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Remarks on Characteristics of Partial Differential Equations of First Order

By

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§1. Introduction

We consider the Cauchy problem for partial differential equations of first order as follows:

$$\frac{\partial u}{\partial t}+f\left(t, x, u, \frac{\partial u}{\partial x}\right)=0 \quad \text{in} \quad \{t>0, x\in \boldsymbol{R}^1\}, \tag{1.1}$$

$$u(0, x)=\phi(x) \quad \text{on} \quad \{t=0, x\in \boldsymbol{R}^1\} \tag{1.2}$$

where f and ϕ are two times differentiable. Let $x=x(t, y)$ be a characteristic curve for (1.1)–(1.2) passing a point $(t, x)=(0, y)$. The definition of $x=x(t, y)$ will be given by (3.3)–(3.4) and (4.3)–(4.4). Assume I) $\partial x/\partial y(t^0, y^0)=0$ and II) $\partial x/\partial y(t, y)\neq 0$ for any $t\in[0, t^0)$ and $y\in \boldsymbol{R}$. Then we have proved in [2] that, when (t, x) tends to (t^0, x^0) where $x^0=x(t^0, y^0)$, first or second derivatives of classical solutions can not remain bounded. By this fact, we treat a weak solution for (1.1) after the time t^0. The most typical singularity of weak solution is shock, and it appears by the collision of characteristics. Therefore we discuss whether the characteristic curves meet or not for $t>t^0$, i.e., whether or not there exist points y_1 and y_2 $(y_1\neq y_2)$ satisfying $x(t, y_1)=x(t, y_2)$ for $t>t^0$. In §2, we give an example of an equation whose characteristic curves do not meet for any $t>t^0$. In §3 and §4, we will give sufficient conditions so that the characteristic curves collide after the time t^0.

§2. Example

We consider the Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}+a(t, u)\dfrac{\partial u}{\partial x}=u \quad \text{in} \quad \{t>0, x\in \boldsymbol{R}^1\}, \\ u(0, x)=x \quad \text{on} \quad \{t=0, x\in \boldsymbol{R}^1\} \end{cases} \tag{2.1}$$

where $a(t, u)=\alpha'(t)\exp(-t)u+\beta'(t)\exp(-3t)u^3$ and the functions $\{\alpha(t), \beta(t)\}$ satisfy the following conditions:

1) $\alpha(t)$ and $\beta(t)$ are in $C^1(\boldsymbol{R}^1)$,
2) $\alpha(0)=1$, $\alpha(t)\geqq 0$ and $\alpha(t)=0$ for any $t\geqq K=\text{constant}>0$,
3) $\beta(0)=0$, $\beta(t)\geqq 0$ and $\beta(t)\neq 0$ for any $t\geqq K$,
4) $\alpha(t)+\beta(t)\neq 0$ for any $t\in\boldsymbol{R}^1$.

Then the characteristic curves for (2.1) are written by

$$x=x(t,y)=\alpha(t)y+\beta(t)y^3\,, \tag{2.2}$$

and the value $v(t,y)$ of solution of (2.1) restricted on the curve (2.2) is given by $v(t,y)=\exp(t)\,y$. Then it follows

$$\frac{\partial x}{\partial y}(t,y)=\alpha(t)+3\beta(t)y^2 \quad\text{and}\quad \frac{\partial x}{\partial y}(t,0)=0 \quad \text{for any } t\geqq K\,.$$

But one can easily see that the characteristic curves (2.2) do not meet for any $t\in\boldsymbol{R}^1$. In this case the solution $u(t,x)$ is represented as

$$u(t,x)=\exp(t)\beta(t)^{-1/3}x^{1/3} \quad\text{for}\quad t\geqq K\,.$$

This representation says that the solution contains algebraic singularity at $x=0$, and that the singularity of shock type does not appear though the Jacobian vanishes.

§3. Quasilinear equation

In this section we treat the quasilinear equations of first order as follows:

$$\frac{\partial u}{\partial t}+a_1(t,x,u)\frac{\partial u}{\partial x}=a_0(t,x,u) \quad\text{in}\quad \{t>0,\,x\in\boldsymbol{R}^1\}\,, \tag{3.1}$$

$$u(0,x)=\phi(x) \quad\text{on}\quad \{t=0,\,x\in\boldsymbol{R}^1\} \tag{3.2}$$

where $a_i(t,x,u)\in C^1(\boldsymbol{R}^3)$ and $\phi(x)\in C^1(\boldsymbol{R}^1)$. The characteristic curves for (3.1)–(3.2) are defined as solution curves of the following system of equations:

$$\frac{dx}{dt}=a_1(t,x,v)\,,\qquad \frac{dv}{dt}=a_0(t,x,v) \tag{3.3}$$

with the initial data

$$x(0)=y\,,\qquad v(0)=\phi(y)\,. \tag{3.4}$$

As in [2], we assume the condition:

(H) For any $y\in\boldsymbol{R}^1$, the Cauchy problem (3.3)–(3.4) always has a unique global C^1-solution $x=x(t,y)$ and $v=v(t,y)$ on $t\geqq 0$.

B. Doubnov [1] gives the sufficient conditions to guarantee the above condition (H). The assumption (H) assures that $x = x(t, y)$ defines a C^1-mapping from $\boldsymbol{R}^1$ to $\boldsymbol{R}^1$ depending on a parameter t. Our aim of this section is to look for the condition so that the characteristic curves meet after the Jacobian vanishes.

Theorem 1. *Under the condition* (H), *we assume* (I) $\partial x/\partial y(t^0, y^0) = 0$ *and* (II) $\partial x/\partial y(t, y) \neq 0$ *for any* $t \in [0, t^0)$ *and* $y \in \boldsymbol{R}^1$. *If* $\partial a_1/\partial u(t^0, x^0, v^0) \neq 0$ *where* $x^0 = x(t^0, y^0)$ *and* $v^0 = v(t^0, y^0)$, *then the characteristic curves meet for* $t > t^0$.

Proof. This theorem follows immediately if the monotonicity of $x = x(t, y)$ with respect to y is violated. At first we prove that $\partial x/\partial y(t, y^0)$ changes the sign at $t = t^0$. As $\partial x/\partial y(0, y) = 1$, we have $\partial x/\partial y(t, y) > 0$ for any $t < t^0$ and $y \in \boldsymbol{R}^1$. Therefore we show that $\partial x/\partial y(t, y^0) < 0$ for $t > t^0$ where $t - t^0$ is small. Differentiating (3.3) with respect to y, we get

$$(3.5)\qquad \begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial x}{\partial y}(t, y)\right) = \dfrac{\partial a_1}{\partial x}(t, x, v)\dfrac{\partial x}{\partial y} + \dfrac{\partial a_1}{\partial v}(t, x, v)\dfrac{\partial v}{\partial y} \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial v}{\partial y}(t, y)\right) = \dfrac{\partial a_0}{\partial x}(t, x, v)\dfrac{\partial x}{\partial y} + \dfrac{\partial a_0}{\partial v}(t, x, v)\dfrac{\partial v}{\partial y} \end{cases}$$

$$(3.6)\qquad \frac{\partial x}{\partial y}(0) = 1\,, \qquad \frac{\partial v}{\partial y}(0) = \phi'(y)\,.$$

As (3.5) is linear with respect to $\partial x/\partial y$ and $\partial v/\partial y$, the initial condition (3.6) guarantees

$$\left(\frac{\partial x}{\partial y}(t, y), \frac{\partial v}{\partial y}(t, y)\right) \neq (0, 0) \quad \text{for any} \quad t \geqq 0 \quad \text{and} \quad y \in \boldsymbol{R}^1\,.$$

Hence we have $\partial v/\partial y(t^0, y^0) \neq 0$. By the assumption, we get

$$\frac{d}{dt}\left(\frac{\partial x}{\partial y}\right)\bigg|_{(t,y)=(t^0,y^0)} = \frac{\partial a_1}{\partial v}(t^0, x^0, v^0)\frac{\partial v}{\partial y}(t^0, y^0) \neq 0\,.$$

Moreover, as $\partial x/\partial y(t^0, y^0) = 0$ and $\partial x/\partial y(t, y^0) > 0$ for $t < t^0$, we have

$$\frac{d}{dt}\left(\frac{\partial x}{\partial y}\right)\bigg|_{(t,y)=(t^0,y^0)} \leqq 0\,, \quad \text{i.e.,} \quad \frac{d}{dt}\left(\frac{\partial x}{\partial y}\right)\bigg|_{(t,y)=(t^0,y^0)} < 0\,.$$

Therefore we get $\partial x/\partial y(t, y^0) < 0$ for $t > t^0$ where $t - t^0$ is small.

On the other hand, there exists a point $y = y^1$ satisfying $\partial x/\partial y(t, y^1) > 0$ for $t > t^0$. The proof is as follows. The function $h(y)$ is defined by $h(y) = \inf\{t; \partial x/\partial y(t, y) = 0\}$ for each y. If $\partial x/\partial y(t, y) \neq 0$ for any $t > 0$, then

$h(y) = \infty$. We see that $h(y)$ is lower semi-continuous at $y = y^0$ and $h(y^0) = t^0$. If $a_i(t, x, u)$ $(i = 0, 1)$ and $\phi(x)$ are two times differentiable, then $h(y)$ is differentiable. By the assumption, we have $h(y) \geqq t^0$. If $h(y) = t^0$ in a neighborhood of $y = y^0$, then $\partial x/\partial y(t^0, y) = 0$, i.e., $x(t^0, y) = x(t^0, y)$ in a neighborhood of $y = y^0$. This means that the characteristic curves meet at the point $(t, x) = (t^0, x^0)$. When $h(y)$ is not constant, there exists a point $y = y^1$ such that $h(y^1) = t^1 > h(y^0) = t^0$. From this, we have $\partial x/\partial y(t, y^1) > 0$ for $t \in (t^0, t^1)$. As $\partial x/\partial y(t, y^0) < 0$ for $t > t^0$, $x = x(t, y)$ is not monotone with respect to y for $t > t^0$. Hence the characteristic curves meet after the time t^0. Q.E.D.

§4. General nonlinear equations of first order

Consider the following Cauchy problem:

$$\frac{\partial u}{\partial t} + f\left(t, x, u, \frac{\partial u}{\partial x}\right) = 0 \quad \text{in} \quad \{t > 0, x \in \boldsymbol{R}^1\}, \tag{4.1}$$

$$u(0, x) = \phi(x) \quad \text{on} \quad \{t = 0, x \in \boldsymbol{R}^1\} \tag{4.2}$$

where $f \in C^2(\boldsymbol{R}^2)$ and $\phi \in C^2(\boldsymbol{R}^1)$. The characteristic curves for (4.1)–(4.2) are the solution curves of the following system of equations:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial f}{\partial p}(t, x, v, p) \\ \dfrac{dv}{dt} = p\dfrac{\partial f}{\partial p}(t, x, v, p) - f(t, x, v, p) \\ \dfrac{dp}{dt} = -\dfrac{\partial f}{\partial x}(t, x, v, p) - p\dfrac{\partial f}{\partial v}(t, x, v, p) \end{cases} \tag{4.3}$$

$$x(0) = y, \qquad v(0) = \phi(y), \qquad p(0) = \phi'(y). \tag{4.4}$$

Here we assume the following condition:

(H)′ For any $y \in \boldsymbol{R}^1$, the Cauchy problem (4.3)–(4.4) has a unique global C^1-solution $x = x(t, y)$, $v = v(t, y)$ and $p = p(t, y)$ for $t \geqq 0$.

Then $x = x(t, y)$ defines a C^1-mapping from $\boldsymbol{R}^1$ to $\boldsymbol{R}^1$ depending on a parameter t.

Theorem 2. *Under the condition* (H)′, *we assume* (I) $\partial x/\partial y(t^0, y^0) = 0$ *and* (II) $\partial x/\partial y(t, y) \neq 0$ *for any* $t < t^0$ *and* $y \in \boldsymbol{R}^1$. *If* $\partial^2 f/\partial p^2(t^0, x^0, v^0, p^0) \neq 0$ *where* $x^0 = x(t^0, y^0)$, $v^0 = v(t^0, y^0)$ *and* $p^0 = p(t^0, y^0)$, *then the characteristic curves meet for* $t > t^0$.

Proof. As in the proof of Theorem 1, we show that $x=x(t, y)$ is not monotone with respect to y for $t>t^0$. As $\partial x/\partial y(t, y)>0$ for $t<t^0$ and $y \in \boldsymbol{R}^1$, we can uniquely solve the equation $x=x(t, y)$ with respect to y, and we denote the solution by $y=y(t, x)$. Then the solution of (4.1)–(4.2) is given by $u(t, x)=v(t, y(t, x))$ and

$$\begin{aligned} p(t, y(t, x)) &= \frac{\partial u}{\partial x}(t, x)=\frac{\partial v}{\partial y}(t, y(t, x)) \frac{\partial y}{\partial x}(t, x) \\ &= \left.\frac{\partial v}{\partial y}(t, y)\left(\frac{\partial x}{\partial y}(t, y)\right)^{-1}\right|_{y=y(t, x)} . \end{aligned}$$

When a point (t, x) tends to (t^0, x^0) along a curve $\{(t, x); x=x(t, y^0)\}$, $\partial x/\partial y(t, y(t, x))$ tends to zero and $p(t, y(t, x))$ remains bounded. Therefore it must be $\partial v/\partial y(t^0, y^0)=0$. Differentiating the system (4.3) with respect to y, we get a system of linear ordinary differential equations for $\partial x/\partial y$, $\partial v/\partial y$ and $\partial p/\partial y$ just like (3.5). As the initial data $(\partial x/\partial y, \partial v/\partial y, \partial p/\partial y)(0)=(1, \phi'(y), \phi''(y)) \neq (0,0,0)$, it follows $(\partial x/\partial y, \partial v/\partial y, \partial p/\partial y)(t) \neq (0,0,0)$ for any $t \geqq 0$. Hence $\partial p/\partial y(t^0, y^0) \neq 0$. The function $\partial x/\partial y(t, y)$ satisfies the following differential equation:

$$\frac{d}{dt}\left(\frac{\partial x}{\partial y}\right)=\frac{\partial^2 f}{\partial x \partial p} \frac{\partial x}{\partial y}+\frac{\partial^2 f}{\partial v \partial p} \frac{\partial v}{\partial y}+\frac{\partial^2 f}{\partial p^2} \frac{\partial p}{\partial y} .$$

Using the assumptions and the above results, we get

$$\left.\frac{d}{dt}\left(\frac{\partial x}{\partial y}\right)\right|_{(t, y)=(t^0, y^0)}=\frac{\partial^2 f}{\partial p^2}(t^0, x^0, v^0, p^0) \frac{\partial p}{\partial y}(t^0, y^0) \neq 0 .$$

Moreover, as $\partial x/\partial y(t^0, y^0)=0$ and $\partial x/\partial y(t, y^0)>0$ for $t<t^0$, it must be

$$\left.\frac{d}{dt}\left(\frac{\partial x}{\partial y}\right)\right|_{(t, y)=(t^0, y^0)}<0, \quad \text{i.e.,} \quad \frac{\partial x}{\partial y}(t, y^0)<0 \quad \text{for} \quad t>t^0 .$$

On the other hand, by the same reasoning as the proof of Theorem 1, we can show that there exists a point $y=y^1$ such that $\partial x/\partial y(t, y^1)>0$ for $t>t^0$ where $t-t^0$ is small. Therefore the function $x=x(t, y)$ is not monotone with respect to y, i.e., the characteristic curves meet after the time t^0, Q.E.D.

Remark. When the condition (H)′ is satisfied, the shocks do not appear. In this case the weak solution is continuous and its singularities are similar to those of generalized solutions for Hamilton-Jacobi equation ([3]).

References

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