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Remetrization and a New Type of Recurrence

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§ 0.

In topological dynamics, various types of recurrence has been defined. Each type of recurrence is a generalization of periodicity ([3], [4]) and is closely related ([1], [11]) to some order relation ([9], [12]).

The aim of this paper is to define a new type of recurrence and to establish its basic properties. Our new type of recurrence is a modification of Conley's chain recurrence ([6]), in fact, it is defined by weakening the requirements in Conley's definition. On the other hand, our new type of recurrence still implies recurrence in the sense of Auslander.

Throughout this paper, let X be a locally compact separable metrizable space and let T: $\boldsymbol{R}\times X\to X$, $(t, x)\to T_t x=T(t, x)$ be a dynamical system on X. The positive semitrajectory through x is denoted by $\gamma^+(x)$.

The first prolongation and the first prolongational limit set of x are defined by $D_1(x)=\{y\in X\mid \text{there exist sequences } \{x_n\}_1^\infty\subset X,\ \{t_n\}_1^\infty\subset \boldsymbol{R}^+ \text{ such that } x_n\to x,\ T_{t_n}x_n\to y\}$ and $J_1(x)=\cap\{D_1(T_t x)\mid t\geqslant 0\}$, respectively. The basic dynamical concepts are defined in terms of a compatible metric d — however, in case of compact metrizable spaces, they do not depend on the choice of this particular metric. This is no longer true if the compactness assumption is dropped ([3]).

Our new type of recurrence is defined in terms of a compatible metric as well. But, even though in the compact case, our new type of recurrence is not independent on the choice of this particular metric.

§ 1. Auslander recurrence

Define a quasi order (i.e. a reflexive and transitive relation) $\dashv$ on X by letting $y\dashv x$ if and only if $y\in\gamma^+(x)$. The transitive closure of $\dashv$ i.e. the intersection of all closed quasi orders containing $\dashv$ is denoted by $\prec_A$ (a quasi order $\prec$ is closed if $\{(x, y)\in X\times X\mid x\prec y\}$ is a closed subset in the product topology). Using the notion of higher prolongations and higher prolongational limit sets, $\prec_A$ can be constructed explicitely via transfinit induction ([1]). For any $x\in X$, define $D(x)=\{y\in X\mid y\prec_A x\}$, $J(x)=\cap\{D(T_t x)\mid t\geqslant 0\}$. It is well-known that $D_1(x)\subset D(x)$, $J_1(x)\subset J(x)$, $D(x)=\gamma^+(x)\cup J(x)$ and that for every t, $s\in\boldsymbol{R}$, $y\in J(x)$

implies $T_t y \in J(T_s x)$. The equivalence relation $\sim_A$ is defined by letting $y \sim_A x$ if and only if $y \prec_A x$ and $x \prec_A y$. Finally, the Auslander recurrent set is defined as $R_A = \{x \in X \mid x \in J(x)\}$. It can be readily verified from the properties of D, J, $\prec_A$ previously mentioned that R_A is closed and invariant and it is the union of the nontrivial equivalence classes under $\prec_A$ and of the set of equilibrium points ([3]).

§ 2. Conley recurrence

In what follows we reformulate some basic properties of Conley recurrence in context of an order relation. The following definitions are taken from [10]. Conley's original concept of chain recurrence was elaborated ([6]) in the compact case.

Let $\varepsilon: X \to \boldsymbol{R}^+$ be a continuous positive function. An (ε, τ)-chain is a finite or countable sequence of points $x_n \in X$ and times $t_n \geqslant \tau$ such that $d(T_{t_n} x_n, x_{n+1}) < \varepsilon(T_{t_n} x_n)$. The points which are (ε, τ)-accessible from x are $P(\varepsilon, \tau, x) = \{y \in X \mid y = x_{k+1}$ for some finite (ε, τ)-chain with $x_0 = x$, $k \in \boldsymbol{N}\}$. Let $P(x) = \bigcap \{P(\varepsilon, \tau, x) \mid \tau > 0$ and $\varepsilon: X \to \boldsymbol{R}^+$ is a continuous positive function$\}$. In Conley's terminology, x and y are chain equivalent if and only if $y \in P(x)$ and $x \in P(y)$. Chain equivalence is a closed transitive relation and for every t, $s \in R$, $y \in P(x)$ implies $T_t y \in P(T_s x)$. Finally, the Conley recurrent set is $R_C = \{x \in X \mid x \in P(x)\}$. It is known that R_C is closed and invariant and that $R_A \subset R_C$ ([6], [10]).

Recurrence in the sense of Auslander as well as recurrence in the sense of Conley are well understood and have found many applications in various fields of topological and differentiable dynamics ([3], [6]).

For any $x \in X$, let $Q(x) = \gamma^+(x) \cup P(x)$ and define a relation $\prec_C$ on X by $y \prec_C x$ if and only if $y \in Q(x)$. It follows easily from the properties of P previously mentioned that $\prec_C$ is a closed quasi order and that $P(x) = \bigcap \{Q(T_t x) \mid t \geqslant 0\}$. We define an equivalence relation $\sim_C$ by letting $y \sim_C x$ if and only if $y \prec_C x$ and $x \prec_C y$. It is not hard to show that R_C is the union of the nontrivial equivalence classes under $\sim_C$ and of the set of equilibrium points. Further, $y \sim_C x$ if and only if y and x are chain equivalent or $y = x$.

§ 3. Σ-recurrence

Let $E: X \to \boldsymbol{R}^+$ be a continuous function. An $(E, \tau)_\Sigma$-chain is a (finite or infinite) sequence of points $x_n \in X$ and times $t_n \geqslant \tau$ such that $\sum_n E(T_{t_n} x_n) \cdot d(T_{t_n} x_n, x_{n+1}) < 1$. The points which are $(E, \tau)_\Sigma$-accessible from x are $M(E, \tau, x) = \{y \in X \mid y = x_{k+1}$ for some finite $(E, \tau)_\Sigma$-chain with $x_0 = x$, $k \in \boldsymbol{N}\}$. Let $M(x) = \bigcap \{M(E, \tau, x) \mid \tau > 0$ and $E: X \to \boldsymbol{R}^+$ is a continuous function$\}$. For any $x \in X$, let $N(x) = \gamma^+(x) \cup M(x)$ and define a relation $\prec_\Sigma$ on X by letting $y \prec_\Sigma x$ if and only if $y \in N(x)$.

Proposition 3.1. *$\prec_\Sigma$ is a closed quasi order.*

Proof. $x \prec_\Sigma x$ is clear. Suppose that $y \prec_\Sigma x$, $z \prec_\Sigma y$. By definition, $y \in \gamma^+(x) \cup M(x)$, $z \in \gamma^+(y) \cup M(y)$. We have to distinguish four cases.

If $y \in \gamma^+(x)$, $z \in \gamma^+(y)$, then $z \in \gamma^+(x)$.

If $y = T_t x$, $t \geqslant 0$ and $y = y_0, y_1, \cdots, y_{k+1} = z$ is an $(E, \tau)_\Sigma$-chain, then $\tilde{y}_0 = x$, $\tilde{y}_1 = y_1, \cdots, \tilde{y}_{k+1} = y_{k+1} = z$ (with $\tilde{t}_0 = t + t_0$, $\tilde{t}_1 = t_1, \cdots, \tilde{t}_k = t_k$) is also an $(E, \tau)_\Sigma$-chain. Consequently, $y \in \gamma^+(x)$, $z \in M(y)$ imply $z \in M(x)$.

Now suppose that $y \in M(x)$ and $z = T_t y$, $t \geqslant 0$. Let $E \colon X \to \boldsymbol{R}^+$ be a continuous positive function. Since X is locally compact, there exists a $\nu > 0$ such that $d(w, z) < \nu$ implies $d(w, z) < 1/E(w)$. By continuity upon initial conditions, there exists a $\mu > 0$ such that $d(T_t v, z) < \nu$ whenever $d(v, y) < \mu$. For $x \in X$, let $\Delta(x) = \max\{E(x), 1/\mu\}$. It is clear that $\Delta \colon X \to \boldsymbol{R}^+$ is a continuous function. Let $x_0 = x, x_1, \cdots, x_{k+1} = y$ be a $(\Delta, \tau)_\Sigma$-chain. Then $\tilde{x}_0 = x_0 = x$, $\tilde{x}_1 = x_1, \cdots, \tilde{x}_k = x_k$, $\tilde{x}_{k+1} = z$ (with $\tilde{t}_0 = t_0, \cdots, \tilde{t}_{k-1} = t_{k-1}$, $\tilde{t}_k = t_k + t$) is an $(E/2, \tau)_\Sigma$-chain. In fact, $d(T_{t_k} x_k, y) \leqslant \mu \cdot \Delta(T_{t_k} x_k) \cdot d(T_{t_k} x_k, y) < \mu$, $d(T_t T_{t_k} x_k, z) < \nu$ and consequently, $\sum_{\ell=0}^{k} E(T_{\tilde{t}_\ell} \tilde{x}_\ell) \cdot d(T_{\tilde{t}_\ell} \tilde{x}_\ell, \tilde{x}_{\ell+1}) \leqslant \sum_{\ell=0}^{k-1} \Delta(T_{t_\ell} x_\ell) \cdot d(T_{t_\ell} x_\ell, x_{\ell+1}) + E(T_t T_{t_k} x_k) \cdot d(T_t T_{t_k} x_k, z) \leqslant 1 + 1 = 2$. Therefore, $z \in M(x)$.

Finally, suppose that $y \in M(x)$, $z \in M(y)$. It is trivial that $x_0 = x, x_1, \cdots, x_{k+1} = y_0 = y, y_1, \cdots, y_{\ell+1} = z$ is an $(E/2, \tau)_\Sigma$-chain whenever $x_0 = x, x_1, \cdots, x_{k+1} = y$ and $y_0 = y, y_1, \cdots, y_{\ell+1} = z$ are $(E, \tau)_\Sigma$-chains and consequently, $z \in M(x)$. Thus, in all cases, $y \prec_\Sigma x$, $z \prec_\Sigma y$ imply $z \prec_\Sigma x$.

Suppose we are given two sequences in X, say $\{x_n\}_1^\infty$, $\{y_n\}_1^\infty$, with the properties that $x_n \to x$, $y_n \to y$ for some $x \in X$, $y \in X$ and that $y_n \prec_\Sigma x_n$ for $n = 1, 2, \cdots$. We have to show that $y \prec_\Sigma x$.

There are essentially two cases according as $y_n \in \gamma^+(x_n)$ or $y_n \in M(x_n)$ for each n. Suppose first that $y_n = T_{t_n} x_n$. There are essentially two cases again according as $t_n \to t \in \boldsymbol{R}^+$ or $t_n \to \infty$. If $t_n \to t \in \boldsymbol{R}^+$, then $y = T_t x$, $y \in \gamma^+(x)$. If $t_n \to \infty$, $\tau > 0$ and $E \colon X \to \boldsymbol{R}^+$ is a continuous function, then $z_{0,n} = x$, $z_{1,n} = T_\tau x_n$, $z_{2,n} = y$ (with $t_{0,n} = \tau$, $t_{1,n} = t_n - \tau\}$ is an $(E, \tau)_\Sigma$-chain for n sufficiently large. In fact, $E(T_{t_{0,n}} z_{0,n}) \cdot d(T_{t_{0,n}} z_{0,n}, z_{1,n}) + E(T_{t_{1,n}} z_{1,n}) \cdot d(T_{t_{1,n}} z_{1,n}, z_{2,n}) = E(T_\tau x) \cdot d(T_\tau x, T_\tau x_n) + E(y_n) \cdot d(y_n, y) \to 0$ as $n \to \infty$. Suppose now that $y_n \in M(x_n)$. Let $E \colon X \to \boldsymbol{R}^+$ be a continuous function. Since X is locally compact, there exist constants μ, $K > 0$ such that $E(w) < K$ whenever $d(w, y) < 2\mu$. Without loss of generality, we may assume that $d(y_n, y) < \mu$. Define a continuous function $\Delta \colon X \to \boldsymbol{R}^+$ by letting $\Delta(x) = \max\{E(x), 1/\mu\}$. For $n = 1, 2, \cdots$, let $z_{0,n} = x_n$, $z_{1,n}, \cdots, z_{k(n)+1,n} = y_n$ be a $(\Delta, 2\tau)_\Sigma$-chain. It is easy to see that $\tilde{z}_{0,n} = x$, $\tilde{z}_{1,n} = T_\tau x_n$, $\tilde{z}_{2,n} = z_{1,n}, \cdots$, $\tilde{z}_{k(n)+1,n} = z_{k(n),n}$, $\tilde{z}_{k(n)+2,n} = y$ (with $\tilde{t}_{0,n} = \tau$, $\tilde{t}_{1,n} = t_{0,n} - \tau$, $\tilde{t}_{2,n} = t_{1,n}, \cdots, \tilde{t}_{k(n)+1,n} = t_{k(n),n}$) is an $(E/2, \tau)_\Sigma$-chain for n sufficiently large. In fact, $d(T_{t_{k(n),n}} z_{k(n),n}, y) \leqslant$

$d(T_{t_{k(n),n}}z_{k(n),n}, y_n)+d(y_n, y)<\mu\cdot\Delta(T_{t_{k(n),n}}z_{k(n),n})\cdot d(T_{t_{k(n),n}}z_{k(n),n}, y_n)+\mu<2\mu$ and, consequently $\sum_{\ell=0}^{k(n)+1} E(T_{\tilde{t}_{\ell,n}}\tilde{z}_{\ell,n})\cdot d(T_{\tilde{t}_{\ell,n}}\tilde{z}_{\ell,n}, \tilde{z}_{\ell+1,n}) = E(T_\tau x)\cdot d(T_\tau x, T_\tau x_n) + \sum_{\ell=0}^{k(n)} E(T_{t_{\ell,n}}z_{\ell,n})\cdot d(T_{t_{\ell,n}}z_{\ell,n}, z_{\ell+1,n}) + E(T_{t_{k(n),n}}z_{k(n),n})\cdot[d(T_{t_{k(n),n}}z_{k(n),n}, y) - d(T_{t_{k(n),n}}z_{k(n),n}, y_n)] \leqslant E(T_\tau x)\cdot d(T_\tau x, T_\tau x_n)+\sum_{\ell=0}^{k(n)} \Delta(T_{t_{\ell,n}}z_{\ell,n})\cdot d(T_{\ell,n}z_{\ell,n}, z_{\ell+1,n})+E(T_{t_{k(n),n}}z_{k(n),n})\cdot d(y_n, y) < E(T_\tau x)\cdot d(T_\tau x, T_\tau x_n) + 1 + K\cdot d(y_n, y)\to 1$ as $n\to\infty$. Thus, in all cases $y\in\gamma^+(x)\cup M(x)$, $y\prec_\Sigma x$. Q. E. D.

Proposition 3.2. *For every t, $s\in\mathbf{R}$, $y\in M(x)$ implies $T_t y\in M(T_s x)$. In particular, $M(x)=M(T_s x)$.*

Proof. The proof follows a similar pattern as the one of the previous proposition. Q. E. D.

Corollary 3.3. *For any $x\in X$, $M(x)=\cap\{N(T_t x)\mid t\geqslant 0\}$.*

Proof. Observe that $\cap\{\gamma^+(T_t x)\mid t\geqslant 0\}=\emptyset$ excepting the case when the trajectory through x is periodic. In this latter case, $\cap\{\gamma^+(T_t x)\mid t\geqslant 0\}=\gamma^+(x)\subset M(x)$.

Thus, $\cap\{N(T_t x)\mid t\geqslant 0\}=\cap\{\gamma^+(T_t x)\cup M(T_t x)\mid t\geqslant 0\}=\cap\{\gamma^+(T_t x)\cup M(x)\mid t\geqslant 0\}=\cap\{\gamma^+(T_t x)\mid t\geqslant 0\}\cup M(x)=M(x)$. Q. E. D.

Define a relation $\sim_\Sigma$ on X by letting $y\sim_\Sigma x$ if and only if $y\prec_\Sigma x$ and $x\prec_\Sigma y$. The Σ-recurrent set is defined as $R_\Sigma=\{x\in X\mid x\in M(x)\}$.

Proposition 3.4. *$\sim_\Sigma$ is a closed equivalence relation. The equivalence classes are closed. The nontrivial equivalence classes are invariant. R_Σ is closed and invariant. R_Σ is the union of the nontrivial equivalence classes under $\sim_\Sigma$ and of the set of equilibrium points.*

Proof. In virtue of Proposition 3.1, $\sim_\Sigma$ is a closed equivalence relation and the equivalence classes are closed. Next we prove that R_Σ is closed. Suppose that $\{x_n\}_1^\infty\subset R_\Sigma$ and $x_n\to x$ for some $x\in X$. By Corollary 3.3, $x_n\in R_\Sigma$ if and only if $x_n\prec_\Sigma T_t x_n$ for all $t\in\mathbf{R}^+$. Hence, applying Proposition 3.1, we have that $x\prec_\Sigma T_t x$ for all $t\in\mathbf{R}^+$. Consequently, $x\in R_\Sigma$.

In virtue of Proposition 3.2, R_Σ is invariant. In particular, if $x\in R_\Sigma$ and the equivalence class containing x is degenerate, then x is an equilibrium point. Conversely, if x is an equilibrium point, then $x\in R_\Sigma$.

Suppose now that $x\neq y$, $y\sim_\Sigma x$ or equivalently, $x\neq y$, $y\in\gamma^+(x)\cup M(x)$, $x\in\gamma^+(y)\cup M(y)$. If $y\in\gamma^+(x)$, $x\in\gamma^+(y)$, then the trajectory through x is periodic and consequently, $x\sim_\Sigma T_t x\in R_\Sigma$, $y\sim_\Sigma T_t y\in R_\Sigma$ for all $t\in\mathbf{R}$. If $y\in M(x)$, $x\in\gamma^+(y)$ or if $y\in\gamma^+(x)$, $x\in M(y)$, then it follows from Proposition 3.2 that $x\in M(x)$, $y\in M(y)$ and that $x\sim_\Sigma T_t x\in R_\Sigma$, $y\sim_\Sigma T_t y\in R_\Sigma$ for all $t\in\mathbf{R}$. As it has been explicitly shown in proving Proposition 3.1, $y\in M(x)$ and $z\in M(y)$ imply $z\in M(x)$.

Hence, if $y \in M(x)$, $x \in M(y)$, then $x \in M(x)$, $y \in M(y)$. Applying Proposition 3.2 again, we conclude that $x \sim_{\Sigma} T_t x \in R_{\Sigma}$, $y \sim_{\Sigma} T_t y \in R_{\Sigma}$ for all $t \in \boldsymbol{R}$. Q. E. D.

Proposition 3.5. *$y \prec_A x$ implies $y \prec_{\Sigma} x$ which, in turn, implies $y \prec_C x$. Similarly, $y \in J(x)$ implies $y \in M(x)$ which, in turn, implies $y \in P(x)$. Further, $R_A \subset R_{\Sigma} \subset R_C$.*

Proof. This is more or less obvious from the previous considerations. Q. E. D.

Example 3.6. Let X be an arbitrary locally compact metrizable space. For any $x \in X$, $t \in \boldsymbol{R}$, define $T_t x = x$. Clearly T is a dynamical system and each point of X is a degenerate equivalence class under $\sim_A$ resp. $\sim_{\Sigma}$. Assume that X is connected. Then any two points of X are $\sim_C$-equivalent.

Example 3.7. Consider the plane with the usual Euclidean topology. The metric is not specified. Consider now the planar dynamical system defined ([10]) by the differential system (in Cartesian coordinates) $\dot{x}_1 = 0$, $\dot{x}_2 = 1 - x_1^2 - x_2^2$. It is obvious that $R_A = R_{\Sigma} = \{(x_1, x_2) \in \boldsymbol{R}^2 \mid x_1^2 + x_2^2 = 1\}$, and $R_C = \{(x_1, x_2) \in \boldsymbol{R}^2 \mid x_1^2 + x_2^2 \leqslant 1\}$ in particular, $R_{\Sigma} \neq R_C$. It will be shown by Example 5.4 that, in general, $R_{\Sigma} \neq R_A$.

§ 4. *c-c* and *c-c-c* property

In order to describe the basic properties of decompositions under $\sim_A$, $\sim_C$ resp. $\sim_{\Sigma}$, we turn our attention to *c-c* and *c-c-c* properties of closed quasi orders. In terms of set-valued maps, *c-c* property has been defined in [2]. In case of recurrence in sense of Auslander as well as of Conley, the results of this Section are known ([1], [2], [3], [6], [10]). *c-c-c* property is a natural generalization of *c-c* property which will enable us to formulate some results of the second author ([7]) in a more general context.

Let X be a locally compact separable metrizable space and let $\prec$ be a closed quasi order on X. We say that $\prec$ has the *c-c* property if for any compact $K \subset X$, for any $x \in K$, $y \notin K$ with $y \prec x$ there exists a $z \in \partial K$ such that $z \prec x$. Here, of course, ∂K denotes the boundary of K. If, in addition, $y \prec z$, then $\prec$ is said to have the *c-c-c* property.

Similarly, we say that $\prec$ has the dual *c-c* property if for any compact $K \subset X$, for any $x \notin K$, $y \in K$ with $y \prec x$ there exists a $z \in \partial K$ such that $z \prec x$. If, in addition, $y \prec z$, then $\prec$ is said to have the dual *c-c-c* property.

Proposition 4.1. a. *$\prec_A$ has the c-c-c and the dual c-c-c properties.*

b. *$\prec_C$ resp. $\prec_{\Sigma}$ have the c-c property. Further, $\prec_C$ resp. $\prec_{\Sigma}$ has the c-c-c and the dual c-c-c properties whenever X is compact.*

Proof. a. Although in terms of set-valued maps, a. has been proved in [7].

b. We restrict ourselves to prove the statements concerning the relation $\prec_\Sigma$ only. With minor modifications, the same proof is valid if $\prec_\Sigma$ is replaced by $\prec_C$.

First we prove that $\prec_\Sigma$ has the *c-c* property. X is endowed with a metric d. Even though in the compact case (cf. Section 5), $\prec_\Sigma$ depends on the particular choice of d. Nevertheless, we prove that $\prec_\Sigma$ has the *c-c* property no matter the particular choice of d.

Let K be a compact subset of X, $x\in K$, $y\notin K$, $y\prec_\Sigma x$. We have to prove that there is a $z\in\partial K$ such that $z\prec_\Sigma x$. The case when $y\in\gamma^+(x)$ is trivial. Therefore, we may assume that $y\in M(x)$. Since K is compact, the local compactness of X implies that there exists a compact neighborhood L of K in X and $\inf\{d(w, v)\mid w\in K,\ v\in X\setminus L\}=\delta_0>0$. Since $y\notin K$, there is no loss of generality in assuming that $y\notin L$. Choose $n_0\in \boldsymbol{N}$ so that $n_0>1/\delta_0$. For $n=n_0,\ n_0+1,\cdots$, let $x_{0,n}=x$, $x_{1,n},\ x_{2,n},\cdots,\ x_{k(n)+1,n}=y$ be an $(n, n)_\Sigma$-chain, i.e. an $(E, \tau)_\Sigma$-chain with $E\equiv n$ and $\tau=n$. For each n, $n=n_0,\ n_0+1,\cdots$, there exists clearly a unique $p(n)\in\{0, 1, 2,\cdots, k(n)\}$ with the following properties:

for $\ell=0, 1,\cdots, p(n)$, there holds $x_{\ell,n}\in K$,

for $\ell=0, 1,\cdots, p(n)$, there holds $T_{t_{\ell,n}}x_{\ell,n}\in K$ and

either (case 1) $x_{p(n)+1,n}\notin K$

or (case 2) $x_{p(n)+1,n}\in K$ but $T_{t_{p(n)+1,n}}x_{p(n)+1,n}\notin K$.

Without loss of generality, we may assume that we have either (case 1) $x_{p(n)+1,n}\notin K$ for each n, $n=n_0,\ n_0+1,\cdots$ or (case 2) $x_{p(n)+1,n}\in K$ but $T_{t_{p(n)+1,n}}x_{p(n)+1,n}\notin K$ for each n, $n=n_0,\ n_0+1,\cdots$.

Consider now the two cases separately. In the first case, let $v_n=x_{p(n)+1,n}$. By the definition of $(E, \tau)_\Sigma$-chains, we have that $n\cdot d(T_{t_{p(n),n}}x_{p(n),n},\ v_n)\leqslant\sum_{\ell=0}^{k(n)} n\cdot d(T_{t_{\ell,n}}x_{\ell,n},\ x_{\ell+1,n})<1$. If follows immediately that $v_n\in L$ and that $\inf\{d(w, v_n)\mid w\in K\}\to 0$ as $n\to\infty$. Consequently, by passing to a subsequence, we may assume that $v_n\to v$ for some $v\in L$ and $v\in\partial K$.

In the second case, let (as above) $v_n=x_{p(n)+1,n}$. Since $\gamma^+(v_n)$ is connected, there exists a $t_n\in[0, t_{p(n)+1,n}]$ such that $T_{t_n}v_n\in\partial K$. Set $z_n=T_{t_n}v_n$. We may assume that $v_n\to v$, $z_n\to z$ for some $v\in K$, $z\in\partial K$. By definition, we have that $z\in D_1(v)$, $z\prec_A v$ and consequently, by Proposition 3.2, we arrive at $z\prec_\Sigma v$.

It is left to prove that $v\prec_\Sigma x$. In fact, the desired result is an easy consequence of $v\prec_\Sigma x$. In the first case, z is simply to be chosen for v. In the second case, $z\prec_\Sigma x$ is implied by $z\prec_\Sigma v$ via the transitivity of $\prec_\Sigma$ established in Proposition 3.1. Thus, it remain to show that given any continuous function $E\colon X\to\boldsymbol{R}^+$ and a number $\tau>0$, there exists a finite $(E, \tau)_\Sigma$-chain with $x_0=x$, $x_{k+1}=v$ for some $k\in\boldsymbol{N}$. In both cases, observe first that $x_{0,n}=x\in L$, $T_{t_{0,n}}x_{0,n}\in L$, $x_{1,n}\in L,\cdots$, $T_{t_{p(n),n}}x_{p(n),n}\in L$, $x_{p(n)+1,n}=v_n\in L$ for each n, $n=n_0,\ n_0+1,\cdots$ and that $v_n\to v\in L$. Recall that L is compact and set $\nu=\max\{E(w)\mid w\in L\}$. Since $v_n\to v$, there is an $n_1\in\boldsymbol{N}$ such that $d(v_n, v)<1/(2\nu)$ whenever $n\geqslant n_1$. Fix an $n_2\in\boldsymbol{N}$ so that $n_2\geqslant$

$\max\{n_0, n_1, \tau, 2v\}$. It is easy to see that $\tilde{x}_{0,n}=x_{0,n}=x$, $\tilde{x}_{1,n}=x_{1,n},\cdots$, $\tilde{x}_{p(n),n}=x_{p(n),n}$, $\tilde{x}_{p(n)+1,n}=v$ (with $\tilde{t}_{0,n}=t_{0,n}$, $\tilde{t}_{1,n}=t_1$, $t_{1,n},\cdots$, $\tilde{t}_{p(n),n}=t_{p(n),n}$) is an $(E, \tau)_\Sigma$-chain whenever $n\geqslant n_2$. In fact, $\min\{\tilde{t}_{\ell,n}\mid \ell=0, 1,\cdots,p(n)\}\geqslant\tau$ and $\sum_{\ell=0}^{p(n)} E(T_{\tilde{t}_{\ell,n}}\tilde{x}_{\ell,n})\cdot d(T_{\tilde{t}_{\ell,n}}\tilde{x}_{\ell,n}, x_{\ell+1,n})\leqslant \sum_{\ell=0}^{p(n)-1} E(T_{t_{\ell,n}}x_{\ell,n})\cdot d(T_{t_{\ell,n}}x_{\ell,n}, x_{\ell+1,n}) + E(T_{t_{p(n),n}}x_{p(n),n})\cdot d(T_{t_{p(n),n}}x_{p(n),n}, x_{p(n)+1,n}) + E(T_{t_{p(n),n}}x_{p(n),n})\cdot d(v_n, v) < 2^{-1}(\sum_{\ell=0}^{p(n)} n\cdot d(T_{t_{\ell,n}}x_{\ell,n}, x_{\ell+1,n})) + v\cdot d(v_n, v) < 2^{-1}+2^{-1}=1$ and this completes the proof of the *c-c* property for $\prec_\Sigma$.

Now we prove that $\prec_\Sigma$ has the *c-c-c* property provided that X is compact. Since the *c-c-c* property is a generalization of the *c-c* property, the proof of the *c-c-c* property is a supplement to the one of the *c-c* property. We have to make use of the compactness assumption in such a way that the improved arguments yield $y\prec_\Sigma z$, the additional requirement in the definition of the *c-c-c* property.

From now on, assume that X is compact. We distinguish the same cases as above. In the first case, let $w_n=x_{p(n)+1,n}$. Recall that (for a subsequence) $w_n\to w=v=z\in\partial K$. For any given continuous function $E: X\to \boldsymbol{R}^+$ and a number $\tau>0$, it is not hard to show that $\hat{w}_{0,n}=w$, $\hat{w}_{1,n}=T_\tau w_n$, $\hat{w}_{2,n}=x_{p(n)+2,n}$, $\hat{w}_{3,n}=x_{p(n)+3,n},\cdots$, $\hat{w}_{k(n)-p(n)+1,n}=x_{k(n)+1,n}=y$ (with $\hat{t}_{0,n}=\tau$, $\hat{t}_{1,n}=t_{p(n)+1,n}-\tau$, $\hat{t}_{2,n}=t_{p(n)+2,n}$, $\hat{t}_{3,n}=t_{p(n)+3,n},\cdots$, $\hat{t}_{k(n)-p(n),n}=t_{k(n),n}$) is an $(E, \tau)_\Sigma$-chain for n sufficiently large. In fact, $\min\{\hat{t}_{\ell,n}\mid \ell=0, 1,\cdots, k(n)-p(n)\}\to\infty$ and, by the boundedness of E, we have that $\sum_{\ell=0}^{k(n)-p(n)} E(T_{\hat{t}_{\ell,n}}\hat{w}_{\ell,n})\cdot d(T_{\hat{t}_{\ell,n}}\hat{w}_{\ell,n}, \hat{w}_{\ell+1,n})=E(T_\tau w)\cdot d(T_\tau w_n, T_\tau w) + \sum_{\ell=p(n)+1}^{k(n)} E(T_{t_{\ell,n}}x_{\ell,n})\cdot d(T_{t_{\ell,n}}x_{\ell,n}, x_{\ell+1,n})\to 0$ as $n\to\infty$. Consequently, $y\prec_\Sigma w=v=z$. In the second case, let $w_n=x_{p(n)+2,n}$. There is no loss of generality in assuming that $w_n\to w$ for some $w\in X$. Repeating the same arguments used in the first case, we obtain that $y\prec_\Sigma w$. On the other hand, by definition, $w\in D_1(z)$, $w\prec_A z$. Therefore, by Proposition 3.2, we have that $w\prec_\Sigma z$. The transitivity of $\prec_\Sigma$ established in Proposition 3.1 implies that $y\prec_\Sigma z$.

Finally, let K be a closed subset of X (X is assumed to be compact), $x\notin X$, $y\in K$ and $y\prec_\Sigma x$. If $y\notin \mathrm{cl}(X\setminus K)$, then, by the *c-c-c* property, there exists a $z\in\partial(\mathrm{cl}(X\setminus K))$ such that $y\prec_\Sigma z$ and $z\prec_\Sigma x$. Since $\partial(\mathrm{cl}(X\setminus K))\subset\partial(X\setminus K)=\partial K$, we have that $z\in\partial K$. If $y\in\mathrm{cl}(X\setminus K)$, then $y\in(\mathrm{cl}(X\setminus K))\setminus(X\setminus K)=\partial(X\setminus K)=\partial K$ and z can be chosen for y and this completes the proof of the dual *c-c-c* property as well as that of the proposition. Q.E.D.

The question whether $\prec_\Sigma$ and/or $\prec_C$ have the dual *c-c* property and/or the *c-c-c* property seems to be rather difficult and remains open.

Proposition 4.2. *Let X be a locally compact separable metrizable space and let $\prec$ be a closed quasi order on X. For any $x\in X$, the set $\{y\in X\mid y\prec x\}$ is denoted by $\Gamma(x)$. Assume that $\prec$ has the c-c property.*

a. ([2]) *Then $\Gamma(x)$ is connected, whenever it is compact.*

b. *Assume that $\prec$ has the dual c-c property either. If $\Gamma(x)$ is not compact, then none of its components is compact.*

Proof. a. Observe first that $\Gamma(x)$ is closed. Let $\Gamma(x)$ be compact and let it be not connected. Then $\Gamma(x)=A\cup B$ where A, B are nonempty, compact, disjoint subsets of X. Since X is locally compact, there exists an open set U with compact closure K such that $A\subset U$, $B\cap K=\emptyset$. Pick an $y\in B$. Since $y\prec x$, there exists a $z\in\partial K$ such that $z\prec x$, a contradiction.

b. Let $\hat{X}$ be the one-point compactification of X, i.e. $\hat{X}=X\cup\{\omega\}$, where ω is the point at infinity. It is well-known that X is a compact metrizable space. Consider the components of $\Gamma(x)\cup\{\omega\}$. The component containing ω is denoted by C_ω. $C_\omega\setminus\{\omega\}$ is the union of the noncompact components of $\Gamma(x)\subset X$ ([5, p. 105]). We have to show that $\Gamma(x)=C_\omega\setminus\{\omega\}$. To the contrary, assume that $\Gamma(x)$ has a compact component, say C. Then C is a component of $\Gamma(x)\cup\{\omega\}$, either. Further, there exists ([5, p. 104]) open sets $U\subset\hat{X}$, $V\subset\hat{X}$ such that $U\cap V=\emptyset$, $C\subset U$, $C_\omega\cup\{\omega\}\subset V$, $\Gamma(x)\cup\{\omega\}\subset U\cup V$. Obviously, U has compact closure $K\subset X$ and $\partial K\cap\Gamma(x)=\emptyset$. The same argument used in the proof of Part a. yields to a contradiction. If the assumption concerning the dual *c-c* property is dropped, one can conclude only that the component containing x is not compact.

Q. E. D.

Example 4.3. Let $X=X^+\cup X^-\subset \boldsymbol{R}^2$ where $X^+=\{(x_1,x_2)\in\boldsymbol{R}^2|x_2=1\}$, $X^-=\{(x_1,x_2)\in\boldsymbol{R}^2\,|\,x_2=-1\}$. The point $(0, 1)\in X^+$ is denoted by y^*. We define a closed quasi order $\prec$ on X by letting $y\prec x$ if and only if $y=x$ or $y\in X^-$, $x\in X^-$ or $y=y^*$, $x\in X^-$. It is easy to show that $\prec$ has the *c-c-c* but not the dual *c-c* property and that for any $x\in X^-$, $\Gamma(x)$ consists of a noncompact component and of an isolated point, $\Gamma(x)=X^-\cup\{y^*\}$.

Remark 4.4. In virtue of Corollary 3.3, Proposition 4.1 and Proposition 4.2 a. imply that $M(x)$ is a non-void compact connected subset of X provided that $N(x)$ is compact.

Proposition 4.5. *Let X be a locally compact separable metrizable space and let $\prec$ be a closed quasi order on X. The corresponding equivalence relation is denoted by $\sim$. The equivalence class containing x is denoted by $\tilde{x}$. Assume that $\prec$ has either the c-c-c or the dual c-c-c property. Then $\tilde{x}$ is connected whenever it is compact. If it is not compact, then none of its components is compact. Assume, in addition, that $\tilde{x}$ is compact for each x in X. Then for each closed subset $F\subset X$, the union of points of equivalence classes that intersect F is closed.*

Proof. The proof may easily be constructed by the methods adopted for the proof of Proposition 4.2 and of a similar statement ([7]) about $\prec_A$. Q. E. D.

§ 5. Remetrization and Σ-recurrence

Let X be a compact metrizable space and let T: $\boldsymbol{R}\times X\to X$ be a dynamical system. We begin with the simple observation that $\prec_A$ and $\prec_C$ do not depend on the choice of the particular metric.

In what follows we show that the analogous statement for $\prec_\Sigma$ is false. More precisely, we consider the two-dimensional dynamical system S on the annulus $\mathscr{A}=\{(x_1, x_2)\in \boldsymbol{R}^2 \mid 1\leqslant x_1^2+x_2^2\leqslant 4\}$ defined (in polar coordinates) by the differential system $\dot{r}=0$, $\dot{\varphi}=1$. S can be defined explicitely by letting $S_t(r, \varphi)=(r, t+\varphi)$ for each $(r, \varphi)\in\mathscr{A}$, $t\in \boldsymbol{R}$. It is obvious that $(r_1, \varphi_1)\prec_A(r_2, \varphi_2)$ if and only if $r_1=r_2$. Further, for each (r_1, φ_1), $(r_2, \varphi_2)\in\mathscr{A}$, there holds $(r_1, \varphi_1)\prec_C(r_2, \varphi_2)$. Observe that $R_A=R_C=\mathscr{A}$. Our aim is to define two metrics on $\mathscr{A}$, say d and ρ, such that $\prec_\Sigma^d\neq\prec_\Sigma^\rho$ (The usual symbol $\prec_\Sigma$ is marked by "d" resp. "ρ" as we want to emphasize the dependendence on the metric. Of course we require that both d and ρ induce the usual Euclidean topology on $\mathscr{A}$). First we need a preliminary result.

Let H be the set of all triadic rationals in [0, 1], i.e. $H=\{k/3^n \mid k, n\in \boldsymbol{Z}^+$ and $k\leqslant 3^n\}$. For brevity, let $c_n^k=k/3^n$.

Lemma 5.1. *There exist continuous strictly increasing functions f, g: $[0, 1]\to[0, 1]$ with $f(0)=g(0)=0$, $f(1)=g(1)=1$ satisfying the following conditions*:

(1) $f(c_n^1)-f(0)<g(c_n^1)-g(0)$ for $n=1, 2,\cdots$

(2) $f(c_n^{k+1})-f(c_n^k)<g(c_n^{k+1})-g(c_n^k)$ if and only if $f(c_n^{k+2})-f(c_n^{k+1})>g(c_n^{k+2})-g(c_n^{k+1})$ for $k=0, 1,\cdots, 3^n-2$; $n=1, 2,\cdots$

(3) $\sum_{k=0}^{3^n-1}\min\{f(c_n^{k+1})-f(c_n^k), g(c_n^{k+1})-g(c_n^k)\}=(7/9)^n$.

Proof. We define f and g on H first. Let $f(0)=0$, $f(1/3)=1/3$, $f(2/3)=2/3$, $f(1)=1$, $g(0)=0$, $g(1/3)=4/9$, $g(2/3)=5/9$, $g(1)=1$. The induction step should now be apparent. Suppose that $f(c_n^k)$, $g(c_n^k)$ satisfying (1), (2), (3) have been defined for all members of H with denominators less than or equal to 3^N. Further, suppose that

(4) $f(c_n^{k+1})>f(c_n^k)$, $g(c_n^{k+1})>g(c_n^k)$ for $k=0, 1,\cdots, 3^n-1$, $n=1, 2,\cdots, N$.

For $k=0, 1,\cdots, 3^N-1$, we define $f(c_{N+1}^{3k+1})$, $f(c_{N+1}^{3k+2})$, $g(c_{N+1}^{3k+1})$, $g(_{N+1}^{3k+2})$ as follows. We distinguish two cases according as $f(c_N^{k+1})-f(c_N^k)>g(c_N^{k+1})-g(c_N^k)$ or $f(c_N^{k+1})-f(c_N^k)<g(c_N^{k+1})-g(c_N^k)$.

If $f(c_N^{k+1})-f(c_N^k)>g(c_N^{k+1})-g(c_N^k)$, let

$$f(c_{N+1}^{3k+1}) = 2^{-1}(f(c_N^k)+f(c_N^{k+1})) - 18^{-1}(g(c_N^{k+1}) - g(c_N^k)),$$

$$f(c_{N+1}^{3k+2}) = 2^{-1}(f(c_N^k)+f(c_N^{k+1})) + 18^{-1}(g(c_N^{k+1})-g(c_N^k)),$$
$$g(c_{N+1}^{3k+1}) = (2g(c_N^k)+g(c_N^{k+1}))/3,$$
$$g(c_{N+1}^{3k+2}) = (g(c_N^k)+2g(c_N^{k+1}))/3.$$

If $f(c_N^{k+1})-f(c_N^k)<g(c_N^{k+1})-g(c_N^k)$, let

$$f(c_{N+1}^{3k+1}) = (2f(c_N^k)+f(c_N^{k+1}))/3,$$
$$f(c_{N+1}^{3k+2}) = (f(c_N^k)+2f(c_N^{k+1}))/3,$$
$$g(c_{N+1}^{3k+1}) = 2^{-1}(g(c_N^k)+g(c_N^{k+1})) - 18^{-1}(f(c_N^{k+1})-f(c_N^k))$$
$$g(c_{N+1}^{3k+2}) = 2^{-1}(g(c_N^k)+g(c_N^{k+1})) + 18^{-1}(f(c_N^{k+1})-f(c_N^k)).$$

It is not hard to check that (1)–(4) are satisfied for all members of H of the form $k/3^{N+1}$, and this completes the induction. Further, it follows from an easy induction argument that

$$(5) \quad \max_{0\leqslant k\leqslant 3^n-1} \{f(c_n^{k+1})-f(c_n^k),\ g(c_n^{k+1})-g(c_n^k)\}<2^{-n}, \text{ for } n=1, 2,\cdots$$

Finally, for $x\in[0, 1]$, let $f(x)=\sup\{f(h)\mid h\in H,\ h\leqslant x\}$. Since the triadic rationals are dense in $[0, 1]$, (4) resp. (5) imply that f is strictly increasing resp. continuous. By the construction, f: $[0, 1]\to[0, 1]$ is a function with $f(0)=0$, $f(1)=1$ satisfying (1)–(3).

Similarly, for $x\in[0, 1]$, let $g(x)=\sup\{g(h)\mid h\in H,\ h\leqslant x\}$. Q. E. D.

Example 5.2. Consider the dynamical system S on $\mathscr{A}$. Let d be the restriction of the usual Euclidean metric onto $\mathscr{A}$. It is easy to show that $(r_1, \varphi_1)\prec_2^d (r_2, \varphi_2)$ if and only if $r_1=r_2$. Thus, $\prec_A=\prec_2^d$.

Example 5.3. Consider the dynamical system S on $\mathscr{A}$. For (r_1, φ_1), $(r_2, \varphi_2)\in\mathscr{A}$, let

$$\rho((r_1, \varphi_1), (r_2, \varphi_2)) = d(h(r_1, \varphi_1), h(r_2, \varphi_2))$$

where h: $\mathscr{A}\to\mathscr{A}$ is a homeomorphism defined by

$$h(r, \varphi) = \begin{cases} \left(1+\frac{\varphi}{\pi}f(r-1)+\frac{\pi-\varphi}{\pi}g(r-1),\ \varphi\right) & \text{if } \varphi(\mathrm{mod}\ 2\pi)\in[0, \pi] \\ \left(1+\frac{\varphi-\pi}{\pi}g(r-1)+\frac{2\pi-\varphi}{\pi}f(r-1),\ \varphi\right) & \text{if } \varphi(\mathrm{mod}\ 2\pi)\in[\pi, 2\pi] \end{cases}$$

It is clear that ρ is a metric on $\mathscr{A}$ topologically equivalent to d.

Fix an $n \in \boldsymbol{N}$ and for $k=0, 1, \cdots, 3^n$, define (in polar coordinates) $x_k=(1+c_n^k, k\pi)$. Observe that $x_0=(1, 0)$ $x_{3^n}=(2, \pi)$. Clearly $x_k \in \mathscr{A}$. For arbitrary $N \in \boldsymbol{N}$, we have

$$\begin{aligned}\rho(S_{(2N+1)\pi}x_k, x_{k+1}) &= d(h(S_{(2N+1)\pi}x_k), h(x_{k+1})) \\ &= d(h(1+c_n^k, (2N+1)\pi+k\pi), h(1+c_n^{k+1}, (k+1)\pi)) \\ &= \begin{cases} d((1+f(c_n^k), \pi), (1+f(c_n^{k+1}), \pi)) & \text{if } k=2m \\ d((1+g(c_n^k), 0), (1+g(c_n^{k+1}), 0)) & \text{if } k=2m+1 \end{cases} \\ &= \begin{cases} f(c_n^{k+1})-f(c_n^k) & \text{if } k=2m \\ g(c_n^{k+1})-g(c_n^k) & \text{if } k=2m+1 \end{cases}\end{aligned}$$

Consequently, in virtue of (1)–(3), we obtain that

$$\textstyle\sum_{k=0}^{3^n-1} \rho(S_{(2N+1)\pi}x_k, x_{k+1}) = (7/9)^n$$

Thus, by definition, $\{(2, \pi)\} \in M(\{(1, 0)\})$.

Similarly, $\sum_{k=0}^{3^n-1} \rho(S_{(2N+1)\pi}y_k, y_{k+1}) = (7/9)^n$, where (in polar coordinates) $y_k=(1+c_n^{3^n-k}, k\pi)$, $k=0, 1, \cdots, 3^n$.
Observe that $y_0=(2, 0)$, $y_{3^n}=(1, \pi)$. Thus, by definition, $\{(1, \pi)\} \in M(\{(2, 0)\})$.

Since $S_\pi(1, \pi)=(1, 0)$ and $S_\pi(2, \pi)=(2, 0)$, we conclude that $\{(1, 0)\} \in M(\{(2, 0)\})$, $\{(2, 0)\} \in M(\{(1, 0)\})$. Therefore, $(1, 0) \sim_\Sigma^\rho (2, 0)$. The results of Section 4 imply that the equivalence classes under $\sim_\Sigma^\rho$ are connected compact invariant subsets of $\mathscr{A}$. It follows easily that there is exactly one equivalence class under $\sim_\Sigma^\rho$. Consequently, $\prec_\Sigma^\rho = \prec_C$, in particulat, $\prec_\Sigma^\rho \neq \prec_\Sigma^d$.

Example 5.4. Concluding this paper, we show that, in general, $R_A \neq R_\Sigma$. The example we give below is a slight modification of Example 5.3.

By a gluing procedure, e.g. by applying Hausdorff's extension theorem for metrics ([8], [13]) it is not hard to construct a compact metric space (X, Δ) and a dynamical system T: $\boldsymbol{R} \times X \to X$ with the following properties:

a. $X = Y \cup \gamma$, $Y \cap \gamma = \emptyset$.

b. γ is homeomorphic to $\boldsymbol{R}$ and consists of a unique trajectory of T.

c. there exists a homeomorphism h of Y onto $\mathscr{A}$ such that $\Delta(x, y)=\rho(h(x), h(y))$, $h(T_t x)=S_t h(x)$ for each $x, y \in Y$, $t \in \boldsymbol{R}$.

d. $\mathrm{cl}(\gamma)=\gamma \cup \{h^{-1}(a) \mid a \in \partial\mathscr{A}\}$.

Here, of course, $\partial\mathscr{A}=\{(x_1, x_2) \in \boldsymbol{R}^2 \mid x_1^2+x_2^2=1 \text{ or } x_1^2+x_2^2=4\}$. By the construction, we have that $R_A = Y$, $R_\Sigma = X$.

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