x) - \sum_{\alpha=0}^{N-1} f_\alpha x^\alpha| \leqq C (N!)^{s-1} A^N |x|^N$$

for any $x \in W$. Hence $f(x)$ is s-Gevrey asymptotically developable to $\hat{f}$ in V.

Now we shall prove the theorem for $n > 1$. Put $\hat{f} = \sum_{\alpha \in \boldsymbol{N}^n} f_\alpha x^\alpha$, $k_i = (s_i - 1)^{-1}$ and $\phi(x) = \sum_{\alpha \in \boldsymbol{N}^n} \dfrac{f_\alpha}{\prod_{i=1}^n \Gamma\left(1 + \frac{\alpha_i}{k_i}\right)} x^\alpha$. Since $\hat{f} \in \hat{\mathcal{O}}_{n,s}$, there exist positive numbers R_i ($i = 1, \dots, n$) such that $\phi(x)$ absolutely converges in the domain $\{|x_1| < R_1\} \times \cdots \times \{|x_n| < R_n\}$. In this case, if we put, for any proper subset J of $[1, n]$ and for any $\alpha_J \in N^J$,

$$\phi_{\alpha_J}(x_{J^c}) = \sum_{\alpha_{J^c} \in N^{J^c}} \frac{f_{\alpha_{J \cup J^c}}}{\prod_{j \in J^c} \Gamma\left(1 + \frac{\alpha_j}{k_j}\right)} x_{J^c}^{\alpha_{J^c}},$$

it absolutely converges in $\prod_{j \in J^c} \{|x_j| < R_j\}$.

For each i take a positive number r_i such that $0 < r_i < R_i$ and fix it. Define functions $f(x)$ and $f_{\alpha_J}(x_{J^c})$ for each α_J by

$$f(x) = \prod_{i=1}^{n} \left(\int_0^{r_i} F(t_i; x_i, k_i) dt_i \right) \phi(t),$$

$$f_{\alpha_J}(x_{J^c}) = \prod_{j \in J^c} \left(\int_0^{r_j} F(t_j; x_j, k_j) dt_j \right) \phi_{\alpha_J}(t_{J^c}),$$

where we put $F(t; x, k) = \frac{k}{x^k} \exp\left(-\frac{t^k}{x^k} \right) t^{k-1}$. Then they are holomorphic respectively in V and $\prod_{j \in J^c} V_j$. We prove that $f(x)$ is s-Gevrey strongly asymptotically developable in V with $TA_{\alpha_J}(f) = f_{\alpha_J}(x_{J^c})$ and $FA(f) = \hat{f}$. Namely for any closed subpolysector W of V we show the existence of constants $C_W, A_{W,1}, \dots, A_{W,n}$ such that

$$|f(x) - \mathrm{App}_N(x, f)| \leqq C_W (N!)^{s-1} A_W |x|^N$$

for any $x \in W$, where $\mathrm{App}_N(x, f)$ is defined by (A.4) by using $f_{\alpha_J}(x_{J^c})$ introduced above. Similarly to the case of $n = 1$, we see

$$\begin{aligned} f_{\alpha_J}(x_{J^c}) x_J^{\alpha_J} &= \prod_{j \in J} \left(\Gamma\left(1 + \frac{\alpha_j}{k_j}\right) x_j^{\alpha_j} \right) \prod_{j \in J^c} \left(\int_0^{r_j} F(t_j; x_j, k_j)\, dt_j \right) \\ &\quad \times \sum_{\alpha_{J^c} \in N^{J^c}} f_{\alpha_{J \cup J^c}} t_{J^c}^{\alpha_{J^c}} \\ &= \prod_{j \in J} \left(\int_0^{\infty} F(t_j; x_j, k_j) t_j^{\alpha_j} dt_j \right) \\ &\quad \times \prod_{i \in J^c} \left(\int_0^{r_i} F(t_i; x_i, k_i) dt_i \right) \sum_{\alpha_{J^c}} \hat{f}_{\alpha_{J \cup J^c}} t_{J^c}^{\alpha_{J^c}} \\ &= \prod_{j \in J} \left(\int_0^{\infty} F(t_j; x_j, k_j) dt_j \right) \prod_{i \in J^c} \left(\int_0^{r_i} F(t_i; x_i, k_i) dt_i \right) \\ &\quad \times \sum_{\alpha_{J^c}} \hat{f}_{\alpha_{J \cup J^c}} t^{\alpha_{J \cup J^c}} \end{aligned}$$

$$= \prod_{j\in J}\left(\int_0^{r_j} F(t_j; x_j, k_j)dt_j + \int_{r_j}^{\infty} F(t_j; x_j, k_j)\, dt_j\right)$$

$$\times \prod_{i\in J^c}\left(\int_0^{r_i} F(t_i; x_i, k_i)dt_i\right) \sum_{\alpha_{J\cup J^c}} \hat{f}_{\alpha_{J\cup J^c}} t^{\alpha_{J\cup J^c}}$$

$$= \sum_{I\subset J} \prod_{j\in I}\left(\int_{r_j}^{\infty} F(t_j; x_j, k_j)dt_j\right)$$

$$\times \prod_{i\in I^c}\left(\int_0^{r_i} F(t_i; x_i, k_i)dt_i\right) \sum_{\alpha_{J^c}} \hat{f}_{\alpha_{J\cup J^c}} t^{\alpha_{J\cup J^c}}$$

$$= \sum_{j\in J^c} \sum_{\alpha_j=0}^{\infty} \sum_{I\subset J} \prod_{j\in I}\left(\int_{r_j}^{\infty} F(t_j; x_j, k_j)dt_j\right)$$

$$\times \prod_{i\in I^c}\left(\int_0^{r_i} F(t_i; x_i, k_i)dt_i\right) \hat{f}_{\alpha_{J\cup J^c}} t^{\alpha_{J\cup J^c}}.$$

From this formula, $\mathrm{App}_N(x, f)$ can be written as follows;

$$\mathrm{App}_N(x, f) = \sum_{J\subset[1,n], J\neq\phi} (-1)^{\#J+1} \sum_{j\in J, i\in J^c} \sum_{\alpha_j=0}^{N_j-1} \sum_{\alpha_i=N_i}^{\infty}$$

$$\times \prod_{j\in J}\left(\int_{r_j}^{\infty} F(t_j; x_j, k_j)dt_j\right)$$

$$\times \prod_{i\in J^c}\left(\int_0^{r_i} F(t_i; x_i, k_i)dt_i\right) \hat{f}_{\alpha_{J\cup J^c}} t^{\alpha_{J\cup J^c}}$$

$$+ \sum_{J\subset[1,n], J\neq\phi} \sum_{j\in J, i\in J^c} \sum_{\alpha_j=0}^{N_j-1} \sum_{\alpha_i=N_i}^{\infty}$$

$$\times \prod_{h=1}^{n}\left(\int_0^{r_h} F(t_h; x_h, k_h)dt_h\right) \hat{f}_{\alpha_{J\cup J^c}} t^{\alpha_{J\cup J^c}},$$

here we used the following fact; for any subsets I and J of $[1, n]$ such that $I\cap J = \phi$,

$$\sum_{L\neq\phi, I\subset L\subset J^c} (-1)^{\#L} = \begin{cases} (-1)^{\#I+1}, & \text{if } \#I + \#J = n, \\ 0, & \text{otherwise}. \end{cases}$$

On the other hand $f(x)$ is written as follows;

$$f(x) = \sum_{J\subset[1,n]} \sum_{j\in J, i\in J^c} \sum_{\alpha_j=0}^{N_j-1} \sum_{\alpha_i=N_i}^{\infty} \prod_{h=1}^{n}\left(\int_0^{r_h} F(t_h; x_h, k_h)dt_h\right)$$

$$\times \hat{f}_{\alpha_{J\cup J^c}} t^{\alpha_{J\cup J^c}}.$$

Thus we have

$$f(x)-\mathrm{App}_N(x,f)=\sum_{J\subset[1,n]}(-1)^{\# J}\sum_{j\in J, i\in J^c}\sum_{\alpha_j=0}^{N_j-1}\sum_{\alpha_i=N_i}^{\infty}$$
$$\times\prod_{j\in J}\left(\int_{r_j}^{\infty}F(t_j;x_j,k_j)dt_j\right)\prod_{i\in J^c}\left(\int_0^{r_i}F(t_i;x_i,k_i)dt_i\right)$$
$$\times\hat{f}_{\alpha_{J\cup J^c}}t^{\alpha_{J\cup J^c}}.$$

In the manner similar to the case of $n=1$, we can show for each $J\subset[1,n]$ that

$$\Bigg|\sum_{j\in J, i\in J^c}\sum_{\alpha_j=0}^{N_j-1}\sum_{\alpha_i=N_i}^{\infty}\prod_{j\in J}\left(\int_{r_j}^{\infty}F(t_j;x_j,k_j)dt_j\right)$$
$$\times\prod_{i\in J^c}\left(\int_0^{r_i}F(t_i;x_i,k_i)\,dt_i\right)\hat{f}_{\alpha_{J\cup J^c}}t^{\alpha_{J\cup J^c}}\Bigg|$$
$$=\Bigg|\prod_{j\in J}\left(\int_{r_j}^{\infty}F(t_j;x_j,k_j)dt_j\right)\prod_{i\in J^c}\left(\int_0^{r_i}F(t_i;x_i,k_i)dt_i\right)$$
$$\times\sum_{j\in J, i\in J^c}\sum_{\alpha_j=0}^{N_j-1}\sum_{\alpha_i=N_i}^{\infty}\hat{f}_{\alpha_{J\cup J^c}}t^{\alpha_{J\cup J^c}}\Bigg|$$
$$\leqq C_{(J)}(N!)^{s-1}A_{(J)}^N|x|^N,$$

where $C_{(J)}$ and $A_{(J)}=(A_{(J),1},\dots,A_{(J),n})$ are a constant and a constant vector respectively depending only on J and W. Put

$$C=\sum_{J\subset[1,n]}C_{(J)},\ A_i=\max_{J\subset[1,n]}\{A_{(J),i}\}$$

and put $A=(A_1,\dots,A_n)$. Then we obtain

$$|f(x)-\mathrm{App}_N(x,f)|\leqq C(N!)^{s-1}A^N|x|^N$$

for any $x\in W$.

Hence $f(x)$ is s-Gevrey strongly asymptotically developable in V with $FA(f)=\hat{f}$.

(2) Let $\hat{f}=\sum_{\alpha\in N^n}f_\alpha x^\alpha$ be in $\hat{\mathcal{O}}'_{n,s}(u)$. Put $\hat{f}_{\alpha_J}=\sum_{\alpha_{J^c}}f_{\alpha_{J\cup J^c}}x^{\alpha_{J^c}}$. Since $\hat{f}\in\hat{\mathcal{O}}'_n(u)$, $\hat{f}_{\alpha_J}$ converges in V_{J^c}.

We have shown in the proof of (1) that there is a function $\tilde{f}$ which is s-Gevrey strongly asymptotically developable in V with $FA(\tilde{f})=\hat{f}$. Now we need to obtain an s-Gevrey strongly asymptotically developable function f such that $FA(f)=\hat{f}$ and $TA_{\alpha_J}(f)=\hat{f}_{\alpha_J}$. It will be constructed inductively by using $\tilde{f}$ and $\hat{f}_{\alpha_J}$.

Take any closed subpolysector W of V and fix it. Since $\tilde{f}$ is s-Gevrey strongly asymptotically developable in V, there are a constant C_0 and a constant vector $A_0 = (A_{0,1}, \dots, A_{0,n})$ such that

$$|\tilde{f} - \mathrm{App}_N(x, \tilde{f})| \leqq C_0(N!)^{s-1} A_0^N |x|^N$$

for any $x \in W$ and for any $N \in \boldsymbol{N}^n$. If we put $TA_{\alpha_J}(\tilde{f}) = \tilde{f}_{\alpha_J}$, we see from the formula (1.4) that

$$(2.2) \qquad |\tilde{f}_{\alpha_J}(x_{J^c}) - \mathrm{App}_{N_{J^c}}(x_{J^c}, \tilde{f}_{\alpha_J})| \leqq C_0(\alpha_J!)^{s_J-1}(N_{J^c}!)^{s_{J^c}-1} A_{0,J}^{\alpha_J} A_{0,J^c}^{N_{J^c}} |x_{J^c}|^{N_{J^c}}$$

for $J \subset [1, n]$, $x_{J^c} \in W_{J^c}$, $\alpha_J \in \boldsymbol{N}^J$ and $N_{J^c} \in \boldsymbol{N}^{J^c}$.

First we consider $J \subset [1, n]$ such that $\# J = n - 1$. For any $\alpha_J \in \boldsymbol{N}^J$, $\tilde{f}_{\alpha_J}(x_{J^c})$ is asymptotically developable to $\tilde{f}_{\alpha_J}$ as a function of one variable. Then, if we put

$$g_{\alpha_J}(x_{J^c}) = \hat{f}_{\alpha_J}(x_{J^c}) - \tilde{f}_{\alpha_J}(x_{J^c}),$$

it is asymptotically developable to 0. Furthermore, from (2.2), we see that there are a constant C_1 and a constant vector $A_1 = (A_{1,1}, \dots, A_{1,n})$ such that

$$(2.3) \qquad |g_{\alpha_J}(x_{J^c})| \leqq C_1(\alpha_J!)^{s_J-1} A_{1,J}^{\alpha_J}(N_{J^c}!)^{s_{J^c}-1} A_{1,J^c}^{N_{J^c}} |x_{J^c}|^{N_{J^c}}$$

for any $x_{J^c} \in W_{J^c}$ and for any $N_{J^c} \in \boldsymbol{N}^{J^c}$. Put

$$g_1(x) = \sum_{J \subset [1,n],\, \# J = n-1} \prod_{j \in J} \left(\int_0^{r_j} F(t_j, x_j, k_j) dt_j \right) \times \sum_{\alpha_J \in \boldsymbol{N}^J} \frac{g_{\alpha_J}(x_{J^c})}{\prod_{j \in J} \Gamma\left(1 + \frac{\alpha_j}{k_j}\right)} t_J^{\alpha_J}.$$

By using (2.3) we can prove in the same way as the proof of (1) that g_1 is s-Gevrey strongly asymptotically developable in V with

$$TA_{\alpha_J}(g_1) = g_{\alpha_J}$$

for any $J \subset [1, n]$ such that $\# J = n - 1$ and for any $\alpha_J \in \boldsymbol{N}^J$. Put $f_1 = g_1 + \tilde{f}$. Then f_1 becomes an s-Gevrey strongly asymptotically developable function in V such that

$$TA_{\alpha_J}(f_1) = f_{\alpha_J} \quad (J \subset [1, n],\ \# J = n - 1,\ \alpha_J \in \boldsymbol{N}^J).$$

Next we consider $J \subset [1, n]$ such that $\# J = n - 2$. For any $\alpha_J \in \boldsymbol{N}^J$, put

$$g_{\alpha_J}(x_{J^c}) = \hat{f}_{\alpha_J}(x_{J^c}) - TA_{\alpha_J}(f_1),$$

then it is strongly asymptotically developable to 0.

Furthermore we see that there are a constant C_2 and a constant vector $A_2 = (A_{2,1}, \dots, A_{2,n})$ such that

$$|g_{\alpha_J}(x_{J^c})| \leqq C_2(\alpha_J!)^{s_J-1} A_{2,J}^{\alpha_J}(N_{J^c}!)^{s_{J^c}-1} A_{2,J^c}^{N_{J^c}} |x_{J^c}|^{N_{J^c}}$$

for any $x_{J^c} \in W_{J^c}$ and for any $N_{J^c} \in \boldsymbol{N}^{J^c}$. Put

$$g_2(x) = \sum_{J \subset [1,n],\, \#J = n-2} \prod_{j \in J} \left(\int_0^{r_j} F(t_j; x_j, k_j) dt_j \right)$$
$$\times \sum_{\alpha_J \in \boldsymbol{N}^J} \frac{g_{\alpha_J}(x_{J^c})}{\prod_{j \in J} \Gamma\left(1 + \frac{\alpha_j}{k_j}\right)} t_J^{\alpha_J} .$$

and put $f_2 = f_1 + g_2$. Then f_2 is s-Gevrey strongly asymptotically developable in V with

$$TA_{\alpha_J}(f_2) = \hat{f}_{\alpha_J}(x_{J^c}) \quad (J \subset [1, n],\ \#J = n - 2,\ \alpha_J \in \boldsymbol{N}^J) .$$

Thus we have

$$TA_{\alpha_J}(f_2) = \hat{f}_{\alpha_J}(x_{J^c}) \quad (J \subset [1, n],\ \#J \geqq n - 2,\ \alpha_J \in \boldsymbol{N}^J) .$$

In a similar way, for any $n' \in [1, n-1]$, we can construct a function $f_{n'}$ which is s-Gevrey strongly asymptotically developable in V with

$$TA_{\alpha_J}(f_{n'}) = \hat{f}_{\alpha_J}(x_{J^c}) \quad (J \subset [1, n],\ \#J \geqq n - n',\ \alpha_J \in \boldsymbol{N}^J) .$$

Put $f = f_{n-1}$, then f is s-Gevrey strongly asymptotically developable in V with

$$TA_{\alpha_J}(f) = \hat{f}_{\alpha_J}(x_{J^c})$$

for any non-empty subset J of $[1, n]$ and for any $\alpha_J \in \boldsymbol{N}^J$. Namely f is s-Gevrey strongly asymptotically developable in V to $\hat{f}$. This completes the proof.

Theorem 1 implies that, whenever V satisfies the condition in the theorem, $FA(*)$ is a surjective homomorphism from $\boldsymbol{C}$-algebra $A_s(V)$ (resp. $A'_s(V)$) to $\boldsymbol{C}$-algebra $\hat{\mathcal{O}}_{n,s}$ (resp. $\hat{\mathcal{O}}'_{n,s}(u)$).

§3. Theorems of Sibuya type

$D(r_i)$ denotes a disk in x_i-plane centered at the origin with radius r_i. Let $\{V(\tau_{-i,h_i}, \tau_{+i,h_i}; r_i)\}_{h_i=1,\dots,g_i}$ be an open sectorial finite covering of $D(r_i) - \{0\}$ for $i = 1, \dots, n$. We use the following notation;

$$D(r)^{n'-1} = D(r_1) \times \cdots \times D(r_{n'-1}) ,$$

$$[n', n''] = \{n', n' + 1, \dots, n''\} ,$$

$$h_{[n',n'']} = (h_{n'}, \dots, h_{n''}) ,$$

$$G_{[n',n'']} = \{h_{[n',n'']};\ h_{n'} = 1,\dots, g_{n'},\dots, h_{n''} = 1,\dots, g_{n''}\}$$

$$V_{h_{[n',n'']}}(r) = \prod_{j=n'}^{n''} V(\tau_{-j,h_j}, \tau_{+j,h_j}; r_j)\,,$$

$$V_{h_{[n',n'']},h'_{[n',n'']}}(r) = V_{h_{[n',n'']}}(r) \cap V_{h'_{[n',n'']}}(r)$$

$$V_{h_{[n',n'']},h'_{[n',n'']},h''_{[n',n'']}}(r) = V_{h_{[n',n'']},h'_{[n',n'']}}(r) \cap V_{h''_{[n',n'']}}(r)$$

for $1 \leqq n' \leqq n'' \leqq n$. If $n' = n''$, we replace $[n', n'']$ by n', and if $n' = 1$ and $n'' = n$, we use the notation without $[n', n'']$.

Let

$$\{D(r)^{n'-1} \times V_{h_{n'}}(r_{n'}) \times V_{h_{[n'+1,n]}}(r)\}_{h_{n'}=1,\dots,g_{n'}}$$

by an open covering of

$$D(r)^{n'-1} \times (D(r_{n'}) - \{0\}) \times V_{h_{[n'+1,n]}}(r)\,.$$

Then we have

Theorem 2. *Let* $\{U_{h_{n'},h'_{n'}}\}_{h_{n'},h'_{n'}=1,\dots,g_{n'}}$ *be a family of m-by-m′ matricial functions satisfying the following conditions;*

(3.1) $U_{h_{n'},h'_{n'}}$ *is holomorphic and strongly asymptotically developable in* $D(r)^{n'-1} \times V_{h_{n'},h'_{n'}}(r_{n'}) \times V_{h_{[n'+1,n]}}(r)$,

$$TA_{\alpha_J}(U_{h_{n'},h'_{n'}}) = O_{m,m'}\,, \tag{3.2}$$

for any non-empty subset J *of* $[n', n]$ *and for any* $\alpha_J \in N^J$,

(3.3) $U_{h_{n'},h'_{n'}}$ *is of Gevrey order* $(s_{n'},\dots,s_n)$ *in* $(x_{n'},\dots,x_n)$ *uniformly in* $(x_1,\dots,x_{n'-1})$,

$$U_{h_{n'},h'_{n'}} + U_{h_{n'},h''_{n'}} = U_{h_{n'},h''_{n'}} \tag{3.4}$$

in $D(r)^{n'-1} \times V_{h_{n'},h'_{n'},h''_{n'}}(r_{n'}) \times V_{h_{[n'+1,n]}}(r)$.

Then there exists a family $\{U_{h_{n'}}\}_{h_{n'}=1,\dots,g_{n'}}$ *of m-by-m′ matricial functions satisfying the following conditions;*

(3.5) $U_{h_{n'}}$ *is holomorphic and strongly asymptotically developable in* $D(r')^{n'-1} \times V_{h_{[n',n]}}(r')$,

$$TA_{\alpha_J}(U_{h_{n'}}) = O_{m,m'}\,, \tag{3.6}$$

for any non-empty subset J *of* $[n'+1, n]$ *and for any* $\alpha_J \in N^J$,

(3.7) $U_{h_{n'}}$ *is of Gevrey order* $(s_{n'},\dots,s_n)$ *in* $(x_{n'},\dots,x_n)$ *uniformly in* $(x_1,\dots,x_{n'-1})$,

$$U_{h_{n'}} + U_{h_{n'},h'_{n'}} = U_{h_{n'}} \tag{3.8}$$

in $D(r')^{n'-1} \times V_{h_{n'},h'_{n'}}(r') \times V_{h_{[n'+1,n]}}(r')$, *where* $r'_i < r_i$ *for* $i = 1,\dots,n$.

Proof. We may assume without loss of generality that

$$\tau_{-n',h_{n'}} < \tau_{+n',h_{n'}-1} < \tau_{-n',h_{n'}+1} < \tau_{+n',h_{n'}}$$

for $h_{n'} = 1, \ldots, g_{n'}$, where $\tau_{+n',0} = \tau_{+n',g_{n'}}$ and $\tau_{+n',g_{n'}+1} = \tau_{-n',1}$.

For each $i = 1, \ldots, n'$, take positive numbers r_i' and r_i'' such that $0 < r_i' < r_i'' < r_i$ and fix them. And take a closed subpolysector $W_{h_{[n'+1,n]}}$ of $V_{h_{[n'+1,n]}}(r)$. We denote a multi-radius of $W_{h_{[n'+1,n]}}$ by $(r'_{n'+1}, \ldots, r'_n)$ and put $r' = (r'_1, \ldots, r'_{n'}, r'_{n'+1}, \ldots, r'_n)$. Put

$$\varepsilon = 8^{-1} \min \{\tau_{+n',h_{n'}} - \tau_{-n',h_{n'}+1}; h_{n'} = 1, \ldots, g_{n'}\},$$

$$\tau_{n',h_{n'}} = 2^{-1}(\tau_{-n',h_{n'}} + \tau_{+n',h_{n'}-1}),$$

$$\tilde{r}_i = 2^{-1}(r_i' + r_i'') \qquad \text{for} \quad i = 1, \ldots, n'.$$

We use the following notation;

$$D[r_i^*] = \{x_i; |x_i| \leqq r_i^*\},$$

$$D[r^*]^{n'-1} = D[r_1^*] \times \cdots \times D[r_{n'-1}^*], \; * = ', '',$$

$$V'_{h_{n'}} = V[\tau_{-n',h_{n'}} + 3\varepsilon, \tau_{+n',h_{n'}} - 3\varepsilon; r'_{n'}],$$

$$V''_{h_{n'}} = V[\tau_{-n',h_{n'}} + \varepsilon, \tau_{+n',h_{n'}} - \varepsilon; r''_{n'}],$$

$$\gamma_{n',h_{n'}} = (t\tilde{r}_{n'} \exp(\sqrt{-1}\tau_{n',h_{n'}}))_{t \in (0,1]},$$

$$\gamma_{-n',h_{n'}} = (t\tilde{r}_{n'} \exp(\sqrt{-1}(\tau_{-n',h_{n'}} + 2\varepsilon)))_{t \in (0,1]}$$
$$\cup (\tilde{r}_{n'} \exp(\sqrt{-1}((1-t)(\tau_{-n',h_{n'}} + 2\varepsilon) + t\tau_{n',h_{n'}})))_{t \in (0,1]},$$

$$\gamma_{+n',h_{n'}} = (t\tilde{r}_{n'} \exp(\sqrt{-1}(\tau_{+n',h_{n'}} - 2\varepsilon)))_{t \in (0,1]}$$
$$\cup (\tilde{r}_{n'} \exp(\sqrt{-1}((1-t)(\tau_{+n',h_n} - 2\varepsilon) + t\tau_{n',h_{n'}})))_{t \in [0,1]}$$

and for each $i = 1, \ldots, n'-1$,

$$\gamma_i = (\tilde{r}_i \exp(2\pi\sqrt{-1}t))_{t \in [0,1]}.$$

It is easy to see that

$$V'_{h_{n'}} \Subset V''_{h_{n'}} \subset V_{h_{n'}}(r),$$

$$\gamma_{*n',h_{n'}} \Subset V''_{h_{n'}},$$

$$\gamma_{*n',h_{n'}} \cap V'_{h_{n'}} = \phi, \; * = +, -.$$

For a family of functions $\{b_{h_{n'}}\}_{h_{n'}=1,\ldots,g_{n'}}$ such that $b_{h_{n'}}$ is defined in $D[r^*]^{n'-1} \times$

$V^*_{h_{n'},h_{n'}+1} \times V_{h_{[n'+1,n]}}(r)$ or in $V^*_{h_{n'},h_{n'}+1} \times V_{h_{[n'+1,n]}}(r)$, where $* = '$ or $''$, we put

$$\text{Integ}\,(h_{n'}; (b_j)) = \frac{1}{2\pi\sqrt{-1}} \sum_{j=1, j\neq h_{n'}, h_{n'}+1}^{g_{n'}} \int_{\gamma_{n'},h_{n'}} b_{j-1} d\zeta_{n'}$$
$$+ \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_{-n'},h_{n'}} b_{h_{n'}-1} d\zeta_{n'} + \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_{+n'},h_{n'}} b_{h_{n'}} d\zeta_{n'}$$

for $h_{n'} = 1, \ldots, g_{n'}$. By this definition we can prove the following lemma.

Lemma 1.

$$|\text{Integ}\,(h_{n'}; |\zeta_{n'}|^2 |\zeta_{n'} - x_{n'}|^{-1})| \leqq K$$

for $x_{n'} \in V'_{h_{n'}}$, where K is a constant independent of $x_{n'}$.

We often write $y = (x_1, \ldots, x_{n'-1})$ and $z = (x_{n'+1}, \ldots, x_n)$.

Now we proceed to the proof of the theorem. By the assumption (3.2) and (3.3), we see that there are a constant C and a constant vector $A = (A_{n'}, \ldots, A_n)$ such that

$$(3.9) \qquad |U_{h_{n'},h_{n'}+1}(x)| \leqq C(N!)^{s_{n'}-1}(L)^{s'-1} A_{n'}^N A'^L |x_{n'}|^N |z|^L$$

for any $N \in \boldsymbol{N}$, any $L \in \boldsymbol{N}^{n-n'}$ and any $x \in D[r'']^{n'-1} \times V''_{h_{n'},h_{n'}+1} \times W_{h_{[n'+1,n]}}$, where $s' = (s_{n'+1}, \ldots, s_n)$ and $A' = (A_{n'+1}, \ldots, A_n)$. Since $U_{h_{n'},h_{n'}+1}(y, x_{n'}, z)$ is holomorphic in y, it has power series expansion. Thus we have

$$(3.10) \qquad U_{h_{n'},h_{n'}+1}(y, x_{n'}, z) = \sum_{\beta \in \boldsymbol{N}^{n-n'}} u_{h_{n'},\beta}(x_{n'}, z) y^\beta .$$

We see by (3.9) that the coefficients are governed by

$$(3.11) \qquad |u_{h_{n'},\beta}(x_{n'}, z)| \leqq r'^{-\beta} C(N!)^{s_{n'}-1}(L!)^{s'-1} A_{n'}^N A'^L |x_{n'}|^N |z|^L$$

for any $N \in \boldsymbol{N}$, any $L \in \boldsymbol{N}^{n-n'}$ and any $(x_{n'}, z) \in V''_{h_{n'},h_{n'}+1} \times W_{h_{[n'+1,n]}}$. If we put

$$(3.12) \qquad E_{h_{n'},M} = U_{h_{n'},h_{n'}+1}(x) - \sum_{\beta \ngeq M} u_{h_{n'},\beta}(x_{n'}, z) y^\beta$$
$$= \sum_{\beta \geqq M} u_{h_{n'},\beta}(x_{n'}, z) y^\beta$$

for any $M \in \boldsymbol{N}^{n-n'}$, it has an integral representation

$$(3.13) \qquad E_{h_{n'},M}(x) = \frac{1}{(2\pi\sqrt{-1})^{n'-1}} \int_{\gamma_1} \cdots \int_{\gamma_{n'-1}} U_{h_{n'},h_{n'}+1}(\zeta, x_{n'}, z)$$
$$\times \prod_{i=1}^{n'-1} \frac{x_i^{M_i} d\zeta_1 \ldots d\zeta_{n'-1}}{\zeta_i^{M_i}(\zeta_i - x_i)} ,$$

so that

$$(3.14) \qquad |E_{h_{n'},M}(x)| \leqq K' r'^{-M} C(N!)^{s_{n'}-1}(L!)^{s'-1} A_{n'}^N A'^L |y|^M |x_{n'}|^N |z|^L$$

for any $M \in N^{n'-1}$, any $N \in N$, any $L \in N^{n-n'}$ and any $(y, x_{n'}, z) \in D[r']^{n'-1} \times V''_{h_{n'}, h_{n'}+1} \times W_{h_{[n'+1,n]}}$, where K' is a constant. Now we define a holomorphic matricial function $F_{h_{n'}}$ by

$$F_{h_{n'}}(x) = \mathrm{Integ}\left(h_{n'}; \frac{U_{j,j+1}(y, \zeta_{n'}, z)}{\zeta_{n'} - x_{n'}}\right) \tag{3.15}$$

for any $(y, x_{n'}, z) \in D[r']^{n'-1} \times V'_{h_{n'}} \times V_{h_{[n'+1,n]}}(r)$. By using Lemma 1 and (3.9), we have

$$|F_{h_{n'}}(x)| \leqq KC(2!)^{s_{n'}-1}(L!)^{s'-1}A_{n'}^2 A'^L |z|^L \tag{3.16}$$

for any $L \in N^{n-n'}$ and any $x \in D[r']^{n'-1} \times V'_{h_{n'}} \times W_{h_{[n'+1,n]}}$. We shall show that $F_{h_{n'}}$ is strongly asymptotically developable. Define $f_{h_{n'},\beta}(x_{n'}, z)$, $f_{h_{n'},\gamma,\beta}(z)$ and $e_{h_{n'},N,M}(y, x_{n'}, z)$ by

$$f_{h_{n'},\beta}(x_{n'}, z) = \mathrm{Integ}\left(h_{n'}; \frac{u_{j,\beta}(\zeta_{n'}, z)}{\zeta_{n'} - x_{n'}}\right) \tag{3.17}$$

for $(x_{n'}, z) \in V'_{h_{n'}} \times V_{h_{[n'+1,n]}}(r)$ and $\beta \in N^{n'-1}$,

$$f_{h_{n'},\gamma,\beta}(z) = \mathrm{Integ}\left(h_{n'}; \frac{u_{j,\beta}(\zeta_{n'}, z)}{\zeta_{n'}^{\gamma+1}}\right) \tag{3.18}$$

for $z \in V_{h_{[n'+1,n]}}(r)$, $\beta \in N^{n'-1}$ and $\gamma \in N$, and

$$\begin{aligned} e_{h_{n'},N,M}(x) = F_{h_{n'}}(x) &- \textstyle\sum_{\beta \ngeqq M} f_{h_{n'},\beta}(x_{n'}, z) y^\beta \\ &- \textstyle\sum_{\gamma=0}^{N-1} \sum_{\beta \geqq M} f_{h_{n'},\gamma,\beta}(z) y^\beta x_{n'}^\gamma \end{aligned} \tag{3.19}$$

for any $N \in N$ and any $M \in N^{n-n'}$. Then $e_{h_{n'},N,M}$ has an integral representation

$$e_{h_{n'},N,M}(x) = \mathrm{Integ}\left(h_{n'}; \frac{x_{n'}^N E_{j,M}(y, \zeta_{n'}, z)}{\zeta_{n'}^N(\zeta_{n'} - x_{n'})}\right). \tag{3.20}$$

By using (3.14), (3.20) and Lemma 1, we have

$$|e_{h_{n'},N,M}(x)| \leqq KK'r'^{-M}C((N+2)!)^{s_{n'}-1}(L!)^{s'-1} |y|^M |x_{n'}|^N |z|^L \tag{3.21}$$

for any $x \in D[r']^{n'-1} \times V'_{h_{n'}} \times W_{h_{[n'+1,n]}}$ and any $(M, N, L) \in N^n$. Hence $F_{h_{n'}}$ is strongly asymptotically developable and, by (3.16), satisfies (3.6). Moreover, since $(N+1)(N+2) \leqq 2^{N+2}$, we have

$$|e_{h_{n'},N,0}(x)| \leqq KK'\tilde{C}(N!)^{s_{n'}-1}(L!)^{s'-1} \tilde{A}_{n'}^N A'^L |x_{n'}|^N |z|^L \tag{3.22}$$

for any $N \in N$, any $L \in N^{n-n'}$ and any $x \in D[r']^{n'-1} \times V'_{h_{n'}} \times W_{h_{[n'+1,n]}}$, where $\tilde{C} = 4^{s_{n'}-1}A_{n'}^2 C$ and $\tilde{A}_{n'} = 2^{s_{n'}-1}A_{n'}$. Then, by the definition and Proposition 3, $F_{h_{n'}}$ is of Gevrey order $(s_{n'}, \dots, s_n)$ in $(x_{n'}, \dots, x_n)$ uniformly in y. From the definition (3.15) and Cauchy's theorem, we obtain

$$F_{h_{n'}} + U_{h_{n'},h_{n'}+1} = F_{h_{n'}+1} \tag{3.23}$$

for any $x \in D[r']^{n'-1} \times V'_{h_{n'},h_{n'}+1} \times V_{h_{[n'+1,n]}}(r)$.

Thus the family $\{F_{h_{n'}}\}$ satisfies (3.5), (3.6), (3.7) and (3.8) in narrowed domains. Put

$$U_{h_{n'}}(x) = \begin{cases} F_{h_{n'}-1}(x) + U_{h_{n'}-1,h_{n'}}(x)\,, \\ \qquad \text{if} \quad x \in D[r']^{n'-1} \times (V'_{h_{n'}-1} \cap V_{h_{n'}}) \times V_{h_{[n'+1,n]}}(r') \\ F_{h_{n'}}(x)\,, \quad \text{if} \quad x \in D[r']^{n'-1} \times V'_{h_{n'}} \times V_{h_{[n'+1,n]}}(r') \\ F_{h_{n'}+1}(x) - U_{h_{n'},h_{n'}+1}(x)\,, \\ \qquad \text{if} \quad x \in D[r']^{n'-1} \times (V_{h_{n'}} \cap V'_{h_{n'}+1}) \times V_{h_{[n'+1,n]}}(r')\,, \end{cases}$$

then the family $\{U_{h_{n'}}\}_{h_{n'}=1,\dots,g_{n'}}$ of matrices satisfies the conditions (3.5), (3.6), (3.7) and (3.8) in the full domains stated in the theorem. This completes the proof.

Let $\{V_h(r)\}_{h\in G}$ be an open polysectorial finite covering of $(D(r_1) - \{0\}) \times \cdots \times (D(r_n) - \{0\})$. Then we have

Theorem 3. *Let $\{U_{hh'}\}_{h,h'\in G}$ be a family of m-by-m' matricial functions satisfying the following conditions;*

(3.24) *$U_{hh'}$ is holomorphic in $V_{h,h'}(r)$,*

(3.25) *$U_{hh'}$ is s-Gevrey strongly asymptotically developable to $O_{m,m'}$* in *$V_{h,h'}(r)$,*

(3.26) *$U_{hh'} + U_{h'h''} = U_{hh''}$ in $V_{h,h',h''}(r)$ for h, h' and $h'' \in G$.*

Then there exists a family $\{U_h\}_{h\in G}$ of m-by-m' matricial functions satisfying the following conditions;

(3.27) *U_h is holomorphic in $V_h(r')$,*

(3.28) *there exists a formal series $\hat{U} \in M(m, m'; \hat{\mathcal{O}}'_{n,s}(r'))$ of Gevrey order s such that U_h is s-Gevrey strongly asymptotically developable to $\hat{U}$ in $V_h(r')$ for all $h \in G$,*

(3.29) *$U_h + U_{hh'} = U_{h'}$ in $V_{h,h'}(r')$ for h and $h' \in G$, where $r' \in (\mathbf{R}_+)^n$, $r' \leqq r$.*

Proof. We shall construct U_h by induction. First we put $U(1, h, h') = U_{hh'}$ for $h, h' \in G$.

Suppose, for a positive integer $n' \leqq n$, that there is a family of matricial functions

$$\{U(n', h_{[n',n]}, h'_{[n',n]})\}_{h_{[n',n]}, h'_{[n',n]} \in G_{[n',n]}}$$

corresponding to the covering

$$\{D(r)^{n'-1} \times V_{h_{[n',n]}}(r)\}_{h_{[n',n]} \in G_{[n',n]}}$$

of $D(r)^{n'-1}\times(D(r_{n'})-\{0\})\times\cdots\times(D(r_n)-\{0\})$ such that

(A, n') $U(n', h_{[n',n]}, h'_{[n',n]})$ is holomorphic and strongly asymptotically developable in $D(r)^{n'-1}\times V_{h_{[n',n]},h'_{[n',n]}}(r)$,

(B, n') $TA_{\alpha_J}(U(n', h_{[n',n]}, h'_{[n',n]}))=O_{m,m'}$ for any non-empty subset J of $[n', n]$ and any $\alpha_J\in N^J$,

(C, n') $U(n', h_{[n',n]}, h'_{[n',n]})$ is of Gevrey order $(s_{n'},\dots,s_n)$ in $(x_{n'},\dots,x_n)$ uniformly in $(x_1,\dots,x_{n'-1})$,

$$(D, n')\quad U(n', h_{[n',n]}, h'_{[n',n]})+U(n', h'_{[n',n]}, h''_{[n',n]})=U(n', h_{[n',n]}, h''_{[n',n]})$$

in $D(r)^{n'-1}\times V_{h_{[n',n]},h'_{[n',n]},h''_{[n',n]}}(r)$.

Take an $h_{[n'+1,n]}\in G_{[n'+1,n]}$ and fix it. Apply Theorem 2 to the family

$$\{U(n', (h_{n'}, h_{[n'+1,n]}), (h'_{n'}, h_{[n'+1,n]}))\}_{h_{n'},h'_{n'}}\in G_{n'}$$

corresponding to the covering

$$\{D(r)^{n'-1}\times V_{h_{n'}}(r_{n'})\times V_{h_{[n'+1,n]}}(r)\}_{h_{n'}\in G_{n'}}$$

of $D(r)^{n'-1}\times(D(r_{n'})-\{0\})\times V_{h_{[n'+1,n]}}(r)$, then we obtain a family of matricial functions

$$\{U(n', (h_{n'}, h_{[n'+1,n]}))\}_{h_{n'}\in G_{n'}}$$

satisfying (3.5), (3.6), (3.7) and (3.8). When $h_{[n'+1,n]}$ and $h'_{[n'+1,n]}$ are fixed, we have, for $h_{n'}$, $h'_{n'}$, $h''_{n'}$ and $h'''_{n'}\in[1, g_{n'}]$,

$$\begin{aligned}&U(n', (h_{n'}, h_{[n'+1,n]}))+U(n', (h_{n'}, h_{[n'+1,n]}), (h'_{n'}, h'_{[n'+1,n]}))\\&\quad-U(n', (h'_{n'}, h'_{[n'+1,n]}))\\&=U(n', (h''_{n'}, h'_{[n'+1,n]}))+U(n', (h''_{n'}, h_{[n'+1,n]}), (h'''_{n'}, h'_{[n'+1,n]}))\\&\quad-U(n', (h'''_{n'}, h'_{[n'+1,n]}))\end{aligned}$$

in $D(r')^{n'-1}\times V_{h_{n'},h'_{n'},h''_{n'},h'''_{n'}}(r')\times V_{h_{[n'+1,n]},h'_{[n'+1,n]}}(r')$ by using (C, n') and (3.8). Therefore, if we define $U(n'+1, h_{[n'+1,n]}, h'_{[n'+1,n]})$ by

$$\begin{aligned}&U(n'+1, h_{[n'+1,n]}, h'_{[n'+1,n]})\\&\quad=U(n', h_{[n',n]})+U(n', h_{[n',n]}, h'_{[n',n]})-U(n', h'_{[n',n]})\end{aligned}$$

for $x\in D(r')^{n'-1}\times V_{h_{n'},h'_{n'}}(r'_{n'})\times V_{h_{[n'+1,n]},h'_{[n'+1,n]}}(r')$, it is well defined in $D(r')^{n'-1}\times(D(r'_{n'})-\{0\})\times V_{h_{[n'+1,n]},h'_{[n'+1,n]}}(r')$. By Riemann's theorem of removable singularities, $U(n'+1, h_{[n'+1,n]}, h'_{[n'+1,n]})$ can be extended to a holomorphic function in $D(r')^{n'}\times V_{h_{[n'+1,n]},h'_{[n'+1,n]}}(r')$. We denote this extended one by the same letter. Then it is easy to see that, corresponding to the covering

$$\{D(r')^{n'}\times V_{h_{[n'+1,n]}}(r')\}_{h_{[n'+1,n]}\in G_{[n'+1,n]}}$$

of $D(r')^{n'} \times (D(r'_{n'+1}) - \{0\}) \times \cdots \times (D(r_n) - \{0\})$, the family

$$\{U(n'+1, h_{[n'+1,n]}, h'_{[n'+1,n]})\}_{h_{[n'+1,n]}, h'_{[n'+1,n]} \in G_{[n'+1,n]}}$$

of matricial functions satisfies the condition $(A, n'+1)$, $(B, n'+1)$, $(C, n'+1)$ and $(D, n'+1)$. Hence we can construct inductively a family of functions

$$\{U(n', h_{n'}, \dots, h_n)\}_{(h_{n'}, \dots, h_n) \in G_{[n',n]}} \quad (n' = 1, \dots, n)$$

satisfying (A, n'), (B, n'), (C, n') and (D, n'). For any $h = (h_1, \dots, h_n) \in G$, put

$$U_h = U(n, h_n) + \cdots + U(n', h_{n'}, \dots, h_n) + \cdots + U(1, h),$$

then the family $\{U_h\}_{h \in G}$ satisfies the condition (3.27) and (3.29). Furthermore, by (3.29), we have

$$FA_J(U_h) = FA_J(U_{h'})$$

for any non-empty subset J of $[1, n]$ and any $h, h' \in G$. Then we have

$$TA_{\alpha_J}(U_h) = TA_{\alpha_J}(U_{h'})$$

for any $\alpha_J \in N^J$ in $\prod_{i \in J^c} V_{h_i, h'_i}(r''_i)$, so that we can define a holomorphic function U_{α_J} in $\prod_{i \in J^c}(D(r''_i) - \{0\})$ by

$$U_{\alpha_J} = TA_{\alpha_J}(U_h)$$

in $\prod_{i \in J^c} V_{h_i}(r''_i)$, $(h_i)_{i \in J^c} \in \prod_{i \in J^c}[1, g_i]$. Then, again by Riemann's theorem, U_{α_J} can be extended to a holomorphic function in $\prod_{i \in J^c} D(r''_i)$, hence U_{α_J} is a convergent power series. Thus U_h is strongly asymptotically developable to $\hat{U} = FA(U_h)$ in $V_h(r'')$. On the other hand, U_h is of Gevrey order s by its definition. Thus, from Propositions 2 and 3, $\{U_h\}_{h \in G}$ satisfies (3.28). This completes the proof.

Now we can show a non-abelian version of Theorem 3.

Theorem 4. *Let $\{V_h(r)\}_{h \in G}$ be an open polysectorial finite covering of $(D(r_1) - \{0\}) \times \cdots \times (D(r_n) - \{0\})$, and let $\{P_{hh'}\}_{h,h' \in G}$ be a family of m-by-m matricial functions such that*

(3.30) *$P_{hh'}$ is holomorphic and invertible in $V_{h,h'}(r)$,*

(3.31) *$P_{hh'}$ is s-Gevrey strongly asymptotically developable to I_m in $V_{h,h'}(r)$,*

(3.32) $$P_{hh'} P_{h'h''} = P_{hh''}$$

in $V_{h,h',h''}(r)$ for h, h' and $h'' \in G$.

Then there exists a family $\{P_h\}_{h \in G}$ of m-by-m matricial functions satisfying the following conditions;

(3.33) *P_h is holomorphic and invertible in $V_h(r')$,* (3.34) *there exists a formal series* $\hat{P} \in GL(m; \hat{\mathcal{O}}'_{n,s}(r'))$ *of Gevrey order* s *such that* P_h *is* s*-Gevrey strongly asymptotically developable to* $\hat{P}$ *in* $V_h(r')$ *for all* $h \in G$,

$$(3.35) \qquad P_h P_{hh'} = P_{h'}$$

in $V_{h,h'}(r')$ *for all* h *and* $h' \in G$, *where* $r' \in (\boldsymbol{R}_+)^n$, $r' \leqq r$.

Proof. Since the family $\{P_{hh'}\}_{h,h' \in G}$ satisfies (3.30), (3.32) and of course

(3.31)′ $P_{hh'}$ is strongly asymptotically developable to I_m in $V_{h,h'}(r)$,

there exists a family $\{P_h\}_{h \in G}$ of m-by-m matricial functions satisfying (3.33), (3.35) and

(3.34)′ there exists a formal series $\hat{P} \in GL(m; \hat{\mathcal{O}}'_n(r'))$ such that P_h is strongly asymptotically developable to $\hat{P}$ in $V_h(r')$ for all $h \in G$.

(see Majima [4], Theorem 3.) We need only to show that

(3.36) P_h is of Gevrey order s in $V_h(r')$ for all $h \in G$

when the family $\{P_{hh'}\}_{h,h' \in G}$ satisfies (3.31), since we see that (3.34)′ and (3.36) yield the condition (3.34) by Propositions 2 and 3.

Put

$$P_{hh'} = I_m + Q_{hh'}$$

for all h, $h' \in G$. Since $P_{hh'}$ is s-Gevrey strongly asymptotically developable to I_m in $V_{h,h'}(r)$, $Q_{hh'}$ is s-Gevrey strongly asymptotically developable to O_m there. For $\{P_h)_{h \in G}$ satisfies (3.35), we can write

$$P_{h'} = P_h + P_h Q_{hh'}$$

in $V_{h,h'}(r')$. P_h is strongly asymptotically developable and $Q_{hh'}$ is of Gevrey order s, then, by Proposition 4, $P_{h'} - P_h = P_h Q_{hh'}$ is of Gevrey order s. Hence, if we put

$$U_{hh'} = P_{h'} - P_h$$

in $V_{h,h}(r')$, $U_{hh'}$ is s-Gevrey strongly asymptotically developable to O_m. Moreover, the family $\{U_{hh'})_{h,h' \in G}$ clearly satisfies (3.26). Thus $\{U_{hh'}\}_{h,h' \in G}$ satisfies all the assumptions of Theorem 3, so that there exists a family $\{U_h\}_{h \in G}$ satisfying (3.27), (3.28) and (3.29), and we have

$$(3.37) \qquad U_{h'} - U_h = U_{hh'} = P_{h'} - P_h$$

in $V_{h,h''}(r'')$, where $r'' \in (\boldsymbol{R}_+)^n$, $r'' \leqq r'$. Therefore, if we define a holomorphic function F by

$$F = P_h - U_h$$

in $V_{h,h'}(r''_1)$, it is well defined in $(D(r'') - \{0\}) \times \cdots \times (D(r''_n) - \{0\})$. By Riemann's theorem of removable singularities (for functions of several variables), F can be extended to the full domain $D(r'')^n$. Then

$$P_h = U_h + F$$

is of Gevrey order s in $V_h(r'')$. This completes the proof.

§4. Theorems of Malgrange type

We identify $(\boldsymbol{C}-\{0\})^n$ with $(\boldsymbol{R} \bmod 2\pi\boldsymbol{Z})^n \times (\boldsymbol{R}_+)^n$ by the mapping

$$\begin{array}{ccc} (\boldsymbol{C}-\{0\})^n & \longrightarrow & (\boldsymbol{R} \bmod 2\pi\boldsymbol{Z})^n \times (\boldsymbol{R}_+)^n \\ \cup\!\!\!| & & \cup\!\!\!| \\ x=(x_1,\dots,x_n) & \longrightarrow & (\arg x_1,\dots,\arg x_n, |x_1|,\dots,|x_n|) \end{array}$$

and further we identify $(\boldsymbol{R} \bmod 2\pi\boldsymbol{Z})^n$ with a real n-dimensional torus T^n. Now we define several sheaves of germs of functions over T^n.

Let $c=\prod_{i=1}^n(\tau_{-i},\tau_{+i})$ be an open set of T^n and $r=(r_1,\dots,r_n)$ be in $(\boldsymbol{R}_+)^n$. We denote by $A_s(c,r)$ the $\boldsymbol{C}$-algebra of all functions s-Gevrey strongly asymptotically developable in the open polysector $V(c,r)=\prod_{i=1}^n V(\tau_{-i},\tau_{+i};r_i)$. By taking the inductive limit of $A_s(c,r)$ with respect to r, we have a presheaf $A_s(c)$ with the natural restriction mapping. Define A_s as the associated sheaf and call it the sheaf of germs of functions s-Gevrey strongly asymptotically developable over T^n.

We denote by $M(m,m';A_s(c,r))$ and $M(m,m';A_s(c,r))_0$ abelian groups of m-by-m' matricial functions s-Gevrey strongly asymptotically developable in $V(c,r)$, and m-by-m' matricial functions s-Gevrey strongly asymptotically developable to $O_{m,m'}$ in $V(c,r)$ respectively. Similarly to the construction of A_s, we define a sheaf $M(m,m';A_s)$ (resp. $M(m,m';A_s)_0$) of germs of m-by-m' matricial functions s-Gevrey strongly asymptotically developable (resp. s-Gevrey strongly asymptotically developable to $O_{m,m'}$) over T^n from the group $M(m,m';A_s(c,r))$ (resp. $M(m,m';A_s(c,r))_0$). In the case of non-abelian groups, we obtain a sheaf $GL(m;A_s)$ (resp. $GL(m;A_s)_I$) of germs of m-by-m invertible matricial functions s-Gevrey strongly asymptotically developable (resp. s-Gevrey strongly asymptotically developable to I_m) over T^n.

By the theorem of Borel-Ritt type (Theorem 1), we see that the short sequence of sheaves

$$0 \to M(m,m';A_s)_0 \to M(m,m';A_s) \to M(m,m';\underline{\hat{\mathcal{O}}'_{n,s}}) \to 0 \tag{4.1}$$

and

$$I \to GL(m;A_s)_I \to GL(m;A_s) \to GL(m;\underline{\hat{\mathcal{O}}'_{n,s}}) \to I \tag{4.2}$$

are exact, where $\underline{\hat{\mathcal{O}}'_{n,s}}$ denotes the constant sheaf of the $\boldsymbol{C}$-algebra $\hat{\mathcal{O}}'_{n,s}$ over T^n. From (4.1) we obtain the long exact sequence

$$\begin{aligned} 0 \to 0 \to M(m,m';\mathcal{O}_n) &\to M(m,m';\hat{\mathcal{O}}'_{n,s}) \to H^1(T^n, M(m,m';A_s)_0) \\ &\to H^1(T^n, M(m,m';A_s)) \to H^1(T^n, M(m,m';\hat{\mathcal{O}}'_{n,s})) \to \cdots \end{aligned} \tag{4.3}$$

Now the theorem of Sibuya type (Theorem 3) implies

Theorem 3. *There exists an isomorphism*

$$M(m, m'; \hat{\mathcal{O}}'_{n,s})/M(m, m'; \mathcal{O}_n) \simeq H^1(T^n, M(m, m'; A_s)_0)\,.$$

Similarly we have from (4.2) the long exact sequence

$$(4.4) \qquad I \to I \to GL(m; \mathcal{O}_n) \to GL(m; \hat{\mathcal{O}}'_{n,s}) \to H^1(T^n, GL(m; A_s)_I)$$

$$\to H^1(T^n, GL(m; A_s)) \to H^1(T^n, GL(m; \hat{\mathcal{O}}'_{n,s}))\,,$$

and we see that Theorem 4 implies

Theorem 6. *There exists an isomorphism*

$$GL(m; \hat{\mathcal{O}}'_{n,s})/GL(m; \mathcal{O}_n) \simeq H^1(T^n, GL(m; A_s)_I)\,.$$

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