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Oscillatory Behavior of a Class of Nonlinear Three Term Recurrence Equations

By

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1. Introduction

Sequences defined by three-term recurrence relations have long been studied in many areas of mathematics [1, 6, 7, 15, 16]. In this paper, we are concerned with the oscillatory properties of sequences that are defined by nonlinear three-term recurrence equations of the form

$$c_n f(x_{n+1}, x_n) + c_{n-1} g(x_{n-1}, x_n) = b_n x_n, \quad n = 1, 2, 3, \cdots \tag{1.1}$$

where $c_n > 0$ for $n = 0, 1, 2, \cdots$, $f(u, v)$ is a QL function and $g(u, v)$ is a WQL function. Here a function $h(u, v)$ is called a QL function (WQL function) if it satisfies the following conditions:

(i) $h(\alpha u, \alpha v) = \alpha h(u, v)$ for every α, u, v;

(ii) $\operatorname{sign} h(u, v) = \operatorname{sign} u$;

(iii) for any $v = v_0$, the range of $h(u, v_0)$ is $\boldsymbol{R}$; and

(iv) $h(u, v)$ is strictly increasing (respectively increasing) in u.

An immediate example of a QL function is the linear function $h(u, v) = u$. A less trivial QL function is a special class of Cobb-Douglas production functions [4, pp. 414–416] widely used in economic analysis:

$$h(u, v) = \begin{cases} u & \text{if } v = 0 \\ u^r r^s & \text{if } v \neq 0 \end{cases} \tag{1.2}$$

where $r + s = 1$, and r is a quotient of positive odd integers.

To see a WQL function, we cite the following

$$h(u, v) = \begin{cases} u & \text{if } v = 0 \\ u & \text{if } v \neq 0, \quad 0 \leq |u| \leq |v| \\ |v| \operatorname{sign} u & \text{if } v \neq 0, \quad |v| \leq |u| \leq 2|v| \\ (|u| - |v|) \operatorname{sign} u & \text{if } v \neq 0, \quad |u| \geq 2|v|. \end{cases} \tag{1.3}$$

A solution of (1.1) is a sequence $\{x_n\} = \{x_n\}_{n=0}^{\infty}$ which satisfies (1.1). Equation

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(1.1) has many properties in common with the linear equation [5, 9, 10, 12, 13]

$$(1.4) \qquad c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \quad c_{n-1} > 0, \quad n = 1, 2, \cdots.$$

In particular, we can easily verify by direct substitution that if $\{x_n\}$ is a solution of (1.1), so is any of its constant multiple. Also given x_i and x_{i+1} for $i \geq 0$, it is clear that x_{i+2} is uniquely determined by the recurrence relation (1.1), while x_{i-1} can also be determined but it may not be unique unless $g(u, v)$ is also a QL function.

Theorem 1.1. *Given a nonnegative integer* i *and real numbers* α, β, *a solution* $\{x_n\}$ *of* (1.1) *exists satisfying* $x_i = \alpha$ *and* $x_{i+1} = \beta$. *If* $\{y_n\}$ *is another solution of* (1.1) *satisfying* $y_i = \alpha$ *and* $y_{i+1} = \beta$, *then* $y_k = x_k$ *for* $k \geq i$. *Moreover,* $y_k = x_k$ *for* $k < i$ *provided* $g(u, v)$ *is a QL function.*

Other properties of (1.1) similar to those of (1.4) are related to oscillatory behavior of its solutions and will be discussed in later sections. For related studies which provide background material and motivation for the present paper, we refer the readers to our references [1, 2, 3, 6, 9, 10, 14].

2. Preliminaries

We first introduce the concept of a node. Let $\{z_k\}_0^\infty$ be a sequence. If the points (k, z_k), $k \geq 0$, are joined by straight line segments to form a broken line, then this broken line represents a piecewise continuous function $z^*(t)$ called the linear spline of $\{z_k\}$. The zeros of $z^*(t)$ are called the nodes of $\{z_k\}$.

Lemma 2.1. *If* $g(u, v)$ *is QL, then two linearly independent solutions of* (1.1) *cannot have a common node.*

This follows easily from the existence and uniqueness Theorem 1.1 mentioned in the last section.

We remark that Lemma 2.1 is not true in general if $g(u, v)$ is not QL. For instance, the equation

$$x_{n+1} + h(x_{n-1}, x_n) = x_n, \quad n = 1, 2, 3, \cdots$$

where $h(u, v)$ is given by (1.3) has two linearly independent solutions $\{1, 1, 0, \cdots\}$ and $\{2, 1, 0, \cdots\}$ which have a common node 2.

Lemma 2.2. ([5, Lemma 1]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences such that* $x_{i+1} > 0$, $x_i \leq 0$ *and* $y_{i+1} > 0$ *for some* $i \geq 0$. *If* $x_{n+1} y_n - x_n y_{n+1} \leq 0$ *at* $n = i$, *then* $\{y_n\}$ *has a node in* $[\alpha, i+1)$, *where* α *is the node of* $\{x_k\}$ *in* $[i, i+1)$.

Lemma 2.3. *Suppose* $f(u, v)$ *is a QL function,* $g(u, v)$ *is a WQL function,* *and the sequence* $\{c_n\}_0^\infty$ *satisfies* $c_n>0$ *for* $n\geq 0$. *Suppose further that the sequences* $\{x_n\}_0^\infty$ *and* $\{y_n\}_0^\infty$ *satisfy*

$$c_n[f(x_{n+1}y_n, x_ny_n)-f(x_ny_{n+1}, x_ny_n)] \geq c_{n-1}[g(x_ny_{n-1}, x_ny_n)-g(x_{n-1}y_n, x_ny_n)]$$

for $0<a+1\leq n\leq b-1$. *Let* $W_n=x_{n+1}y_n-x_ny_{n+1}$ *for* $n\geq 0$. *If* W_n *is nonnegative at* $n=a$, *then* $W_n\geq 0$ *for* $n=a+1, a+2,\cdots, b-1$.

Proof. Since $g(u, v)$ is increasing in u, thus $W_a\geq 0$ implies $g(x_{a+1}y_a, x_{a+1}y_{a+1})-g(x_ay_{a+1}, x_{a+1}y_{a+1})\geq 0$. As a consequence, $f(x_{a+2}y_{a+1}, x_{a+1}y_{a+1})-f(x_{a+1}y_{a+2}, x_{a+1}y_{a+1})\geq 0$. Since $f(u, v)$ is strictly increasing in u, we have

$$W_{a+1} = x_{a+2}y_{a+1} - x_{a+1}y_{a+2} \geq 0.$$

Inductively, we may show that $W_k\geq 0$ for $k=a+2,\cdots, b-1$.

The following is a Sturm-type comparison theorem which extends Lemma 2 in [5].

Theorem 2.4. *Suppose* $\{x_k\}$ *and* $\{y_k\}$ *are respectively nontrivial solutions of the equations*

$$c_kf(x_{k+1}, x_k) + c_{k-1}g(x_{k-1}, x_k) = q_kx_k, \quad k = 1, 2,\cdots \tag{2.1}$$

and

$$c_kf(y_{k+1}, y_k) + c_{k-1}g(y_{k-1}, y_k) = p_ky_k, \quad k = 1, 2,\cdots \tag{2.2}$$

where $c_k>0$ *for* $k\geq 0$, $f(u, v)$ *is QL and* $g(u, v)$ *is WQL. Let* a, b $(a<b)$ *be two nonnegative integers such that* $[a, b]$ *contains two consecutive nodes* α *and* β *of* $\{x_k\}$. *If* $q_k\geq p_k$ *for* $a+1\leq k\leq b-1$, *then* $\{y_k\}$ *has a node in* $(\alpha, \beta]$.

Proof. Since any constant multiple of a solution of (1.1) is also a solution, we may assume that $x^*(t)$, the linear spline of $\{x_k\}$, is positive for $\alpha<t<\beta$. Without loss of generality, we may assume that $a\leq\alpha<a+1$ and $b-1<\beta\leq b$. Suppose to the contrary that $y^*(t)$, the linear spline of $\{y_k\}$, is positive for $\alpha<t\leq\beta$. Then $x_ky_k>0$ for $k=a+1, a+2,\cdots, b-1$. Let $W_k=x_{k+1}y_k-x_ky_{k+1}$ for $k\geq 0$. From (2.1) and (2.2), we have

$$\begin{aligned} &c_k[f(x_{k+1}y_k, x_ky_k)-f(x_ky_{k+1}, x_ky_k)] \\ &\quad = c_{k-1}[g(x_ky_{k-1}, x_ky_k)-g(x_{k-1}y_k, x_ky_k)] + (q_k-p_k)x_ky_k \\ &\quad \geq c_{k-1}[g(x_ky_{k-1}, x_ky_k)-g(x_{k-1}y_k, x_ky_k)] \end{aligned}$$

for $a+1\leq k\leq b-1$. Furthermore, by Lemma 2.2, $W_k\geq 0$ for $k=a$. As a consequence, we have from Lemma 2.3 that $W_k\geq 0$ for $k=b-1$. By Lemma 2.2 again, $\{y_k\}$ would then have a node in $(b-1, \beta]$. This contradicts our assumption that $y^*(t)>0$ for $\alpha<t\leq\beta$. The proof is complete.

In particular, if $g(u, v)$ is QL and $p_k=q_k=b_k$ for $k\geq 1$, then Theorem 2.4 together with Lemma 2.1 imply the following Sturm-type separation theorem.

Corollary 2.5. *If $g(u, v)$ is QL, then the nodes of any two linearly independent solutions of* (1.1) *separate each other.*

3. Oscillatory equations and Riccati transformation

A nontrivial sequence $\{x_k\}_0^\infty$ is said to be nonoscillatory if there is an integer N such that $x_k x_{k+1}>0$ for $k\geq N$. It is said to be oscillatory otherwise. It is clear that $\{x_k\}$ is nonoscillatory if and only if $\{x_k\}$ does not have any nodes for k larger than some integer T.

Theorem 3.1. (cf. [9, Th. 1]). *If $\{b_k\}_1^\infty$ has a subsequence $\{b_{k(i)}\}$ with non-positive terms, then all solutions of equation* (1.1) *are oscillatory.*

Proof. Suppose to the contrary that $\{x_k\}$ is a nonosciallatory solution of (1.1). We may assume without loss of generality that $x_k>0$ for k larger than some integer N. Since $f(u, v)$ is QL and $g(u, v)$ is WQL, we have from (1.1) that

$$b_k x_k = c_k f(x_{k+1}, x_k) + c_{k-1} g(x_{k-1}, x_k) > 0$$

for $k>N+1$. This implies $b_k>0$ for $k>N+1$, which is the desired contradiction.

Theorem 3.2. *If equation* (1.1) *has an oscillatory solution, then all solutions of* (1.1) *are oscillatory.*

The proof follows easily from the comparison Theorem 2.4.

We say that equation (1.1) is oscillatory if all its solutions are oscillatory, and nonoscillatory if all its solutions are nonoscillatory. In view of Theorem 3.2, oscillatory and nonoscillatory equations (1.1) are mutually exclusive and exhaustive.

We next consider a transformation which is known as a Riccati-type transformation in the literature [9, 10, 14]. Let $\{x_k\}$ be a solution of (1.1) such that $x_k\neq 0$ for k larger than or equal to some integer N. The substitution of $r_k=x_{k+1}/x_k$, $k\geq N$, into (1.1) then leads to the following nonlinear two-term recurrence relation

$$c_k f(r_k, 1) + c_{k-1} g(r_{k-1}^{-1}, 1) = b_k \tag{3.1}$$

for $k>N$. We shall call the equation

(3.2) $$c_k f(x_k, 1) + c_{k-1} g(x_{k-1}^{-1}, 1) = b_k, \quad k \geq 1$$

the Riccati equation corresponding to (1.1). A solution of (3.1) is a sequence $\{r_k\}$ where $a \leq k < b$, b may be finite or infinite, such that $r_k \neq 0$ for $a \leq k < b$ and that (3.1) is satisfied by $\{r_k\}$ for $a+1 \leq k < b$.

Theorem 3.3. (cf. [9, Th. 2]). *Equation* (1.1) *is nonoscillatory if and only if equation* (3.2) *has a positive solution* $\{r_k\}$ *for* k *larger than or equal to some integer* N.

Proof. The fact that we can construct a positive solution $\{r_k\}_N^\infty$ of (3.2) is clear from the above discussions. Now if $\{r_k\}_N^\infty$ is a positive solution of (3.2), let $x_N = 1$ and let $x_{N+1} = r_N$, then by Theorem 1.1, the solution $\{x_k\}$ satisfying these conditions exists. We assert that $x_k > 0$ for $k \geq N+2$. Indeed, by (1.1), we have

$$c_{N+1} f(x_{N+2}, r_N) + c_N g(1, r_N) = b_{N+1} r_N$$

or

$$c_{N+1} f(x_{N+2}/r_N, 1) + c_N g(r_N^{-1}, 1) = b_{N+1}.$$

Since $f(u, v)$ is strictly increasing in u, and since

$$c_{N+1} f(r_{N+1}, 1) + c_N g(r_N^{-1}, 1) = b_{N+1},$$

we must have $x_{N+2}/r_N = r_{N+1}$ or $x_{N+2} = r_N r_{N+1}$. Similarly, we have

$$x_k = r_N r_{N+1} \cdots r_{k-1}$$

which is positive for $k \geq N+2$ as desired.

4. Comparison theorems

A Sturm-type comparison Theorem has already been derived in Section two. In this section, we shall derive another comparison theorem which will be useful for deriving various oscillation criteria for equation (1.1).

Lemma 4.1. *Let* $F(x, y), f(x, y)$ *be QL and* $G(x, y), g(x, y)$ *be WQL functions such that*

(4.1) $$\inf \{F(v, 1)/f(v, 1) \mid v > 0\} = m \neq 0$$

and

(4.2) $$\sup \{g(v, 1)/G(v, 1) \mid v > 0\} = M < \infty.$$

Suppose the sequences $\{c_k\}_0^\infty$ *and* $\{d_k\}_0^\infty$ *satisfy* $c_k>0$, $d_k>0$ *for* $k\geq 0$ *and*

$$d_k/c_k \geq \max\{m^{-1}, M\}, \quad k \geq 0. \tag{4.3}$$

Suppose further that u_N, u_{N+1}, v_N *and* v_{N+1} *satisfy* $v_N\geq u_N>0$, $u_{N+1}>0$ *and*

$$c_{N+1}f(v_{N+1}, 1) - d_{N+1}F(u_{N+1}, 1) \geq d_N G(u_N^{-1}, 1) - c_N g(v_N^{-1}, 1),$$

then $v_{N+1}\geq u_{N+1}$.

Proof. We first assert that

$$d_N G(u_N^{-1}, 1) \geq c_N g(v_N^{-1}, 1).$$

For otherwise we would have

$$M \leq \frac{d_N}{c_N} < \frac{g(v_N^{-1}, 1)}{G(u_N^{-1}, 1)} \leq \frac{g(u_N^{-1}, 1)}{G(u_N^{-1}, 1)} \leq M,$$

which is a contradiction. Thus we have

$$c_{N+1}f(v_{N+1}, 1) - d_{N+1}F(u_{N+1}, 1) \geq 0.$$

Since $u_{N+1}>0$, sign $F(u_{N+1}, 1)=$sign u_{N+1} and sign $f(v_{N+1}, 1)=$sign v_{N+1}, thus $v_{N+1}>0$. Suppose to the contrary that $v_{N+1}<u_{N+1}$, then $F(u_{N+1}, 1)>F(v_{N+1}, 1)$ so that $c_{N+1}f(v_{N+1}, 1)>d_{N+1}F(v_{N+1}, 1)$. But then

$$m \geq \frac{c_{N+1}}{d_{N+1}} > \frac{F(v_{N+1}, 1)}{f(v_{N+1}, 1)} \geq m$$

which is a contradiction. Hence $v_{N+1}\geq u_{N+1}$ as desired.

Theorem 4.2. *Let* $F(x, y)$, $f(x, y)$ *be QL and* $G(x, y)$, $g(x, y)$ *be WQL functions such that* (4.1) *and* (4.2) *hold. Suppose the sequences* $\{c_k\}_0^\infty$ *and* $\{d_k\}_0^\infty$ *satisfy* $c_k>0$, $d_k>0$ *for* $k\geq 0$, *and that* (4.3) *holds. Suppose further that* $q_k\geq p_k$ *for* $k\geq 1$. *If the equation*

$$d_k F(x_{k+1}, x_k) + d_{k-1}G(x_{k-1}, x_k) = p_k x_k, \quad k = 1, 2, 3,\cdots \tag{4.4}$$

is nonoscillatory, so is the equation

$$c_k f(x_{k+1}, x_k) + c_{k-1}g(x_{k-1}, x_k) = q_k x_k, \quad k = 1, 2, 3,\cdots. \tag{4.5}$$

Proof. If (4.4) is nonoscillatory, then

$$d_k F(r_k, 1) + d_{k-1}G(r_{k-1}^{-1}, 1) = p_k, \quad k = 1, 2, 3,\cdots$$

has a positive solution $\{r_k\}$ for k larger than or equal to some integer T. Let v_T

be a number larger than or equal to r_T. Since $f(x, 1)$ is strictly incresaing and onto, we can determine v_{T+1} uniquely from

$$c_{T+1}f(v_{T+1}, 1) + c_T g(v_T^{-1}, 1) = q_{T+1}.$$

Since $q_{T+1} \geq p_{T+1}$, we then have

$$c_{T+1}f(v_{T+1}, 1) + c_T g(v_T^{-1}, 1) \geq d_{T+1}F(r_{T+1}, 1) + d_T G(r_T^{-1}, 1).$$

According to Lemma 4.1, $v_{T+1} \geq r_{T+1} > 0$. We may now repeat the same argument inductively to conclude the existence of a positive solution $\{v_k\}$ of

$$c_k f(v_k, 1) + c_{k-1} g(v_{k-1}^{-1}, 1) = q_k, \quad k = 1, 2, 3, \cdots$$

for $k > T$. By Theorem 3.3, equation (4.5) is nonoscillatory. The proof is complete.

Corollary 4.3. *Suppose* $f(x, y) \leq x$ *and* $g(x, y) \leq x$ *for* $x \geq 0$. *If equation* (1.4) *is nonoscillatory, so is equation* (1.1).

Corollary 4.4. *Suppose* $f(x, y) \geq x$ *and* $g(x, y) \geq x$ *for* $x \geq 0$. *If equation* (1.4) *is oscillatory, so is equation* (1.1).

Oscillatory criteria for the linear equation (1.4) have been derived by a number of authors [8, Th. 4; 9, Th. 5, Th. 6, Th. 8; 10, Th. 2.3, Th. 3.2; 12, Th. 1, Th. 3]. In view of Corollaries 4.3 and 4.4, we can use these criteria to obtain corresponding oscillation criteria for equation (1.1).

5. Oscillation criteria

In this section, we shall establish a number of additional oscillation criteria which extend those pertaining to equation (1.4).

Theorem 5.1. (cf. [12, Lemma 2]). *If there is a positive number* r *such that* $b_n \geq c_n f(r, 1) + c_{n-1} g(r^{-1}, 1)$ *for* n *larger than or equal to some integer* N, *then equation* (1.1) *is nonoscillatory.*

Proof. Let v_N be a number larger than or equal to r. Since $f(x, 1)$ is strictly increasing and onto, we can determine v_{N+1} from

$$c_{N+1}f(v_{N+1}, 1) + c_N g(v_N^{-1}, 1) = b_{N+1}.$$

Since $b_{N+1} \geq c_{N+1} f(r, 1) + c_N g(r^{-1}, 1)$, we have, by Lemma 4.1, that $v_{N+1} \geq r > 0$. We may now repeat the same argument inductively to conclude the existence of a positive solution $\{v_k\}$ of equation (3.2) for $k \geq N+1$. Thus equation (1.1) is nonoscillatory by Theorem 3.3.

As an illustration of Theorem 5.1, consider the equation

$$(5.1) \qquad f(x_{n+1}, x_n) + g(x_{n-1}, x_n) = 2x_n, \quad n = 1, 2, 3, \cdots$$

where $f(x, y)$ and $g(x, y)$ are special Cobb-Douglas function of the form

$$(5.2) \qquad f(x, y) = \begin{cases} x & \text{if } y = 0 \\ x^s y^t & \text{if } y \neq 0 \end{cases}$$

and

$$(5.3) \qquad g(x, y) = \begin{cases} x & \text{if } y = 0 \\ x^p y^q & \text{if } y \neq 0 \end{cases}$$

where $s+t=1$, $s=b/a$, a, b are positive odd integers, $p+q=1$, $p=d/c$, and c, d are positive odd integers. Since

$$f(1, 1) + g(1, 1) = 2,$$

thus this equation is nonoscillatory. We remark that if we let

$$r = \left(\frac{p}{s}\right)^{\frac{1}{p+s}}$$

then

$$2 \geq f(r, 1) + g(r^{-1}, 1),$$

which also yields the same conclusion.

Theorem 5.2. (cf. [12, Th. 7]). *If there exists a positive number r such that $b_n \leq \min\{c_n f(r, 1), c_{n-1} g(r^{-1}, 1)\}$ for n larger than some integer N, then equation* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that equation (3.2) has a positive solution $\{r_k\}$ for $k \geq N$. Then $b_k > c_k f(r_k, 1)$ and $b_k > c_{k-1} g(r_{k-1}^{-1}, 1)$ for $k > N$. Furthermore, either $r_i \geq r$ for some integer $i > N$ or $r_k < r$ for all $k > N$. If $r_i \geq r$, then

$$b_i > c_i f(r_i, 1) \geq c_i f(r, 1).$$

If $r_k < r$, then

$$b_k > c_{k-1} g(r_{k-1}^{-1}, 1) \geq c_{k-1} g(r^{-1}, 1).$$

Since both cases contradict our hypothesis, hence (1.1) is oscillatory. The proof is complete.

Let

$$\tau = \inf\{f(v, 1)+g(v^{-1}, 1) \mid v>0\}. \tag{5.4}$$

In general, $\tau \geq 0$. Note moreover that if f and g are given by (5.2) and (5.3) respectively, then

$$2 \geq \tau = \left(\frac{p}{s}\right)^{\frac{s}{p+s}} + \left(\frac{s}{p}\right)^{\frac{p}{p+s}} > 0. \tag{5.5}$$

Theorem 5.3. (cf. [10, Lemma 3.5]). *Suppose there exist two sequences* $\{\alpha_k\}_1^\infty$ *and* $\{\beta_k\}_0^\infty$ *such that* $\alpha_k+\beta_k \leq \tau$ *(where* τ *is given by* (5.4)*) for all large* k, *that*

$$\sum_{k=1}^{\infty} b_k - \alpha_k c_k - \beta_{k-1} c_{k-1} = -\infty, \tag{5.6}$$

and that for any nonnegative integer N,

$$\alpha_j c_j + \sum_{k=N}^{j} (b_k - \alpha_k c_k - \beta_{k-1} c_{k-1}) \leq 0 \tag{5.7}$$

for all large j, *then equation* (1.1) *is oscillatory.*

Proof. Assume to the contrary that $\{x_k\}$ is a nonoscillatory solution of (1.1) such that $x_k>0$ for $k \geq T$. Let $r_k = x_{k+1}/x_k$ for $k \geq T$. Then $\{r_k\}$ is a positive solution of equation (3.2) for $k>T$. Without loss of generality, let us assume that $\alpha_k+\beta_k \leq \tau$ for $k \geq T$. Then

$$\begin{aligned} c_k[f(r_k, 1)-\alpha_k] + c_{k-1}[g(r_{k-1}^{-1}, 1)-\beta_{k-1}] \\ = b_k - \alpha_k c_k - \beta_{k-1} c_{k-1}, \quad k > T. \end{aligned} \tag{5.8}$$

Summing these equations for $T+1 \leq k \leq j$ and rearranging the subsequent equation, we obtain

$$\begin{aligned} & c_j[\alpha_j - f(r_j, 1)] \\ & = c_T[g(r_T^{-1}, 1)-\beta_T] + \sum_{k=T+1}^{j-1} c_k[f(r_k, 1)+g(r_k^{-1}, 1)-\alpha_k-\beta_k] \\ & \quad - \sum_{k=T+1}^{j} (b_k - \alpha_k c_k - \beta_{k-1} c_{k-1}). \end{aligned}$$

Since

$$f(r_k, 1) + g(r_k^{-1}, 1) \geq \tau \geq \alpha_k + \beta_k, \quad k > T$$

and since (5.6) holds, the right hand side of the above equation will be positive for j larger than or equal to some integer N. This implies $\alpha_j > f(r_j, 1)$ which in turn implies $g(r_j^{-1}, 1) > \beta_j$ for $j \geq N$. We now sum equations (5.8) for $N+1 \leq k \leq j$ to obtain

$$\alpha_j c_j + \sum_{k=N+1}^{j} (b_k - \alpha_k c_k - \beta_{k-1} c_{k-1})$$
$$= c_j f(r_j, 1) + c_N[g(r_N^{-1}, 1) - \beta_N] + \sum_{k=N+1}^{j-1} c_k[f(r_k, 1) + g(r_k^{-1}, 1) - \alpha_k - \beta_k].$$

The right hand side of the above equation is positive, but the left hand side is non-positive for all large j, which is a contradiction.

As an illustration of the above Theorem, consider the equation

$$(5.9) \qquad f(x_{n+1}, x_n) + g(x_{n-1}, x_n) = x_n, \quad n = 1, 2, 3, \cdots$$

where $f(x, y)$ and $g(x, y)$ are given by (5.2) and (5.3) respectively. This equation has the solution $\{x_k\}_0^\infty$ where $x_0 = -1$, $x_{3n+1} = 0$, $x_{3n+2} = (-1)^n$, and $x_{3n+3} = (-1)^n$ for $n = 0, 1, 2, \cdots$. Since this solution is oscillatory, so is equation (5.9). We can also show the same conclusion as follows. Take

$$\alpha_n = \left(\frac{s}{p}\right)^{\frac{p}{p+s}} \quad \text{and} \quad \beta_n = \left(\frac{p}{s}\right)^{\frac{s}{p+s}},$$

note that (see (5.5))

$$\tau = \alpha_n + \beta_{n-1} = \alpha_n c_n + \beta_{n-1} c_{n-1} \geq 1 + \min\left\{\left(\frac{s}{p}\right)^{\frac{p}{p+s}}, \left(\frac{p}{s}\right)^{\frac{s}{p+s}}\right\},$$

thus

$$b_n - \alpha_n c_n - \beta_{n-1} c_{n-1} \leq -\min\left\{\left(\frac{s}{p}\right)^{\frac{p}{p+s}}, \left(\frac{p}{s}\right)^{\frac{s}{p+s}}\right\}$$

which implies (5.6) holds. We may now see from Theorem 5.3 that equation (5.9) is oscillatory.

As an immediate consequence of Theorem 5.3, we have

Corollary 5.4. *Suppose there exist two sequences $\{\alpha_k\}_1^\infty$ and $\{\beta_k\}_0^\infty$ such that $\alpha_k + \beta_k \leq \tau$ for all large k (where τ is given by* (5.4)*), that $\{\alpha_k c_k\}$ is bounded above, and that* (5.6) *holds, then equation* (1.1) *is oscillatory.*

In particular, if we take $\alpha_k = 0$ and $\beta_k = \tau$ in Corollary 5.4, then we have

Corollary 5.5. (cf. [8, Th. 3]). *Let τ be given by* (5.4). *Suppose*

$$\sum_{k=1}^{\infty} b_k - \tau c_{k-1} = -\infty$$

holds, then equation (1.1) *is oscillatory.*

As an illustration of the above Corollary, consider the equation

$$cf(x_{n+1}, x_n) + cg(x_{n-1}, x_n) = 2x_n, \quad n = 1, 2, 3, \cdots \tag{5.10}$$

where $f(x, y)$ and $g(x, y)$ are given by (5.2) and (5.3) respectively and

$$c = \left(\frac{p}{s}\right)^{\frac{p}{p+s}} + \left(\frac{s}{p}\right)^{\frac{s}{p+s}}. \tag{5.11}$$

In view of (5.5), $b_n - \tau c_{n-1}$ is equal to

$$2 - \left[\left(\frac{p}{s}\right)^{\frac{s}{p+s}} + \left(\frac{s}{p}\right)^{\frac{p}{p+s}}\right]\left[\left(\frac{p}{s}\right)^{\frac{p}{p+s}} + \left(\frac{s}{p}\right)^{\frac{s}{p+s}}\right] = -\left[\frac{p}{s} + \frac{s}{p}\right]$$

which is less than or equal to -2. By Corollary 5.5, equation (5.10) is oscillatory.

Note that

$$\alpha_j c_j + \sum_{k=N}^{j} (b_k - \alpha_k c_k - \beta_{k-1} c_{k-1}) = b_j - \beta_{j-1} c_{j-1} + \sum_{k=N}^{j-1} (b_k - \alpha_k c_k - \beta_{k-1} c_{k-1})$$

from which we deduce the following

Corollary 5.6. *Suppose there exist two sequences* $\{\alpha_k\}_1^\infty$ *and* $\{\beta_k\}_0^\infty$ *such that* $\alpha_k + \beta_k \le \tau$ *(where* τ *is given by* (5.4)*) for all large* k*, that* $\{b_k - \beta_{k-1} c_{k-1}\}$ *is bounded above, and that* (5.6) *holds, then equation* (1.1) *is oscillatory.*

As an example, the equation

$$(n+1) f(x_{n+1}, x_n) + n g(x_{n-1}, x_n) = x_n, \quad n = 1, 2, 3, \cdots$$

where $f(x, y)$ and $g(x, y)$ are given by (5.2) and (5.3) respectively, can be shown to be oscillatory if we take

$$\alpha_n = \frac{1}{n+1}\left(\frac{s}{p}\right)^{\frac{p}{p+s}} \quad \text{and} \quad \beta_n = \frac{1}{n+1}\left(\frac{p}{s}\right)^{\frac{s}{p+s}}$$

and apply Corollary 5.6.

Our last result is related to the following number

$$\lambda = \inf\{f(v, 1) g(v^{-1}, 1) \mid v > 0\}.$$

Theorem 5.7. (cf. [9, Th. 7]). *If* $b_{k(i)} b_{k(i)+1} \le \lambda c_{k(i)}^2$ *for some sequence* $k(i)$, $k(i) \to \infty$ *as* $i \to \infty$*, then equation* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $\{r_k\}$ is a positive solution of (3.2) for $k \geq N$. Then

$$b_k b_{k+1} = [c_k f(r_k, 1) + c_{k-1} g(r_{k-1}^{-1}, 1)][c_{k+1} f(r_{k+1}, 1) + c_k g(r_k^{-1}, 1)]$$
$$> c_k^2 f(r_k, 1) g(r_k^{-1}, 1) \geq c_k^2 \lambda, \quad k > N$$

which contradicts our assumption as desired.

As a final example, consider the functions $g(x, y) = x$ and

$$f(x, y) = \begin{cases} x & \text{if } y = 0 \\ \dfrac{3x^3 + 2xy^2}{x^2 + y^2} & \text{if } y \neq 0. \end{cases}$$

It is not difficult to see that λ is equal to 2. As a consequence, the following equation

$$f(x_{n+1}, x_n) + g(x_{n-1}, x_n) = x_n, \quad n = 1, 2, 3, \cdots$$

is oscillatory by Theorem 5.7.

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References

[1] Atkinson, F. V., *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.

[2] Bihari, I., On the second order half-linear differential equations, Studia Sci. Math. Hung., **3** (1968), 411–437.

[3] Cheng, S. S., Sturmian comparison theorems for three-term recurrence equations, J. Math. Anal. Appl., **111** (1985), 465–474.

[4] Chiang, A. C., *Fundamental Methods of Mathematical Economics*, 3rd edition, McGraw-Hill Book Co., 1984.

[5] Cheng, S. S. and Cho, A. M., Convexity of nodes of discrete Sturm-Liouville functions, Hokkaido Math. J., **11** (1982), 8–14.

[6] Fort, T., *Finite Differences and Difference Equations in the Real Domain*, Oxford University Press, London, 1948.

[7] Gautschi, W., Computational aspects of three-term recurrence relations, SIAM Rev., **9** (1967), 24–82.

[8] Hinton, D. and Lewis, R., Spectral analysis of second difference equations, J. Math. Anal. Appl., **63** (1978), 421–438.

[9] Hooker, J. W. and Patula, W. T., Riccati type transformations for second order linear difference equations, J. Math. Anal. Appl., **82** (1981), 451–462.

[10] Kwong, M. K., Hooker, J. W. and Patula, W. T., Riccati type transformations for second-order difference equations, J. Math. Anal. Appl., **107** (1985), 182–196.

[11] McCarthy, P. J., Note on oscillation of solutions of second-order linear difference equa-

tions, Port. Math., **18** (1959), 203–205.
[12] Patula, W. T., Growth and oscillation properties of second order linear difference equations, SIAM J. Math. Anal., **10** (1979), 55–61.
[13] Patula, W. T., Growth, oscillation and comparison theorem for second order linear difference equations, SIAM J. Math. Anal., **10** (1979), 1272–1279.
[14] Reid, W. T., *Riccati Differential Equations*, Academic Press, New York, 1972.
[15] Samuelson, P. A., Interactions between the multiplier analysis and the principles of acceleration, Review of Economic Statistics, May (1939), 75–78.
[16] Wouk, A., Difference equations and *j*-matrices, Duke Math. J., **20** (1953), 141–159.

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