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On Non-Local Problems for Elliptic Linear Equations

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In this article we investigate the existence and uniqueness theorems for a class of non-local problems for elliptic linear equations. The first general formulation of a non-local problem can be found in [1]. Since then several authors have developed the ideas of this paper in many ways for parabolic and elliptic equations (see [2], [3], [4], [7], [8], [9] and [11]). The main purpose of this work is to extend the recent results of [11]. In particular we derive the maximum principle and some apriori bounds for solutions of non-local problems in bounded domains. The existence of solutions then follows from the Fredholm theory of the integral equations. In the final part of this paper we discuss some non-local problems in the half space.

1. Uniqueness results

Let $Q \subset \boldsymbol{R}^n$ be a bounded domain with the boundary ∂Q. In Q we consider the equation

(1) $$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f(x)\,.$$

We make the following assumptions:

(A) there exists a positive constant a such that

$$a|\lambda|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\lambda_i\lambda_j$$

for all $x \in Q$ and $\lambda \in \boldsymbol{R}^n$. Moreover the coefficients a_{ij}, b_i, c and f are bounded in Q with $c(x) \leq 0$ on Q.

Let Γ_0 and Γ be open subsets in ∂Q and let $\Gamma_0 \cap \Gamma = \emptyset$, $\partial\Gamma_0 = \partial\Gamma = \gamma$ and $\Gamma \cup \Gamma_0 \cup \gamma = \partial Q$.

To formulate a non-local problem associated with (1) we introduce a mapping $F: \bar{\Gamma} \times C(\bar{Q}) \to \boldsymbol{R}$. In this paper we investigate the following non-local problem: given functions f, ϕ and Ψ defined on Q, $\bar{\Gamma}_0$ and $\bar{\Gamma}$, respectively, find a solution $u \in C^2(Q) \cap C(\bar{Q})$ of (1) satisfying the conditions

(2) $$u(x) = \phi(x) \qquad \text{on } \bar{\Gamma}_0\,,$$

and

$$(3) \qquad u(x) + F(x, u(\cdot)) = \Psi(x) \qquad \text{on } \bar{\Gamma}.$$

The condition (3) determines the non-local character of the problem (1), (2), (3) in which the values of a solution on the part Γ of the boundary are connected with values of u on Q.

We commence with the following uniqueness results.

Proposition 1. *Suppose the mapping F is linear in u and has the following property*:

(A_1) *for every $x \in \Gamma$ and $u \in C(\bar{Q})$ with $u \not\equiv 0$ on Q there exists a point $\tilde{x} \in \bar{Q}$ such that*

$$(4) \qquad |F(x, u(\cdot))| < |u(\tilde{x})|.$$

Then the problem (1), (2) *and* (3) *admits at most one solution in $C^2(Q) \cap C(\bar{Q})$.*

Proof. Let $u \in C^2(Q) \cap C(\bar{Q})$ be a solution of the homogeneous problem

$$Lu = 0 \qquad \text{on } Q,$$

$$u = 0 \qquad \text{on } \bar{\Gamma}_0,$$

and

$$u(x) + F(x, u(\cdot)) = 0 \qquad \text{on } \bar{\Gamma}.$$

Suppose that $u \not\equiv 0$. We may assume that u takes on a negative value at certain point of $\bar{Q}$. By the strong maximum principle ([5] Theorem 3.5 p. 35) there exists a point $x^0 \in \Gamma$ such that $u(x^0) = \min_{\bar{Q}} u < 0$. By ($A_1$) there exists a point $x^1 \in \bar{Q}$ such that

$$|u(x^0)| = |F(x^0, u(\cdot))| < |u(x^1)|$$

If $u(x^1) < 0$, then $u(x^1) < u(x^0) = \min_{\bar{Q}} u$ and we get a contradiction. Hence $u(x^1) > 0$ and consequently there exists a point $x^2 \in \bar{Q}$ such that $u(x^2) = \max_{\bar{Q}} u > 0$. Again by the strong maximum principle $x^2 \in \Gamma$. It follows from (A_1) that there exists a point $x^3 \in \bar{Q}$ such that

$$u(x^2) = |F(x^2, u(\cdot))| < |u(x^3)|.$$

If $u(x^3) > 0$ we get a contradiction, hence $u(x^3) < 0$. Now we must distinguish two cases

$$|u(x^0)| < u(x^2) \quad \text{or} \quad |u(x^0)| \geq u(x^2).$$

In the first case $|u(x^0)| < u(x^2) < |u(x^3)|$, since both values $u(x^0)$ and $u(x^3)$ are negative we obtain a contradiction. Similarly in the second case

$$u(x^2) \leq |u(x^0)| < |u(x^1)| = u(x^1)$$

and again we obtain a contradiction.

As an example of mapping F having property (A_1) we can give

$$F(x, u) = \int_Q u(y)\mu^x(dy)\,, \qquad x \in \bar{\Gamma}\,, \tag{6}$$

where for each $x \in \bar{\Gamma}$ $\mu^x(\cdot)$ is a Borel signed measure on $\bar{Q}$ with property $|\mu^x| < 1$ for each $x \in \bar{\Gamma}$ (here $|\mu^x|$ denotes the total variation of μ^x).

Proposition 2. *Suppose the mapping F is linear in u and that*
(A_2) *for each $x \in \Gamma$ and $u \in C(\bar{Q})$ there exists a point $\tilde{x} \in Q$ such that*

$$|F(x, u(\cdot)| \leq |u(\tilde{x})|\,. \tag{4'}$$

Then the problem (1), (2) *and* (3) *has at most one solution in $C^2(Q) \cap C(\bar{Q})$.*

The proof is similar to that of Proposition 1 and is therefore omitted. The mapping (5) fulfills condition (A_2) provided $\operatorname{supp} \mu^x \subset Q$ and $|\mu^x| \leq 1$ for each $x \in \Gamma$.

We now establish the following variant of the maximum principle.

Proposition 3. *Suppose that*

(a) $-1 \leq F(x, 1)$ *for every $x \in \Gamma$,*

(b) *for every point $x^0 \in \Gamma$ such that $F(x^0, 1) > -1$, $F(x^0, \cdot)$ is decreasing and $F(x^0, l) = l\, F(x^0, l)$ for every constant l,*

(c) *for every point $x^0 \in \Gamma$ such that $F(x^0, 1) = -1$ and every $u \in C(\bar{Q})$ there exists a point $\tilde{x} \in Q$ such that*

$$-F(x^0, u(\cdot)) \leq u(\tilde{x})\,.$$

Let $u \in C^2(Q) \cap C(\bar{Q})$. If $Lu \geq 0$ (≤ 0) in Q, $u(x) \leq 0$ (≥ 0) on $\bar{\Gamma}_0$ and $u(x) + F(x, u(\cdot)) \leq 0$ (≥ 0) on $\bar{\Gamma}$, then $u(x) \leq 0$ (≥ 0) on $\bar{Q}$.

Proof. It suffices to prove the first part of the theorem. We may assume that there exists $x^0 \in \Gamma$ such that $0 < u(x^0) = \max_{\bar{Q}} u$. Now we distinguish two cases

$$F(x^0, 1) > -1 \quad \text{and} \quad F(x_0, 1) = -1\,.$$

In the first case it follows from (b) that

$$u(x^0) + F(x^0, 1)u(x^0) \leq u(x^0) + F(x^0, u) \leq 0$$

and consequently $u(x^0) \leq 0$ and we get a contradiction. In the second case according to (c) there exists a point $x^1 \in Q$ such that

$$u(x^0) \leq -F(x^0, u) \leq u(x^1)$$

and u takes on a positive maximum at $x^1 \in Q$ and we get a contradiction.

2. Apriori bounds

Proposition 3 can be employed to derive apriori estimates for solutions of the problem (1), (2) and (3).

Theorem 1. *Let $c(x) \leq -d$ on Q for some positive constant d and suppose that for each $x \in \Gamma$, $F(x, u)$ is linear in u and that there exists a constant $0 < \delta < 1$ such that $F(x, 1) \geq -\delta$ for all $x \in \Gamma$. If $u \in C^2(Q) \cap C(\bar{Q})$ is a solution of the problem* (1), (2) *and* (3), *then*

$$|u(x)| \leq \sup_{\Gamma} |\Psi| + \frac{1}{d} \sup_{Q} |f| + \frac{1}{1-\delta} \sup_{\Gamma_0} |\phi| \tag{7}$$

for all $x \in \bar{Q}$.

Proof. We may assume that the right side of (7) is finite since otherwise there is nothing to prove. Let

$$w(x) = u(x) + \frac{M}{d} + M_1 + \frac{M_2}{1-\delta} \qquad \text{for } x \in Q,$$

where $M = \sup_Q |f|$, $M_1 = \sup_{\Gamma_0} |\phi|$ and $M_2 = \sup_\Gamma |\Psi|$. Then

$$L(w) = f + \frac{cM}{d} + cM_1 + \frac{cM_2}{1-\delta} \leq f - M \leq 0 \qquad \text{on } Q,$$

$$w = \phi + \frac{M}{d} + M_1 + \frac{M_2}{1-\delta} \geq \phi + M_1 \geq 0 \qquad \text{on } \Gamma_0,$$

and

$$\begin{aligned} w + F(x, w) &= \Psi + \left(\frac{M}{d} + M_1 + \frac{M_2}{1-\delta}\right) + F(x, 1)\left(\frac{M}{d} + M_1 + \frac{M_2}{1-\delta}\right) \\ &\geq -M_2 + \left(\frac{M}{d} + M_1 + \frac{M_2}{1-\delta}\right) - \delta\left(\frac{M}{d} + M_1 + \frac{M_2}{1-\delta}\right) \\ &\geq 0 \quad \text{on } \Gamma. \end{aligned}$$

Proposition 3 implies that $w \geq 0$ on Q. Similarly we can establish the inequality

$$u(x) \leq \frac{M}{d} + M_1 + \frac{M_2}{1-\delta} \qquad \text{on } Q$$

considering the auxiliary function

$$z(x) = u(x) - \frac{M}{d} - M_1 - \frac{M_2}{1-\delta} \qquad \text{on } Q\,.$$

Theorem 1 can not be applied to solutions of non-local problems for the Laplace equation. One can establish some apriori estimates for a class of elliptic equations which also cover the Laplace equation provided the domain Q is cylindrical.

Let $Q = \Omega \times (0, T)$, where Ω is a bounded domain $\boldsymbol{R}^{n-1}$ and $T < \infty$. A typical point of Q is denoted by $x = (x', x_n)$ with $x' \in \Omega$ and $0 < x_n < T$. Using notations of Section 1 we put

$$\Gamma = \Omega \times \{x_n = 0\}\,, \quad \gamma = \partial\Omega \times \{x_n = 0\} \quad \text{and} \quad \Gamma_0 = \partial Q - \gamma - \Gamma\,.$$

For the sake of simplicity we only consider the following non-local problem

$$(2') \qquad u(x) = \phi(x) \qquad \text{on } \bar{\Gamma}_0\,,$$

$$(3') \qquad u(x', 0) + \sum_{i=1}^{\infty} \beta_i(x') \int_\Omega u(y', T_i)\mu_i(dy') = \Psi(x') \qquad \text{on } \Omega\,,$$

where μ_i are non-negative Borel measures on Ω, $T_i \in (0, T)$ $(i = 1, 2, \dots)$ with $\inf_i T_i > 0$.

Theorem 2. *Let* $c(x) \le 0$ *and* $b_n(x) \le 0$ *on* Q. *Suppose that* $-1 \le \sum_{i=1}^{\infty} \beta_i(x') \le 0$ *and* $\beta_i(x') \le 0$ $(i = 1, 2, \dots)$ *on* Ω *and that* μ_i *are non-negative Borel measures on* Ω *with* $\mu_i(\Omega) \le 1$ $(i = 1, 2, \dots)$. *If* $u \in C^2(Q) \cap C(\bar{Q})$ *is a solution of the problem* (1), (2′) *and* (3′), *then*

$$|u(x)| \le C \max\,(\sup_{\Gamma_0} |\phi|, \sup_\Omega |\Psi|, \sup_Q |f|)$$

in Q, *where* C *is a positive constant depending on* T.

Proof. We first prove the theorem under the additional hypothesis

$$-1 < -\lambda < \sum_{i=1}^{\infty} \beta_i(x') \le 0 \qquad \text{on } \Omega\,,$$

where λ is a constant. We may also assume that $M = \max\,(\sup_{\Gamma_0} |\phi|, \sup_\Omega |\Psi|) < \infty$ and $M_1 = \sup_\Omega |f| < \infty$. Set

$$v(x) = u(x) - \frac{M}{1-\lambda} - M_1(e^{\alpha T} - e^{\alpha x_n})\,,$$

where a constant $\alpha > 0$ is to be determined. Then

$$Lv = f - \frac{cM}{1-\lambda} + M_1\alpha^2 e^{\alpha x_n} a_{nn} + \alpha M_1 e^{\alpha x_n} b_n - M_1(e^{\alpha T} - e^{\alpha x_n})c$$

Choosing α sufficiently large we may assume that $\alpha^2 a_{nn} + \alpha b_n \geq 1$ on Q and consequently

$$Lv \geq 0 \qquad \text{on } Q\,.$$

On the other hand $v \leq 0$ on $\bar{\Gamma}_0$ and

$$\begin{aligned} v(x', 0) &+ \sum_{i=1}^{\infty} \beta_i(x') \int_{\Omega} v(z', T_i)\mu_i(dz') \\ &= \Psi - \frac{M}{1-\lambda} - M_1(e^{\alpha T} - 1) - \frac{M}{1-\lambda} \sum_{i=1}^{\infty} \beta_i(x')\mu_i(\Omega) \\ &\quad - M_1 \sum_{i=1}^{\infty} \beta_i(x')\mu_i(\Omega)(e^{\alpha T} - e^{\alpha T_i}) \\ &\leq M - \frac{M}{1-\lambda} - M_1(e^{\alpha T} - 1) + \frac{M\lambda}{1-\lambda} + \lambda M_1(e^{\alpha T} - 1) \\ &= M_1(\lambda - 1)(e^{\alpha T} - 1) \leq 0 \qquad \text{on } \Omega \end{aligned}$$

and hence by Proposition 3 $v \leq 0$ on Q. Introducing the auxiliary function

$$w = u + \frac{M}{1-\lambda} + M_1(e^{\alpha T} - e^{\alpha x_n})$$

we prove the inequality

$$u(x) \geq -\frac{M}{1-\gamma} - M_1(e^{\alpha T} - e^{\alpha x_n}) \qquad \text{on } Q\,.$$

The general case can be reduced to the previous situation by means of the transformation

$$u(x) = v(x) \cos \delta x_n$$

where $\delta > 0$ is a constant such that

$$\delta \tan \delta T \leq 1 \quad \text{and} \quad \delta T \leq \pi/2 - \varepsilon$$

for a certain $\varepsilon > 0$. Indeed, let

$$\begin{aligned} L_1 v \equiv \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} &+ \sum_{i=1}^{n} (b_i - 2a_{in} \tan \delta x_n) \frac{\partial v}{\partial x_i} \\ &+ (c - \delta^2 a_{nn} - \delta b_n \tan \delta x_n) v \qquad \text{in } Q\,. \end{aligned}$$

Then v is a solution to the problem

$$L_1 v = \frac{f}{\cos \delta x_n} \qquad \text{on } Q\,,$$

$$v = \frac{\phi}{\cos \delta x_n} \qquad \text{on } \bar{\Gamma}_0\,,$$

and

$$v(x', 0) + \sum_{i=1}^{\infty} \beta_i(x') \cos \delta T_i \int_{\Omega} u(z', T_i)\mu_i(dz) = \Psi(x') \qquad \text{on } \bar{\Omega}\,.$$

Since $0 < \cos \delta T_i < \cos \delta T_0 < 1$ $(i = 1, 2, \dots)$, where $T_0 = \inf_i T_i$, we see that

$$-1 < \cos \delta T_0 \le \sum_{i=1}^{\infty} \beta_i(x') \cos \delta T_i \le 0\,,$$

and by the previous part of the proof we have

$$|v(x)| \le \frac{\bar{M}}{1 - \cos \delta T_0} + \bar{M}_1(e^{\alpha T} - e^{\alpha x_n}) \qquad \text{in } Q$$

for sufficiently large α, where

$$\bar{M} = \sup_Q \frac{|f|}{\cos \delta x_n} \quad \text{and} \quad \bar{M}_1 = \max\left[\sup_{\Gamma_0} \frac{|\phi|}{\cos \delta x_n}, \sup_{\Omega} |\Psi|\right]$$

and the result easily follows.

Theorem 2 is related to Theorem 2.2 in [7], where a similar apriori estimate has been obtained under the assumption $c(x) \le -d$ on Q, $d > 0$.

3. Existence theorem

For the existence theorem we shall need the following assumption

(B) $\dfrac{\partial^2 a_{ij}}{\partial x_i \partial x_j}$ $(i, j = 1, \dots, n)$, $\dfrac{\partial b_i}{\partial x_i}$ $(i = 1, \dots, n)$ and c are Hölder continuous on $\bar{Q}$ (with exponent α).

Moreover we assume that $\partial Q \in C^{2+\alpha}$. We only consider the non-local problem where the functional F is given by

$$F(x, u(\cdot)) = \int_Q \beta(x, y)u(y)dy\,,$$

that is (3) takes the form

(3′) $$u(x) + \int_Q \beta(x, y)u(y)dy = \Psi(x) \qquad \text{on } \bar{\Gamma}\,.$$

We now make the following assumption:
(C) The function $\beta(x, y)$ is continuous on $\bar{\Gamma} \times Q$, $|\beta(x, y)| \leq 1$ on $\bar{\Gamma} \times \bar{Q}$, for each $x \in \Gamma$ supp $\beta(x, \cdot) \subset Q$ and $\beta(x, y) = 0$ for all $(x, y) \in \gamma \times \bar{Q}$.

Theorem 3. *Suppose that* (*A*), (*B*) *and* (*C*) *hold. Let* $c(x) \leq -d$ *on* Q, *where* $d > 0$ *is a constant. If* ϕ *and* Ψ *are continuous functions on* $\bar{\Gamma}_0$ *and* $\bar{\Gamma}$, *respectively, with* $\phi = \Psi$ *on* γ, *then the problem* (1), (2) *and* (3′) *admits a unique solution in* $C^2(Q) \cap C(\bar{Q})$.

Proof. We first assume that $\phi = 0$ on $\bar{\Gamma}_0$. We try to find a solution in the form

$$u(x) = \int_{\partial Q} \frac{dG(x, y)}{d\nu_y} v(y) dS_y - \int_Q G(x, y) f(y) dy, \tag{7}$$

where $v \in C(\partial Q)$ with supp $v \subset \bar{\Gamma}_0$ is to be determined, G is the Green function for the operator L and $dG/d\nu_y$ denotes the conormal derivative. Set $K(x, y) = dG(x, y)/d\nu_y$, then the condition (3′) leads to the Fredholm intergal equation of the second kind

$$\begin{aligned} v(x) + \int_{\partial Q} \left[\int_Q \beta(x, y) K(y, z) dy \right] v(z) dS_z \\ = \Psi(x) + \int_Q \beta(x, y) \left[\int_Q G(y, z) f(z) dz \right] dy \end{aligned} \tag{8}$$

for $x \in \Gamma$. Applying Proposition 2 it is easy to show that the corresponding homogeneous equation only has a trivial solution in $L^2(\Gamma)$. Hence there exists a unique solution v in $L^2(\Gamma)$ of the equation (8). It is clear that every solution v of (8) is continuous on $\bar{\Gamma}$ and vanishes on γ. Extending v by 0 on $\partial Q - \bar{\Gamma}$ we obtain a continuous function v on ∂Q such that the formula (7) gives a solution in this case.

Suppose next $\phi \not\equiv 0$ on $\bar{\Gamma}_0$. Let Φ be a continuous extension of ϕ to ∂Q and let $w \in C^2(Q) \cap C(\bar{Q})$ be a solution to the Dirichlet problem

$$Lw = f \quad \text{on } Q,$$
$$w = \Phi \quad \text{on } \partial Q.$$

We now consider the following non-local problem

$$Lz = 0 \quad \text{in } Q,$$
$$z = 0 \quad \text{on } \bar{\Gamma}_0$$

and

$$z(x) + \int_Q \beta(x, y) z(y) dy = \Psi(x) - \int_Q \beta(x, y) w(y) dy - \Phi(x) \qquad \text{on } \bar{\Gamma}.$$

Observe that $\Psi - \Phi = \Psi - \phi = 0$ on γ. Therefore by the previous part of the proof this problem has a unique solution $z \in C^2(Q) \cap C(\bar{Q})$. It is easily checked that $z + w$ is a solution of the problem (1), (2) and (3').

We point out here that the existence result in [11] (Theorem 1) corresponds to the situation where $|\beta(x, y)| \leq r$ on $\bar{\Gamma} \times \bar{Q}$ with $0 < r < 1$.

5. A class of non-local problems in a half-space

For a point $x \in \boldsymbol{R}^n$ we write $x = (x', x_n)$, where $x' \in \boldsymbol{R}^{n-1}$, $x_n \in (-\infty, \infty)$. Let $\boldsymbol{R}^n_+ = \{x; x' \in \boldsymbol{R}^{n-1}, x_n > 0\}$. We restrict ourselves to the following class of non-local problems for the Laplace equation

$$\Delta u = f(x) \quad \text{in } \boldsymbol{R}^n_+ , \tag{9}$$

$$u(x', 0) + \sum_{i=1}^{\infty} \beta_i(x') u(x', Y_i) = \Psi(x') \qquad \text{on } \boldsymbol{R}^{n-1} , \tag{10}$$

where f, β_i $(i = 1, 2, \dots)$ and Ψ are given functions on $\boldsymbol{R}^n_+$, $\boldsymbol{R}^{n-1}$ and $\boldsymbol{R}^{n-1}$, respectively.

We commence with the uniqueness result for (9), (10) in a class P of functions of polynomial growth in x', that is, $P = \{u; |u(x)| \leq C(|x'|^m + 1)$ on $\boldsymbol{R}^n_+$ for some positive constants m and $C\}$.

Theorem 4. *Let* $\inf_i Y_i > 0$ *and* $\sup_i Y_i = Y < \infty$. *Suppose that* $-\dfrac{1}{1+Y} \leq \sum_{i=1}^{\infty} \beta_i(x') \leq 0$ *and* $\beta_i(x') \leq 0$ $(i = 1, 2, \dots)$ *on* $\boldsymbol{R}^{n-1}$. *Then there exists at most one solution* $u \in C^2(\boldsymbol{R}^n_+) \cap C(\overline{\boldsymbol{R}^n_+}) \cap P$.

Proof. We follow the argument used in the proof of Proposition 1 in [10] (p. 199). Let u be a solution in $C^2(\boldsymbol{R}^n_+) \cap C(\overline{\boldsymbol{R}^n_+}) \cap P$ to the homogeneous problem

$$\Delta u = 0 \qquad \text{in } \boldsymbol{R}^n_+ ,$$

$$u(x', 0) + \sum_{i=1}^{\infty} \beta_i(x') u(x', Y_i) = 0 \qquad \text{on } \boldsymbol{R}^{n-1} .$$

Let $\bar{x}'$ be an arbitrary point in $\boldsymbol{R}^{n-1}$ and for a positive number R we set

$$M = M(R) = \sup_{(|x' - \bar{x}'| < R) \times [0, \infty)} |u(x', x_n)| .$$

For fixed $\varepsilon > 0$ we introduce the auxiliary function v defined by

$$v(x) = u(x) - \varepsilon(x_n + 1)M - \varepsilon \prod_{i=1}^{n-1} \cosh \frac{\varepsilon(x_i - \bar{x}_i)\pi}{4\sqrt{n}} \cos \frac{\varepsilon\pi}{4} x_n ,$$

which is clearly harmonic in $\boldsymbol{R}_+^n$ and continuous on $\overline{\boldsymbol{R}_+^n}$. Let us consider the cylinder $\{x; |x' - \bar{x}'| < R, 0 < x_n < 1/\varepsilon\}$. It is clear that

$$v\left(x', \frac{1}{\varepsilon}\right) = u\left(x', \frac{1}{\varepsilon}\right) - M - \varepsilon M - \varepsilon \prod_{i=1}^{n-1} \cosh \frac{\varepsilon(x_i - \bar{x}_i)\pi}{4\sqrt{n}} \cos \frac{\pi}{4} \leq 0$$

on $|x' - \bar{x}'| < R$ and

$$v(x', x_n) \leq 0 \qquad \text{on } (|x' - \bar{x}'| = R) \times \left[0, \frac{1}{\varepsilon}\right)$$

provided R is sufficiently large. Now

$$\begin{aligned} v(x', 0) &+ \sum_{i=1}^{\infty} \beta_i(x') v(x', Y_i) \\ &= -\varepsilon M - \varepsilon \prod_{i=1}^{n-1} \cosh \frac{\varepsilon(x_i - \bar{x}_i)\pi}{4\sqrt{n}} \\ &\quad - \sum_{i=1}^{\infty} \beta_i(x')(Y_i + 1)M\varepsilon - \sum_{i=1}^{\infty} \beta_i(x')\varepsilon \prod_{i=1}^{n-1} \cosh \frac{\varepsilon(x_i - \bar{x}_i)}{4\sqrt{n}} \cos \frac{\varepsilon\pi}{4} Y_i \\ &\leq -\varepsilon M - \varepsilon \prod_{i=1}^{n-1} \cosh \frac{\varepsilon(x_i - \bar{x}_i)\pi}{4\sqrt{n}} + M\varepsilon + \frac{\varepsilon}{Y+1} \prod_{i=1}^{n} \cosh \frac{\varepsilon(x_i - \bar{x}_i)\pi}{4\sqrt{n}} \leq 0 \end{aligned}$$

on $|x' - \bar{x}'| < R$. Consequently by Proposition 3 we obtain

$$u(x) \leq \varepsilon(x_n + 1)M + \varepsilon \prod_{i=1}^{n-1} \cosh \frac{\varepsilon(x_i - \bar{x}_i)\pi}{4\sqrt{n}} \cos \varepsilon \frac{\pi}{4} x_n$$

on $(|x' - \bar{x}'| < R) \times (0, 1/\varepsilon)$. Similarly we establish the inequality

$$u(x) \geq -\varepsilon(x_n + 1)M - \varepsilon \prod_{i=1}^{n-1} \cosh \frac{\varepsilon(x_i - \bar{x}_i)}{4\sqrt{n}} \cos \frac{\varepsilon\pi}{4} x_n$$

on $(|x' - \bar{x}'| < R) \times (0, 1/\varepsilon)$. Letting $\varepsilon \to 0$ we get $u(\bar{x}', x_n) = 0$ for all $x_n \geq 0$. Since $\bar{x}'$ is an arbitrary point in $\boldsymbol{R}^{n-1}$ the result follows.

In our final result, for the sake of simplicity, we only consider the homogeneous equation (9).

Theorem 5. *Assume that* $-1 < \gamma \leq \sum_{i=1}^{\infty} \beta_i(x') \leq 0$ *on* $\boldsymbol{R}^{n-1}$, *where* γ *is a positive constant and that* $\beta_i(x') \leq 0$ $(i = 1, 2, \dots)$ *on* $\boldsymbol{R}^{n-1}$. *Furthermore suppose that the series* $\sum_{i=1}^{\infty} \beta_i(x')$ *is uniformly convergent on* $\boldsymbol{R}^{n-1}$. *Then for every continuous and bounded function* Ψ *on* $\boldsymbol{R}^{n-1}$ *there exists a unique bounded solution in* $C^2(\boldsymbol{R}_+^n) \cap (\overline{\boldsymbol{R}_+^n})$ *of the problem* (8), (9) (*with* $f \equiv 0$).

Proof. It is easy to show that every bounded solution to our problem satisfies the integral equation

$$u(x) = -\int_{R^{n-1}} \sum_{i=1}^{\infty} \beta_i(z')u(z', Y_i)P_{x_n}(x' - z')dz' + \int_{R^{n-1}} P_{x_n}(x' - z')\Psi(z')dz',$$

where

$$P_{x_n}(x') = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} \frac{x_n}{(|x'|^2 + x_n^2)^{(n+1)/2}}$$

(see [10] p. 61–62). Let $C_b(\overline{R^n_+})$ denote a space of bounded continuous functions on $\overline{R^n_+}$ equipped with the supremum norm. To every $v \in C_b(\overline{R^n_+})$ we assign a function $u = Tv \in C_b(\overline{R^n_+})$ given by the formula

$$u(x) = \int_{R^{n-1}} - \sum_{i=1}^{\infty} \beta_i(z')v(z', Y_i)P_{x_n}(x' - z')dz'$$

$$+ \int_{R^{n-1}} P_{x_n}(x' - z')\Psi(z')dz'.$$

It is clear that u is harmonic on R^n_+, continuous on $\overline{R^n_+}$ and

$$u(x', 0) = -\sum_{i=1}^{\infty} \beta_i(x')v(x', Y_i) + \Psi(x') \qquad \text{on } R^{n-1}.$$

Since T is a contraction on $C_b(\overline{R^n_+})$ the result follows from the Banach fixed point theorem.

References

[1] Bicadze, A. V. and Samarski, A. A., On some simple generalizations of linear elliptic boundary problems, Soviet Math. Dokl., **10** (1969), 398–400.

[2] Chabrowski, J., On non-local problems for parabolic equations, Nagoya Math. J., **93** (1984), 109–131.

[3] Chabrowski, J., On the non-local problem with a functional for parabolic equation, Funkcial. Ekvac., **27** (1984), 101–123.

[4] Chrzanowski, E. M., Nonlocal nonlinear problems for one-dimensional parabolic system, Zeitschrift für Analysis and ihre Anwendungen, **3** (1984), 329–336.

[5] Gilbarg, P. and Trudinger, N. S., *Elliptic partial differential equations of second order*, Grundlehrnen der mathematischen Wissenschaften 224, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983 (Second Edition).

[6] Kerefov, A. A., Non-local boundary value problems for parabolic equation, Differential Equations, **15** (1979), 52–55.

[7] Tsai, Long-yi, Non-local elliptic problems, preprint.

[8] Majchrowski, M., On non-local problems for the parabolic system of partial differential equations, J. Integral Equations, **4** (1982), 339–351.

[9] Paneyakh, B. P., Certain nonlocal boundary value problems for linear differential operators, Math. Notes, **35** (1984), 223–229.

[10] Stein, E. M., *Singular integrals and differentiability properties of functions,* Princeton University Press, Princeton, New Jersey, 1970.

[11] Vabishchevich, P. N., Non-local parabolic problems and the inverse heat-conduction problem, Differential Equations, **17** (1981), 761–765.

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