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On the Existence of Global Solutions of a Coupled Nonlinear Klein-Gordon Equations

By

M. Milla MIRANDA and L. A. MEDEIROS

(Universidade Federal de Rio de Janeiro, Brasil)

Introduction

The motion of charged mesons in an electromagnetic field can be described by the following system:

$$(*)\quad \begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \Delta u + \alpha^2 u + g^2 v^2 u = 0, \\ &\frac{\partial^2 v}{\partial t^2} - \Delta v + \beta^2 v + h^2 v^2 v = 0 \end{aligned}$$

where α, β, g and h are non zero real constants. The system (*) was introduced by I. Segal [12]. A number of authors studied such systems, we can mention K. Jörgens [4] and V. G. Makhankov [7] among others. Recently, L. A. Medeiros–G. Perla Menzala [8] have analysed the existence of weak solutions of the mixed problem for the system (*) in $\Omega \times [0, T]$, where Ω is a bounded domain of $\boldsymbol{R}^n$, $n=1, 2, 3$. These results were generalized by the authors in [9] and [10] for the case where the nonlinear terms are of the form $|v|^{\rho+2}|u|^\rho u$, $|u|^{\rho+2}|v|^\rho v$ and Ω is any domain of $\boldsymbol{R}^n$. The decay of solutions of system (*) can be found in J. Ferreira–G. Perla Menzala [3].

There is no loss of generality if we consider all the constants of system (*) equal to one. In this paper we study the existence and uniqueness of weak solutions of the mixed problem for the system:

$$(**)\quad \begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \Delta u + u - |v|^{\rho+2}|u|^\rho u = f_1, \\ &\frac{\partial^2 v}{\partial t^2} - \Delta v + v - |u|^{\rho+2}|v|^\rho v = f_2, \end{aligned}$$

in $\Omega \times [0, T]$ where ρ is a real number satisfying certain conditions and Ω is any domain of $\boldsymbol{R}^n$.

We observe that the standard energy method to obtain solutions does not work for the system (**) because the nonlinear terms are negative. We determine the existence of weak solutions of the system (**) using the method of potential well introduced by D. H. Sattinger [11] (see also J. L. Lions [5], Y. Ebihara-

M. Nakao-T. Nanbu [2] and Y. Ebihara [1]) and an argument of the authors [10]. The uniqueness is obtained by a method due to M. I. Visik-O. A. Ladysenskaja [13], see also J. Lions-E. Magenes [6] and L. A. Medeiros-M. Milla Miranda [9].

§1. Notations and Main Results

Let Ω be a domain of $\boldsymbol{R}^n$. By $H^m(\Omega)$, m an integer non negative, we denote the usual Sobolev space of order m. For $m=0$, $H^0(\Omega)=L^2(\Omega)$. By $H_0^m(\Omega)$ we represent the closure in $H^m(\Omega)$ of the space $\mathscr{D}(\Omega)$, where $\mathscr{D}(\Omega)$ denotes the space of infinitely differentiable functions with compact support contained in Ω. The inner product and norm of $L^2(\Omega)$ and $H^1(\Omega)$ will be represented by $(\cdot,\cdot)$, $|\cdot|$ and $((\cdot,\cdot))$, $\|\cdot\|$, respectively.

Let $T>0$ be a real number and X a Banach space. We shall represent by $L^p(0, T; X)$, $1\leq p<\infty$, the Banach space of vector-valued functions u: $[0, T]\mapsto X$ which are measurable and $\|u(t)\|_X^p \in L^p(0, T)$ with the norm

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p},$$

and by $L^\infty(0, T; X)$ the Banach space of functions u: $]0, T[\mapsto X$ which are measurable and $\|u(t)\|_X \in L^\infty(0, T)$ with the norm

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X .$$

Finally, $\mathscr{D}'(Q)$ and $\mathscr{D}'(0, T)$, where $Q=\Omega\times[0, T]$, will denote the space of distributions on Ω and $[0, T]$, respectively. All the scalar functions considered in this paper will be real-valued.

We proceed formally. By multiplying the first equation of the system (**) by $\partial u/\partial t$ and the second by $\partial u/\partial t$, integrating on Ω and assuming that u and v vanishing on the boundary Γ of Ω, for each $t\in[0, T]$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left|\frac{\partial u}{\partial t}\right|^2 + \frac{1}{2}\frac{d}{dt}\|u\|^2 - \int_\Omega |v|^{\rho+2}|u|^\rho u \frac{\partial u}{\partial t}\, dx = \int_\Omega f_1 \frac{\partial u}{\partial t}\, dt$$

$$\frac{1}{2}\frac{d}{dt}\left|\frac{\partial v}{\partial t}\right|^2 + \frac{2}{1}\frac{d}{dt}\|v\|^2 - \int_\Omega |u|^{\rho+2}|v|^\rho v \frac{\partial v}{\partial t}\, dx = \int_\Omega f_2 \frac{\partial v}{\partial t}\, dx$$

Adding these expressions:

$$(1)\quad \frac{1}{2}\frac{d}{dt}\left(\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial v}{\partial t}\right|^2\right) + \frac{d}{dt}\left(\frac{1}{2}(\|u\|^2+\|v\|^2) - \frac{1}{\rho+2}\|uv\|_{L^{\rho+2}(\Omega)}^{\rho+2}\right)$$
$$= \int_\Omega \left[f_1 \frac{\partial u}{\partial t} + f_2 \frac{\partial v}{\partial t}\right] dx.$$

Thus the kinetic and potential energies associated with the system (**) are the functionals:

$$K([u, v]) = \frac{1}{2}\left(\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial v}{\partial t}\right|^2\right),$$

$$J([u, v]) = \frac{1}{2}(\|u\|^2 + \|v\|^2) - \frac{1}{\rho+2}\|uv\|^{\rho+2}_{L^{\rho+2}(\Omega)}.$$

We make the following restrictions on the real number ρ:

(2) $$p > -1 \quad \text{if} \quad n = 1, 2, \quad \text{and} \quad -1 < \rho \leq \frac{4-n}{n-2} \quad \text{if} \quad n \geq 3.$$

In this conditions,

(3) $H^1_0(\Omega)$ is continuously embedded in $L^{2(\rho+2)}(\Omega)$.

We define the potential energy $J([w, z])$ as the functional on $Y = H^1_0(\Omega) \times H^1_0(\Omega)$, ρ satisfying (2):

$$J([w, z]) = \frac{1}{2}(\|w\|^2 + \|z\|^2) - \frac{1}{\rho+2}\|wz\|^{\rho+2}_{L^{\rho+2}(\Omega)}$$

and the kinetic energy $K([\varphi, \psi])$ as the functional on $Z = L^2(\Omega) \times L^2(\Omega)$:

$$K([\varphi, \psi]) = \frac{1}{2}(|\varphi|^2 + |\psi|^2).$$

Lemma 1. We have

$$\inf_{\substack{[w,z]\in Y \\ w\neq 0 \text{ or } z\neq 0}} \{\sup_{\lambda\geq 0} J(\lambda[w, z])\} = d > 0.$$

The potential well W is then defined as the set

$$W = \{[w, z] \in Y;\ J(\lambda[w, z]) < d, \quad \forall \lambda \in [0, 1]\}.$$

We will show in §2 that $J(\lambda[w, z)] \to 0$, for all $\lambda \in [0, 1]$, whenever $[w, z] \in W$.

Let us obtain some inequalities. First examine the case $n \geq 3$. Let the real numbers:

(4) $$\theta = \frac{2n(\rho+2)}{(n-2)(\rho+2) + 2n(\rho+1)}, \quad \gamma = \frac{2n(\rho+2)}{(n+2)(\rho+2) - 2n(\rho+1)}$$

and

(5) $$\alpha = \frac{\rho+2}{(\rho+1)\theta}, \quad \beta = \frac{\rho+2}{(\rho+2) - (\rho+)\theta}.$$

By simple calculations we obtain:

$$\theta > 1, \quad \gamma > 1, \quad \frac{1}{\theta} + \frac{1}{\gamma} = 1 \tag{6}$$

and

$$\alpha > 1, \quad \beta > 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad \theta\beta = \frac{2n}{n-2}. \tag{7}$$

Let w, $z \in H_0^1(\Omega)$. Then, by Hölder's inequality we have:

$$\int_\Omega (|w|^{\rho+2}|z|^{\rho+2})^\theta dx \leq \left[\int_\Omega |wz|^{(\rho+1)\theta\alpha} dx\right]^{1/\alpha} \left[\int_\Omega |w|^{\beta\theta} dx\right]^{1/\beta}$$

$$= \|wz\|^{(\rho+1)\theta}_{L^{\rho+2}(\Omega)} \; w\|^\theta_{L^{\beta\theta}(\Omega)}$$

whence by the condition (2) on ρ, Sobolev's embedding theorem, (7) and the Schwarz's inequality it follows

$$\| |w|^{\rho+2}|z|^\rho z\|_{L^\theta(\Omega)} \leq c\|w\|^{\rho+2}_{H_0^1(\Omega)}\|z\|^{\rho+1}_{H_0^1(\Omega)}. \tag{8}$$

For the case $n=1$, 2, using the same above arguments we obtain:

$$\int_\Omega (|w|^{\rho+2}|z|^{\rho+1})^2 dx \leq \left[\int_\Omega |wz|^{2(\rho+2)} dx\right]^{\rho+1/\rho+2} \left[\int_\Omega |v|^{2(\rho+2)} dx\right]^{1/\rho+2}$$

whence

$$\| |w|^{\rho+2}|z|^\rho z\|_{L^2(\Omega)} \leq c\|w\|^{\rho+2}_{H_0^1(\Omega)}\|z\|^{\rho+1}_{H_0^1(\Omega)}, \quad \forall w,\ z \in H_0^1(\Omega). \tag{9}$$

Theorem 1. *Let Ω be any open set of $\mathbf{R}^n$, ρ satisfying the condition* (2) *and d the number determined by Lemma* 1. *Let*

$$[u_0, v_0] \in W, \quad u_1, v_1 \in L^2(\Omega) \quad \text{and} \quad f_1, f_2 \in L^2(0, T; L^2(\Omega)) \tag{10}$$

such that

$$E_0 + 2\left\{2E_0 + \left[2\int_0^T (|f_1| + |f_2|)dt\right]^2\right\}^{1/2} \int_0^T (|f_1| + |f_2|)dt < d \tag{11}$$

where E_0 is the initial total energy

$$E_0 = K([u_1, v_1]) + J([u_0, v_0]).$$

Then there exist functions u, v: $[0, T] \to L^2(\Omega)$ in the class

$$u, v \in L^\infty(0, T; H_0^1(\Omega)) \tag{12}$$

(13) $$u',\ v' \in L^\infty(0,\ T;\ L^2(\Omega)) \quad \left(u' = \frac{du}{dt}\right)$$

satisfying the nonlinear system

(14) $$u'' - \Delta u + u - |v|^{\rho+2}|u|^\rho u = f_1 \quad in \quad L^2(0,\ T;\ H^{-1}(\Omega) + L^\theta(\Omega))$$

(15) $$v'' - \Delta v + v - |u|^{\rho+2}|v|^\rho v = f_2 \quad in \quad L^2(0,\ T;\ H^{-1}(\Omega) + L^\theta(\Omega))$$

(*of* $n=1,\ 2$, *the above equalities are in* $L^2(0,\ T;\ H^{-1}(\Omega))$)

(16) $$u(0) = u_0, \quad v(0) = v_0$$

(17) $$u'(0) = u_1, \quad v'(0) = v_1\,.$$

Theorem 2. *Let* u, v *and* $\hat{u}$, $\hat{v}$ *be functions in the class* (12), (13) *satisfying* (14) *to* (17). *Then,* $u=\hat{u}$ *and* $v=\hat{v}$ *provided that* $\rho\geq 0$ *in case* $n=1$ *or* $n=2$; $u=\hat{u}$ *and* $v=\hat{v}$ *if* $\rho=0$ *in case* $n=3$.

Remark 1. By (12) and (13) we have that $u(0)\in H_0^1(\Omega)$ (see [6]). By (12) and (8) or (9) we obtain from the equation (14) that

$$u'' \in L^2(0,\ T;\ H^{-1}(\Omega) + L^\theta(\Omega))\,.$$

This condition together with (13) implies that $u'(0)\in L^2(\Omega)$. Analogous results for v. We conclude then that the initial conditions (16), (17) make sense.

§ 2. Proof of the Results

We observe by (1) that the standard energy method does not work in our problem because one has a negative term in the left hand side of (1). The potential well method permets us to obtain a priori estimates for the approximate solutions of the problem (14)–(17). The Lemma 1 is fundamental for it.

Proof of the Lemma 1. Let w, $z\in H_0^1(\Omega)$ with $w\neq 0$ or $z\neq 0$. We first examine the case $wz\neq 0$. Let $M(\lambda)$, $\lambda\in \boldsymbol{R}$, be the function

(18) $$M(\lambda) = J(\lambda[w,\ z]) = \frac{\lambda^2}{2}(\|w\|^2 + \|z\|^2) - \frac{1}{\rho+2}|\lambda|^{2\rho+4}\|wz\|^{\rho+2}_{L^{\rho+2}(\Omega)}$$

then all critical points of this function are $\lambda_0=0$ and

(19) $$\pm\lambda_1 = \pm\,\lambda_1([w,\ z]) = \pm\left(\frac{\|w\|^2 + \|z\|^2}{2\|wz\|^{\rho+2}_{L^{\rho+2}(\Omega)}}\right)^{1/2\rho+2}.$$

We have $(d/d\lambda)^2 M(0) > 0$ and

$$\frac{d^2}{d\lambda^2} M(\pm\lambda_1) = (\|w\|^2 + \|z\|^2) - (2\rho+3)(\|w\|^2 + \|z\|^2) < 0.$$

Thus, $\lambda_0 = 0$ and $\pm\lambda_1$ are the unique points where $M(\lambda)$ has a local minimum and a local maximum, respectively, being $M(0) = 0$ and

$$M(\pm\lambda_1) = \left(\frac{\rho+1}{\rho+2}\right)\left(\frac{\|w\|^2 + \|z\|^2}{2\|wz\|_{L^{\rho+2}(\Omega)}}\right)^{\frac{\rho+2}{\rho+1}}. \tag{20}$$

From the Schwarz's inequality and the condition (2) on ρ it follows that

$$\|wz\|_{L^{\rho+2}(\Omega)} \leq c\|w\|\,\|z\| \leq c(\|w\|^2 + \|z\|^2).$$

Therefore

$$\frac{\|w\|^2 + \|z\|^2}{\|wz\|_{L^{\rho+2}(\Omega)}} \geq \frac{1}{c}. \tag{21}$$

Consequently, by (20) and (21) we have

$$\sup_{\lambda\geq 0} J(\lambda[w, z]) = J(\lambda_1([w, z])) \geq c_0 > 0 \tag{22}$$

where c_0 is independent of w, $z \in H_0^1(\Omega)$, $wz \neq 0$ and $w \neq 0$ or $z \neq 0$.

For the case $wz = 0$, $w \neq 0$ or $z \neq 0$, we obtain that $M(\lambda)$ defined by (18) has a minimum in $\lambda_0 = 0$ and

$$\sup_{\lambda\geq 0} M(\lambda) = \infty.$$

From (22) and the above equality it follows the Lemma 1. Q.E.D.

Remark 2. We observe that if $[w, z] \in W$ then

$$M(\lambda) = J(\lambda[w, z]) \geq 0, \quad \forall\lambda \in [0, 1].$$

In fact, for the case $w \neq 0$ or $z \neq 0$, and $wz \neq 0$, we have that $\lambda_1([w, z]) > 1$, therefore $M(\lambda)$ in increasing in $[0, 1]$, which implies the result because $M(0) = 0$. For the other cases, $M(\lambda)$ is non decreasing in $[0, \infty)$ and by the same arguments it follows the result.

Next we obtain some properties of the set W which we will use in the proof of the Theorem 1.

By the continuous embedding (3) and Hölder's inequality we get:

Lemma 2. *Let $r > 0$ be a number satisfying the condition*

$$\frac{1}{2} r^2 + \frac{(C_0 r)^{2\rho+4}}{\rho+2} < d$$

where C_0 *is the constant of the continuous embedding of* $H_0^1(\Omega)$ *in* $L^{2(\rho+2)}(\Omega)$. *Then the open ball*

$$B = \{[w, z] \in 0;\ \|w\|^2 + \|z\|^2 < r^2\}$$

is contained in W.

Lemma 3. *Let*

$$W_* = \{[w, z] \in Y;\ \|w\|^2 + \|z\|^2 - 2\|wz\|_{L^{\rho+2}(\Omega)}^{\rho+2} > 0,\ J([w, z]) < d\}.$$

Then

$$W = W_* \cup \{[0, 0]\}.$$

Proof of the Leema 3. Let $[w, z] \in W$. Then

$$J(\lambda[w, z]) < d, \quad \forall\lambda \in [0, 1]. \tag{23}$$

Suppose that $w \neq 0$ or $z \neq 0$. First examine the case $wz \neq 0$. From (23) and (19) it follows that $\lambda_1([w, z]) > 1$, or

$$\|w\|^2 + \|z\|^2 - 2\|wz\|_{L^{\rho+2}(\Omega)}^{\rho+2} > 0. \tag{24}$$

By (23) and (24) we conclude that $[w, z]$ belongs to W_*. For the case $wz = 0$, it trivially follows that $[w, z] \in W_*$.

Let $[w, z] \in W_*$. First consider the case $wz \neq 0$. Then $\lambda_1([w, z]) > 1$. Therefore $M(\lambda)$ defined as in (18) is increasing in $[0, 1]$ and since $M(1) < d$ we conclude that $[w, z] \in W$. The same reasoning for the case $wz = 0$. Thus the Lemma 3 is proved. Q.E.D.

From the Lemma 2 and 3 it follows that :

Corollary 1. *The set* W *is open in* Y.

Lemma 4. *If* $[w, z] \in \partial W$ *then* $J([w, z]) = d$.

Proof. As $[w, z] \in \partial W$ there exists a sequence $([w_\nu, z_\nu])$ of elements of W that converges to $[w, z]$ in Y, therefore

$$J([w, z]) \leq d.$$

We make the proof by absurd. Suppose that

$$J([w, z]) < d. \tag{25}$$

As $[w, z] \notin W$ we have that $w \neq 0$ or $z \neq 0$. First examine the case $wz \neq 0$. From (19), (25) and the Lemma 3 it follows that

$$\|w\|^2 + \|z\|^2 - 2\|wz\|^{\rho+2}_{L^{\rho+2}(\Omega)} < 0.$$

Hence by Lemma 3 and above inequality, we have that $[w, z]$ belongs to ext W which is an absurd. Examine the case $wz=0$. As $w\neq 0$ or $z\neq 0$, $M(\lambda)=J(\lambda[w, z])$ is increasing in $[0, \infty)$. Then by (25) we have that $[w, z]\in W$ which is an absurd. Thus $J([w, z])=d$. Q. E. D.

Lemma 5. *The set W is bounded in Y.*

Proof. Let $[w, z]\in W$. Then by Remark 2,

$$\frac{\lambda^2}{2}(\|w\|^2+\|z\|^2) - \frac{\lambda^{2\rho+4}}{\rho+2}\|wz\|^{\rho+2}_{L^{\rho+2}(\Omega)} \geq 0\,, \quad \forall\lambda\in[0, 1].$$

Making $\lambda=1$ in this inequality we obtain:

$$\frac{1}{\rho+2}\|wz\|^{\rho+2}_{L^{\rho+2}(\Omega)} \leq \frac{1}{2}\ (\|w\|^2+\|z\|^2)\,. \tag{26}$$

By (26) we have

$$(\lambda^2-\lambda^{2\rho+4})(\|w\|^2+\|z\|^2) < 2d, \quad \forall\lambda\in[0, 1]\,. \tag{27}$$

Noting that $2\rho+4>2$, one obtain $a(\lambda)=\lambda^2-\lambda^{2\rho+4}>0$ for all $\lambda\in[0, 1]$. Then, for example, for $\lambda=1/2$, by (27) we get

$$\|w\|^2 + \|z\|^2 < \frac{2d}{a(1/2)} = c_1$$

and the proof of the Lemma 5 is complete. Q. E. D.

Proof of the Theorem 1. We divide the proof in four parts.

i) *Approximate Solutions.* Let $w_1, w_2,\cdots$ be a basis of the space $V=H^1_0(\Omega)\cap L^\gamma(\Omega)$ (γ defined by (4)). Let

$$u_m(t) = \sum_{j=1}^m g_{jm}(r)w_j, \quad v_m(t) = \sum_{j=1}^m h_{jm}(t)w_j$$

be defined as a solution of the system

$$(u''_m, w_j) + ((u_m, w_j)) - (|v_m|^{\rho+2}|u_m|^\rho u_m, w_j) = (f_1, w_j) \tag{28}$$

$$(v''_m, w_j) + ((v_m, w_j)) - |(u_m|^{\rho+2}|v_m|^\rho v_m, w_j) = (f_2, w_j) \quad j = 1, 2,\cdots, m \tag{29}$$

$$u_m(0) = u_{0m}, \quad u_{0m} \longrightarrow u_0 \quad \text{in} \quad H^1_0(\Omega) \tag{30}$$

$$v_m(0) = v_{0m}, \quad v_{0m} \longrightarrow v_0 \quad \text{in} \quad H^1_0(\Omega) \tag{31}$$

$$u'_m(0) = u_{1m}, \quad u_{1m} \longrightarrow u_1 \quad \text{in} \quad L^2(\Omega) \tag{32}$$

(33) $v_m'(0) = v_{1m}, \quad v_{1m} \longrightarrow v_1 \quad \text{in} \quad L^2(\Omega)$

where u_{0m}, v_{0m}, u_{1m} and v_{1m} belongs to the subspace $[w_1, \cdots, w_m]$ generated by the m first vectors of the basis (w_j). We observe that $u_m(t)$ and $v_m(t)$ are defined in $[0, T_m)$, $T_m > 0$.

ii) *A Priori Estimates.* By multiplying the equation (28) by $g'_{jm}(t)$ and adding from $j=0$ to $j=m$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2 + \frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 - \int_\Omega |v_m|^{\rho+2}|u_m|^\rho u_m u_m' dx = (f_1, u_m').$$

Analogously from (29),

$$\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2 + \frac{1}{2}\frac{d}{dt}\|v_m(t)\|^2 - \int_\Omega |u_m|^{\rho+2}|v_m|^\rho v_m v_m' dx = (f_2, v_m').$$

By adding the two last equalities and integratin from 0 to t, $0 < t < T_m$, we have:

(34)
$$\frac{1}{2}|u_m'(t)|^2 + \frac{1}{2}|v_m'(t)|^2 + J([u_m(t), v_m(t)]) \le$$
$$\le \int_0^t (|f_1||u_m'| + |f_2||v_m'|)d\tau + K([u_{1m}, v_{1m}]) + J([u_{0m}, v_{0m}]).$$

For the other side, as W is open and $[u_0, v_0]$ belongs to W it follows that there exists an integer m_0 such that

(35)
$$[u_{0m}, v_{0m}] \in W, \quad \forall m \ge m_0.$$

Also, by convergences (30)–(33), hypothesis (11) and observing that

$$J([u_{0m}, v_{0m}]) \longrightarrow J([u_0, v_0]),$$

we obtain

(36)
$$E_{0m} + 2\left\{2E_{0m} + \left[2\int_0^T (|f_1| + |f_2|)dt\right]^2\right\}^{1/2} \int_0^T (|f_1| + |f_2|)dt < d, \quad \forall m \ge m_0$$

where

$$E_{0m} = K([u_{1m}, v_{1m}]) + J([u_{0m}, v_{0m}]).$$

We want to prove that for all $m \ge m_0$

(37)
$$[u_m(t), v_m(t)] \in W, \quad \forall t \in [0, T_m).$$

We prove (37). Fix $m \ge m_0$ and use the notation $N(t) = [u_m(t), v_m(t)]$. Make the proof by absurd. Suppose that (37) is false. Then there exists $t^* \in [0, T_m)$ such that

$$N(t^*) \notin W.$$

Let

$$t_1 = \inf\{t^* \in [0, T_m);\ N(t^*) \notin W\}.$$

We have by Corollary 1, (35) and continuity of $N(t)$ on $[0, T_m)$ that $t_1 > 0$. We also obtain by the continuity of $N(t)$ on $[0, T_m)$ that $N(t_1) \in \partial W$, whence by Lemma 4,

$$J(N(t_1)) = d. \tag{38}$$

We note that for $t \in [0, t_1)$, $J(N(t)) \geq 0$ because $N(t) \in W$. Then from (34) for $t \in [0, t_1]$ it follows that

$$\frac{1}{2}|u'_m(t)|^2 + \frac{1}{2}|v'_m(t)|^2 \leq \leq \int_0^{t_1} \| (|f_1||u'_m| + |f_2||v'_m|)dt + E_{0m} \tag{39}$$

and

$$|u'_m(t)|\,|v'_m(t)| \leq \int_0^{t_1} (|f_1||u'_m| + |f_2||v'_m|)dt + E_{0m}. \tag{40}$$

We have

$$\int_0^{t_1} (|f_1||u'_m| + |f_2||v'_m|)dt \leq S\int_0^T (|f_1| + |f_2|)dt \tag{41}$$

where

$$S = \max_{0 \leq T \leq t_1} \{|u'_m(\tau)| + |v'_m(\tau)|\}.$$

By adding the inequalities (39), (40) and using the majoration (41) we obtain

$$\frac{1}{2}\{|u'_m(t)| + |v'_m(t)|\}^2 \leq 2E_{0m} + 2S\int_0^T (|f_1| + |f_2|)dt.$$

By taking the maximum of the left hand side of the above inequality on $[0, t_1]$, we have after simple calculations:

$$S \leq 2\left\{2E_{0m} + \left[2\int_0^T (|f_1| + |f_2|)dt\right]^2\right\}^{1/2}. \tag{42}$$

From (34) and (38) it follows that

$$d = N(t_1) \leq \int_0^{t_1} (|f_1||u'_m| + |f_2||v'_m|)d\tau + E_{0m}. \tag{43}$$

By using the majoration (41) in the right hand side of (43), the estimate (42) and (36) we obtain

$$d \leq E_{0m} + 2\left\{2E_{0m} + \left[2\int_0^T (|f_1|+|f_2|)dt\right]^2\right\}^{1/2} \int_0^T (|f_1|+|f_2|)dt < d$$

which is an absurd. Thus (37) holds.

We have by (37) and Lemma 5 that

(44) $(u_m)_{m\in\boldsymbol{N}}$ and $(v_m)_{m\in\boldsymbol{N}}$ are bounded in $L^\infty(0, T; H_0^1(\Omega))$.

By the above and noting that $J([u_m, v_m]) \geq 0$ we obtain from (34) that

(45) $(u'_m)_{m\in\boldsymbol{N}}$ and $(v'_m)_{m\in\boldsymbol{N}}$ are bounded in $L^\infty(0, T; L^2(\Omega))$.

Using the boundedness (44), (45) and applying the reasoning employed to obtain (8) or (9), we have:

(46) $(|v_m|^{\rho+2}|u_m|^\rho u_m)_{m\in\boldsymbol{N}}$, $(|u_m|^{\rho+2}|v_m|^\rho v_m)_{m\in\boldsymbol{N}}$ are bounded in $L^\infty(0, T; L^\theta(\Omega))$

for $n \geq 3$; and the same sequence are bounded in $L^\infty(0, T; L^2(\Omega))$ for the case $n=1, 2$.

We discuss the case $n \geq 3$. For the case $n=1, 2$ we will obtain similar results.

From the (44)–(46) it follows that there exist a subsequence of $(u_m)_{m\in\boldsymbol{N}}$ and a subsequence of $(v_m)_{m\in\boldsymbol{N}}$, still denoted by the same symbols, such that

(47) $u_m \longrightarrow u$ and $v_m \longrightarrow v$ weak star in $L^\infty(0, T; H_0^1(\Omega))$

(48) $u'_m \longrightarrow u'$ and $v'_m \longrightarrow v'$ weak star in $L^\infty(0, T; L^2(\Omega))$

(49) $|v_m|^{\rho+2}|u_m|^\rho u_m \longrightarrow \chi_1$ weak star in $L^\infty(0, T; L^\theta(\Omega))$

and

(50) $|u_m|^{\rho+2}|v_m|^\rho v_m \longrightarrow \chi_2$ weak star in $L^\infty(0, T; L^\theta(\Omega))$.

iii) *The Nonlinear Term.* We want to prove is this part that

$$\chi_1 = |v|^{\rho+2}|u|^\rho u \quad \text{and} \quad \chi_2 = |u|^{\rho+2}|v|^\rho v.$$

Let $\mathcal{O}$ be an open ball contained in Ω. By $R_{\mathcal{O}}h = Rh$ we shall denote the restriction of a function $h\colon \Omega \mapsto \boldsymbol{R}$ to $\mathcal{O}$.

We observe that

$$(Ru_m)' = Ru'_m \quad \text{and} \quad (Rv_m)' = Rv'_m.$$

From (44) and (45) it follows that

$$(Ru_m)_{m\in\boldsymbol{N}}, (Rv_m)_{m\in\boldsymbol{N}} \quad \text{are bounded in} \quad L^\infty(0, T; H^1(\mathcal{O}))$$

and

$$([Ru_m]')_{m\in N}, ([Rv_m]')_{m\in N} \text{ are bounded in } L^\infty(0, T; L^2(\mathcal{O})).$$

From the above, using Lions–Aubin's theorem and convergence (47), we obtain

$$Ru_m \longrightarrow Ru, \quad Rv_m \longrightarrow Rv \quad \text{in} \quad L^2(0, T; L^2(\mathcal{O}))$$

which implies

(51) $$Ru_m \longrightarrow Ru, \quad Rv_m \longrightarrow Rv \quad \text{a.e. in} \quad \mathcal{O} \times]0, T[.$$

By (46) we have

(52) $$(|Rv_m|^{\rho+2}|Ru_m|^\rho Ru_m)_{m\in N} \quad \text{is bounded in} \quad L^\theta(\mathcal{O}\times]0, T[)$$

and by (51),

(53) $$|Rv_m|^{\rho+2}|Ru_m|^\rho Ru_m \longrightarrow |Rv|^{\rho+2}|Ru|^\rho Ru \quad \text{a.e. in} \quad \mathcal{O} \times]0, T[.$$

From (52) and (53), using the Lions [5], Lemma 1.4, it follows that

$$|Rv_m|^{\rho+2}|Ru_m|^\rho Ru_m \longrightarrow |Rv|^{\rho+2}|Ru|^\rho Ru \quad \text{weak in} \quad L^\theta(\mathcal{O}\times]0, T[).$$

On the other side, by (49) we have that

$$|Rv_m|^{\rho+2}|Ru_m|^\rho Ru_m \longrightarrow R\chi_1 \quad \text{weak in} \quad L^\theta(\mathcal{O}\times]0, T[).$$

We conclude from the above two last convergences and observing that $\mathcal{O}$ was arbitrary that

(54) $$R_{\mathcal{O}}(|v|^{\rho+2}|u|^\rho u) = R_{\mathcal{O}}\chi_1 \quad \text{in} \quad L^\theta(\mathcal{O}\times[0, T])$$

for all open ball contained in Ω.

We observe by (8) that $|v|^{\rho+2}|u|^\rho u$ belongs to $L^\infty(0, T; L^\theta(\Omega))$. Let $\varphi \in \mathcal{D}(Q)$ and K the projection on $\mathbf{R}^n$ of the support of φ. K is a compact subset of Ω. Let $(\mathcal{O}_i)_{1\le i\le \ell}$ be a finite cover of K formed by open balls contained in Ω. Subordinated to $(\mathcal{O}_i)_{1\le i\le \ell}$, we construct a C^∞-partition of unity $(\alpha_i)_{1\le i\le \ell}$. Thus

$$\varphi(x, t) = \sum_{i=1}^{\ell} \alpha_i(x)\varphi(x, t), \quad \forall(x, t)\in K \times [0. T]$$

and

$$\operatorname{supp}(\alpha_i) \subset \mathcal{O}_i, \quad \forall i.$$

By (54) we have then

$$\int_Q |v|^{\rho+2}|u|^\rho u\varphi dxdt = \sum_{i=1}^{\ell}\int_{\mathcal{O}_i\times]0,T[} |v|^{\rho+2}|u|^\rho u\alpha_i\varphi dxdt =$$

$$= \sum_{i=1}^{\ell}\int_{\mathcal{O}_i\times]0,T[} \chi_1\alpha_i\varphi dxdt = \int_Q \chi_1\varphi dxdt$$

which implies that

(55) $$\chi_1 = |v|^{\rho+2}|u|^\rho u.$$

Analogously we obtain

(56) $$\chi_2 = |u|^{\rho+2}|v|^\rho v.$$

iv) *Passage to the Limit.* By multiplying both sides of the equation (28) by $\eta \in \mathscr{D}(0, T)$, integrating from 0 to T and passing to the limit, using the convergences (47)–(49) and the equality (55), we obtain

(57) $$\left(\int_0^T u'\eta' dt,\, w_j\right) + \left(\left(\int_0^T u\eta dt,\, w_j\right)\right) +$$
$$+ \left(\int_0^T |v|^{\rho+2}|u|^\rho u\eta dt,\, w_j\right) = \left(\int_0^T f_1\eta dt,\, w_j\right), \quad \forall j.$$

Let $\zeta \in \mathscr{D}(\Omega)$. Then ζ can be approximated in $H_0^1(\Omega) \cap L^\gamma(\Omega)$, consequently in $H_0^1(\Omega)$ and $L^2(\Omega)$, by finite linear combinations of the w_j. This implies that in (57) we can change w_j by ζ. Thus equation (14) is proved. Analogously by (50) and (56) we obtain the equation (15).

The initial conditions (16) and (17) are obtained from the convergences (47), (48) and the equations (14), (15). Thus the proof of the Theorem 1 is concluded.

Q. E. D.

Proof of the Theorem 2. Let u, v and $\hat{u}$, $\hat{v}$ satisfying the conditions of the Theorem 2. Let

$$U = u - \hat{u} \quad \text{and} \quad V = v - \hat{v}.$$

Then

(58) $$U'' - \Delta U + U + |\hat{v}|^{\rho+2}|\hat{u}|^\rho \hat{u} - |v|^{\rho+2}|u|^\rho u = 0$$

(59) $$V'' - \Delta V + V + |\hat{u}|^{\rho+2}|\hat{v}|^\rho \hat{v} - |u|^{\rho+2}|v|^\rho v = 0$$

$$U(0) = 0, \quad V(0) = 0$$

$$U'(0) = 0, \quad V'(0) = 0.$$

Remark 3. We observe that

$$U''(t) \in H^{-1}(\Omega) + L^\theta(\Omega) \quad \text{and} \quad U'(t) \in L^2(\Omega).$$

Therefore, does not make sense to calculate the duality $\langle U''(t), U'(t)\rangle$. Consequently, the energy method to prove the uniqueness of solutions does not work in this case. We make the proof of the Theorem 2 using a method introduced by M. I. Visik-O. A. Ladysenskaja [13], see also J. L. Lions-E. Magenes [6].

Let

$$\varphi(t) = \begin{cases} -\int_t^s U(\sigma)d\sigma & \text{for } 0 \leq t \leq s \\ 0 & \text{for } s < t \leq T \end{cases}$$

and

$$\psi(t) = \begin{cases} -\int_t^s V(\sigma)d\sigma & \text{for } 0 \leq t \leq s \\ 0 & \text{for } s < t \leq T \end{cases}$$

Then

$$\varphi,\ \psi \in L^\infty(0,\ T;\ H_0^1(\Omega)).$$

We take the inner product of the equation (58) with φ and the inner product of (59) with ψ. Therefore, it is possible to obtain estimates which should permit to have the uniqueness in some interval $[0, s_0]$. The way s_0 is constructed will be able to repeat the same technique up to arrive T. For a detailed proof of the Theorem 2 see [9]. Q. E. D.

Remark 4. If Ω is a bounded open set of $\boldsymbol{R}^n$, we can apply the same technique of the Theorem 1 and 2 to obtain existence and uniqueness of weak solutions of the mixed problem for the system

$$u'' - \Delta u - |v|^{\rho+2}|u|^\rho u = f_1,$$
$$v'' - \Delta v - |u|^{\rho+2}|v|^\rho v = f_2.$$

Remark 5. In natural form it is possible to obtain the same results of the Theorem 1 and 2 for the system (**), changing the nonlinear terms of this by $a(x)|v|^{\rho+2}|u|^\rho u$ and $a(x)|u|^{\rho+2}|v|^\rho v$, where $a \in L^\infty(\Omega)$.

References

[1] Ebihara, Y., On the global Cauchy problem for $\Box u + g^2 u - u^3 = f$, J. Math. Anal. App., **64** (1978), 398–406.

[2] Ebihara, Y., Nakao, M. and Nanbu, T., On the existence of global classical solution fo initial-boundary value problem for $\Box u - u^3 = f$, Pacific J. Math., **60** (1975), 63–70.

[3] Ferreira, J. and Perla Menzala, G., Decay of solutions of a system of nonlinear Klein-Gordon equations, Atas 21° Seminário Brasileiro de Análise, Brasília, May 1985.

[4] Jörgens, K., *Nonlinear wave equations*, University of Colorado, Department of Mathematics, 1970.

[5] Lions, J. L., *Quelques Méthodes de Résolution de Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[6] Lions, J. L. and Magenes, E., *Problèmes aux Limites Non Homogènes et Applications*, Vol. 1, Dunod, Paris, 1968.

[7] Makhankov, V. G., Dynamics of classical solutions in integrable systems, Physics Reports (Secion C of Physic Letters) **35** (1978), 1–128.

[8] Medeiros, L. A. and Perla Menzala, G., On a mixed problem for a class of nonlinear Klein-Gordon equations, Atas 21° Seminário Brasileiro de Análise, Brasília, May 1985.

[9] Medeiros, L. A. and Milla Miranda, M., Weak solutions for a system of nonlinear Klein-Gordon equations, to appear.

[10] Medeiros, L. A. and Milla Miranda, M., Weak solutions for a coupled nonlinear hyperbolic equations, to appear.

[11] Sattinger, D. H., On global solutions of nonlinear hyperbolic equations, Arch. Rational Mech. Anal. **30** (1968), 148–172.

[12] Segal, I., Nonlinear partial differential equations in Quantum Field Theory, Proc. Symp. Appl. Math. A.M.S. **17** (1965), 210–226.

[13] Visik, M. I. and Ladysenskaja, O. A., On boundary value problem for partial differential equations and certain class of operator equations, A.M.S. Translations, Serie 2, **10** (1958).

nuna adreso:
Instituto de Matemática-UFRJ
CP 68530-CEP 21944
Rio de Janeiro, Brasil

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