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Decay Estimates for Some Semilinear Wave Equations with Degenerate Dissipative Terms

By

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1. Introduction

In the present paper we are concerned with the decay estimates of the smooth solutions to the initial boundary value problem for the semilinear wave equations:

$$u_{tt} + Lu + a(x)u_t + f(x, u) = g(x, t) \quad \text{in} \quad \Omega \times (0, \infty) \tag{1}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad D^\gamma u|_{\partial\Omega} = 0, \quad |\gamma| = 0,\cdots, m-1, \tag{2}$$

where Ω is a bounded domain in $\boldsymbol{R}^N$ with C^∞-boundary $\partial\Omega$, and L is a self-adjoint uniformly elliptic operator of order $2m$, i.e.,

$$Lu = \sum_{\substack{|\alpha|\leqq m \\ |\beta|\leqq m}} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u)$$

and

$$\|u\|_H^2 \equiv \int_\Omega a_{\alpha\beta}(x) D^\alpha u D^\beta u dx \geqq k_0 \|u\|_{H_m}^2 \qquad \text{for} \quad u \in \overset{\circ}{H}_m \cap H_{2m}, \quad k_0 > 0,$$

with usual notation.

We assume also $a_{\alpha\beta} \in C^\infty(\bar{\Omega})$, $a \in C^\infty(\bar{\Omega})$ and $g \in C^\infty(\bar{\Omega} \times \boldsymbol{R}^+)$. Throughout this paper we assume $2m \geqq [N/2]+1$, and the functions considered are all real valued.

It is well-known that if $a(x) \geqq \varepsilon_0$ for some positive constant ε_0 and if $g(x, t)$ tends to 0 rapidly in a certain sense the energy of the solution u of (1)–(2):

$$E(u(t)) = \frac{1}{2}(\|u_t(t)\|_2^2 + \|u(t)\|_H^2)$$

(together with the escalated norms) decays exponentially to 0 as $t \to \infty$ under certain monotonicity and growth conditions on $f(u, x)$ (cf. Strauss [9], Nakao [3]). Here we shall investigate the decay propety of the solution of (1)–(2) under the weaker assumption $a(x) \geqq 0$ (possibly $a(x) = 0$ somewhere in Ω). When $L = -\Delta$ and $f(x, u) \equiv 0$ (linear equation) such problem has been considered by one of the authors in [5]. There it is shown under the assumption $a(x)^{-1} \in L^p$, $0 < p < \infty$, that

$$E(u(t)) \leqq C_k(1+t)^{-kp}$$

where k denotes certain index of regularity of solutions. Related result is obtained also by Russell [7] under a different condition on $a(x)$. The object of this paper is to extend the result of [5] to the semilinear case.

Let us state our precise assumptions on $f(x, u)$ and $a(x)$.

A1. $f(x, u) \in C^\infty(\bar{\Omega} \times \boldsymbol{R})$ and

$$\left| \frac{\partial^i}{\partial u^i} f(x, u) \right| \leqq k_1 a(x)(|u|^{\max(\alpha-i,0)}+1) \quad i=0, 1, 2, \tag{3}$$

for some $k_1>0$, and α such that $1 \leqq \alpha \leqq N/(N-2m)$ if $N>2m$ and $1 \leqq \alpha < \infty$ if $1 \leqq N \leqq 2m$. Moreover, f satisfies

$$f(x, u)u \geqq k_2 \int_0^u f(x, \eta)d\eta \geqq 0, \quad (x, u) \in \bar{\Omega} \times \boldsymbol{R} \tag{4}$$

for some $k_2>0$ (note that $f(x, 0) \equiv 0$).

A2. $a(x)$ is nonnegative on $\bar{\Omega}$, $a^{-1}(x)$ is defined a.e. $x \in \bar{\Omega}$ and $a^{-1}(\cdot) \in L^p(\bar{\Omega})$ for some $0<p<\infty$ (Since $a(x)$ is assumed to be smooth the hypotheses A2 is meaningful for $0<p<N/2$).

Under our assumption A1 the problem (1)–(2) admits a unique global smooth solution u for each smooth (u_0, u_1) if the datum $(u_0, u_1, g(x, 0))$ satisfies appropriate compatibility conditions (cf. Sather [8], Inoue [1], von Wahl [10], [11], [11], Nakao [4] etc.), and we treat only sufficiently smooth global solutions.

When $m=N=1$ our method can be applied to more general equations:

$$u_{tt} - u_{xx} + \sigma(x, u_t) + \beta(x, u) = f(x, t).$$

For details see [6]. But, for $N \geqslant 2m$ a new device is required (see Propositions 2–4).

2. Result and the proof

Our result reads as follows.

Theorem 1. *Let $k \geqq 2$ be an integer. Suppose, in addition to* A1, A2, *that*

$$\sum_{i=0}^{k-1} \sup_{t \in \boldsymbol{R}^+} \left\| \frac{\partial^i}{\partial t^i} g(x, t) \right\|_2 + \sum_{i=0}^{k} \int_0^\infty \int_\Omega a^{-1}(x) \left| \frac{\partial^i}{\partial t^i} g(x, t) \right|^2 dxdt < \infty$$

and

$$\delta_0(t)^2 \equiv \int_t^{t+1} \int_\Omega a^{-1}(x) |g(x, s)|^2 dxds = o(t^{-1-2kmp/N})$$

as $t \to \infty$.

Then for a smooth solution $u(x, t)$ *of* (1)–(2) *we have*

$$E(u(t)) \equiv \frac{1}{2}(\|u_t(t)\|_2^2 + \|u(t)\|_H^2) + \int_\Omega \int_0^u f(x, \eta)\, dxd\eta$$
$$\leqq C_k(1+t)^{-2kmp/N}, \tag{5}$$

where C_k *denotes a constant depending on* $\|u_0\|_{H_{(k+1)m}}$ *and* $\|u_1\|_{H_{km}}$.

For the proof we utilize the following lemmas.

Lemma 1 (Gagliard & Nirenberg). *It holds that*

$$\|u\|_q \leqq \text{const}\, \|u\|_{H_{jm}}^\theta \|u\|_r^{1-\theta} \qquad \text{for} \quad u \in H_{jm} \tag{6}$$

with $\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{jm}{N} + \frac{1}{r} - \frac{1}{2}\right)^{-1}$, *where* j *is a positive integer,*

$$1 \leqq r < q < 2N/(N-2jm) \quad \text{if} \quad N > 2jm$$
$$1 \leqq r < q < \infty \qquad\qquad \text{if} \quad N = 2jm$$

and

$$1 \leqq r \leqq q \leqq \infty \quad \text{if} \quad 1 \leqq N < 2jm.$$

($\|\cdot\|_r$ *denotes* $L^r(\Omega)$*-norm*).

Lemma 2 ([2]). *Let* $\phi(t)$ *be a nonnegative bounded function on* $\boldsymbol{R}^+ = [0, \infty)$ *satisfying*

$$\sup_{t \leqq s \leqq t+1} \phi(s)^{1+\alpha} \leqq C(\phi(t) - \phi(t+1)) + g(t) \tag{7}$$

with $\alpha > 0$, $C > 0$ *and* $g(t) = o(t^{-1-1/\alpha})$ *as* $t \to \infty$.
Then we have

$$\phi(t) \leqq C'(1+t)^{-1/\alpha}$$

where C' *is a constant depending on* $\phi(0)$ *and other known constants.*

The proof of Theorem 1 will be given after a series of propositions. Let us begin with :

Proposition 1. *Under* A1, A2 *a smooth solution of* (1)–(2) *satisfies*

$$\sup_{t \leqq s \leqq t+1} E(u(s)) \leqq C\{D(t)^{2p/(p+1)} \sup_{t \leqq s \leqq t+1} \|u_t(s)\|_\infty^{2/(p+1)} + D(t)^2 + \delta_0(t)^2\} \tag{8}$$

where we set

$$D(t)^2 = C\{E(u(t)) - E(u(t+1))\} + C\delta_0(t)^2, \quad C > 0.$$

Proof. The proof is essentially included in [6]. For completeness, however, we reproduce it briefly.

Multiplying the equation (1) by u_t and integrating over $[t, t+1] \times \Omega$ we have

$$\int_t^{t+1} \int_\Omega a(x)\,|u_t|^2 dxds \leqq 2\{E(u(t)) - E(u(t+1))\} + \delta_0(t)^2 \equiv D(t)^2. \tag{9}$$

Hence,

$$\begin{aligned}
\int_t^{t+1} \int_\Omega |u_t|^2\, dxds &\leqq \int_t^{t+1} \left(\int_\Omega a^{-p}(x) dx \right)^{1/(p+1)} \\
&\quad \times \left(\int_\Omega a(x)|u_t|^{2(p+1)/p} dx \right)^{p/(p+1)} ds \\
&\leqq \|a^{-1}\|_p^{p/(p+1)} \left(\int_t^{t+1} \int_\Omega a(x)|u_t|^2 dxds \right)^{p/(p+1)} \\
&\quad \times \sup_{t \leqq s \leqq t+1} \|u_t(s)\|_\infty^{2/(p+1)} \\
&\leqq CD(t)^{2p/(p+1)} \sup_{t \leqq s \leqq t+1} \|u_t(s)\|_\infty^{2/(p+1)} \\
&\equiv A(t)^2.
\end{aligned} \tag{10}$$

By (10) we see that there exist $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that $\|u_t(t_i)\|_2 \leqq 2A(t)$, $i=1, 2$. Then, multiplying the equation by u and integrating over $[t_1, t_2] \times \Omega$ we have

$$\begin{aligned}
\int_{t_1}^{t_2} \left[\|u(s)\|_H^2 + \int_\Omega f(x, u) u dx \right] ds &\leqq (u_t(t_1), u(t_1)) - (u_t(t_2), u(t_2)) \\
&\quad + \int_{t_1}^{t_2} \|u_t\|_2^2 ds + \int_{t_1}^{t_2} \int_\Omega a(x)\,|u_t|\,|u| dxds \\
&\quad + \int_{t_1}^{t_2} \int_\Omega |gu| dxds \\
&\leqq C\{ \sup_{t \leqq s \leqq t+1} \sqrt{E(u(s))}\, A(t) + A(t)^2 + \delta_0(t)^2 \} \\
&\quad + \frac{1}{2} \int_{t_1}^{t_2} \|u(s)\|_H^2 ds.
\end{aligned} \tag{11}$$

It follows from (4), (10) and (11) that

$$\int_{t_1}^{t_2} E(u(s)) ds \leqq C\{ \sup_{t \leqq s \leqq t+1} \sqrt{E(u(s))}\, A(t) + A(t)^2 + \delta_0(t)^2 \}. \tag{12}$$

Therefore, there exists $t^* \in [t_1, t_2]$ such that $E(u(t^*))$ is bounded by the right hand side of (12) (with a larger C) and hence, by the energy identity,

$$\sup_{t\leqq s\leqq +1} E(u(s)) \leqq E(u(t^*)) + \int_t^{t+1}\int_\Omega a(x)\,|u_t|^2 dxds + \int_t^{t+1}\int_\Omega |g|\,|u_t| dxds$$
$$\leqq C\{\sup_{t\leqq s\leqq t+1}\sqrt{E(u(s))}\,A(t) + A(t)^2 + D(t)^2 + \delta_0(t)^2\}$$

which yields immediately (8).

Proposition 2. *In addition to* A1, A2, *suppose that*

$$\sup_{t\in \boldsymbol{R}^+} \|g(x, t)\|_2 < \infty \quad \textit{and} \quad \int_0^\infty\int_\Omega a^{-1}(x)(|g|^2 + |g_t|^2) dxds < \infty.$$

Then

$$E(u_t(t)) \equiv \frac{1}{2}(\|u_{tt}(t)\|_2^2 + \|u_t(t)\|_{\mathring{H}}^2) \leqq C_1 < \infty \qquad \textit{for} \quad t\in \boldsymbol{R}^+$$

and

$$\sup_{t\in \boldsymbol{R}^+} \|u(t)\|_{H_{2m}} + \int_0^\infty\int_\Omega a(x)\,|u_{tt}|^2 dxdt \leqq C_1 < \infty$$

where C_1 is a constant depending on $\|u_0\|_{H_{2m}}$ and $\|u_1\|_{\mathring{H}_m}$.

Proof. We consider the case $N>2m$; the other case can be proved in a similar and simpler manner.

First we observe that

$$E(u(t)) + \frac{1}{2}\int_0^t\int_\Omega a(x)u_t^2 dxds \leqq E(u(0)) + \int_0^\infty\int_\Omega a^{-1}(x)\,|g(s)|^2 dxds$$
$$\leqq C_0 < \infty. \tag{14}$$

Next, consider the differentiated equation

$$u_{ttt} + Lu_t + a(x)u_{tt} + f_u(x, u)u_t = g_t(x, t). \tag{15}$$

Multiplying (15) by u_{tt} yields

$$E(u_t(t)) + \frac{1}{2}\int_0^t\int_\Omega a(x)\,|u_{tt}|^2 dxds$$
$$\leqq E(u_t(0)) + C\int_0^t\int_\Omega a(x)(|u|^{\alpha-1}+1)\,|u_t|\,|u_{tt}| dxds$$
$$+ \frac{1}{2}\int_0^t\int_\Omega a(x)^{-1}|g_t(x, s)|^2 dxds$$

$$\leqq E(u_t(0)) + C\left(\int_0^t \int_\Omega a(x)|u_{tt}|^2 dxds\right)^{1/2}$$
$$\times\left(\int_0^t \int_\Omega a(x)(|u|^{2(\alpha-1)}+1)\,|u_t|^2 dxds\right)^{1/2}$$
$$+\frac{1}{2}\int_0^\infty \int_\Omega a(x)^{-1}|g_t|^2 dxds$$

and hence, by (14),

$$E(u_t(t)) + \frac{1}{4}\int_0^t \int_\Omega a(x)\,|u_{tt}|^2 dxds$$
$$\leqq E(u_t(0)) + C\int_0^t \|u(s)\|_\infty^{2(\alpha-1)}\int_\Omega a(x)\,|u_t(t)|^2 dxds + C_0. \tag{16}$$

$\|u(s)\|_\infty$ in the integrand of (16) is treated as follows. By Lemma 1 and (14),

$$\|u(s)\|_\infty \leqq C\|u(s)\|_{H_{2m}}^{\theta_0}\|u(s)\|_{2N/(N-2m)}^{1-\theta_0} \quad (\theta_0=(N-2m)/2m)$$
$$\leqq C\|u(s)\|_{H_{2m}}^{\theta_0}E(u(s))^{(1-\theta_0)/2} \leqq C\|u(s)\|_{H_{2m}}^{\theta_0}. \tag{17}$$

Furthermore by virture of elliptic regularity theorem,

$$\|u(s)\|_{H_{2m}} \leqq C(\|Lu\|_2+\|u\|_2)$$
$$\leqq C(\|u_{tt}\|_2+\|u_t\|_2+\|u\|_{2\alpha}^\alpha+1+\|g(s)\|_2)$$
$$\leqq C_0(\sqrt{E(u_t(t))}+1) \tag{18}$$

where we have used again (14) and the fact that $\|u(s)\|_{2\alpha}\leqq C\|u(s)\|_{\mathring{H}_m}$.

Thus we have from (16)–(18) that

$$E(u_t(t)) + \frac{1}{4}\int_0^t \int_\Omega a(x)\,|u_{tt}|^2 dxds$$
$$\leqq E(u_t(0)) + C_0\int_0^t E(u_t(s))^{\theta_0(\alpha-1)}\int_\Omega a(x)\,|u_t|^2 dxds + C_0. \tag{19}$$

Since $\theta_0(\alpha-1)\leqq 1$ by our assumption on α, the inequality (19) together with (14) implies

$$E(u_t(t)) \leqq C(E(u_t(0)),\, C_0) < \infty,$$

and consequently

$$\|u(t)\|_{H_{2m}} + \int_0^\infty \int_\Omega a(x)\,|u_{tt}|^2 dxds \leqq C(E(u_t(0),\, C_0)) < \infty.$$

Noting that $E(u_t(0))\leqq C(\|u_0\|_{H_{2m}},\ \|u_1\|_{\mathring{H}_m})$, the proof is complete.

Proposition 3. *Suppose, in addition to* A1, A2, *that*

$$\sup_{t\in\mathbf{R}^+}(\|g(t)\|_2+\|g_t(t)\|_2)+\sum_{i=0}^2\int_0^\infty\int_\Omega a^{-1}(x)\left|\frac{\partial^i}{\partial t^i}g(x,s)\right|^2 dxds<\infty.$$

Then, we have for a smooth solution u of (1)–(2),

$$E(u_{tt}(t))\leqq C_2<\infty,\qquad \textit{for any}\quad t\in\mathbf{R}^+$$

and

$$\sup_{t\in\mathbf{R}^+}\|u_t(t)\|_{H_{2m}}+\int_0^\infty\int_\Omega a(x)|D_t^3u|^2dxds\leqq C_2<\infty\quad\left(D_t^3=\frac{\partial^3}{\partial t^3}\right)$$

where C_2 denotes a constant depending on $\|u_0\|_{H_{3m}}$ and $\|u_1\|_{H_{2m}}$.

Proof. Differentiating the equation (1) twice with respect to t we have

$$D_t^4u+LD_t^2u+a(x)D_t^3u+f_{uu}(x,u)u_t^2+f_u(x,u)u_{tt}=g_{tt}(x,t)\tag{20}$$

$\left(D_t^i\equiv\dfrac{\partial^i}{\partial t^i}\right)$. The proof will be given by applying a similar argument as in the proof of Proposition 2 to (20). We sketch it briefly.

Multiplication by D_t^3u yields

$$\begin{aligned}E(u_{tt}(t))&+\frac{1}{2}\int_0^t\int_\Omega a(x)|D_t^3u|^2dxds\\&\leqq E(u_{tt}(0))+C\left\{\int_0^t\int_\Omega a(x)(|u|^{2\max(\alpha-2,0)}+1)|u_t|^4dxds\right.\\&\quad\left.+\int_0^t\int_\Omega a(x)|u_{tt}|^2|u|^{2(\alpha-1)}dxds+\int_0^t\int_\Omega a^{-1}(x)g_{tt}^2\,dxds\right\}.\end{aligned}\tag{21}$$

Here

$$\begin{aligned}\int_0^t\int_\Omega a(x)|u_{tt}|^2|u|^{2(\alpha-1)}dxds&\leqq\sup_{t\in\mathbf{R}^+}\|u(t)\|_\infty^{2(\alpha-1)}\\&\qquad\times\int_0^\infty\int_\Omega a(x)|u_{tt}|^2dxds\\&\leqq C_1<\infty,\ \text{(by Proposition 2)}\end{aligned}$$

and

$$\begin{aligned}&\int_0^t\int_\Omega a(x)(|u|^{2\max(\alpha-2,0)}+1)|u_t|^4dxds\\&\qquad\leqq C_1\int_0^t\|u_t(s)\|_\infty^2\int_\Omega a(x)|u_t|^2dxds.\end{aligned}$$

Moreover we see (cf. (17), (18))

$$\begin{aligned}\|u_t(s)\|_\infty &\leqq C_0(\|Lu_t(s)\|_2+\|u_t(s)\|_2)^{\theta_0} \quad (\theta_0=(N-2m)/2m)\\ &\leqq C_0\{\|D_t^3u(s)\|_2+\|u_{tt}(s)\|_2+(\|u(s)\|_\infty^{\alpha-1}+1)\|u_t(s)\|_2\\ &\quad+\|g_t(x,s)\|_2\}^{\theta_0}\\ &\leqq C_1(1+\sqrt{E(u_{tt}(s))})^{\theta_0}.\end{aligned}$$

Thus we obtain

$$E(u_{tt}(t)) \leqq E(u_{tt}(0)) + C_1\left\{1+\int_0^t E(u_{tt}(s))^{\theta_0}\int_\Omega a(x)|u_t|^2dxds\right\}$$

which implies (note that $0<\theta_0<1$)

$$E(u_{tt}(t)) \leqq C(E(u_{tt}(0)),\, C_1) \leqq C_2 < \infty,$$

hence the estimates for $\|u_t(t)\|_{H_{2m}}$ and $\int_0^\infty\int_\Omega a(x)|D_t^3u|^2dxds$.

Since $\|u_t(s)\|_\infty \leqq C\|u_t(s)\|_{H_{2m}}^{\theta_1}\|\,\|u_t(s)\|_2^{1-\theta_1}$ with $\theta_1=N/4m<1$, we have, by Proposition 3,

$$\|u_t(s)\|_\infty \leqq C_2E(u(s))^{(1-\theta_1)/2}.$$

Substituting this into the inequality (8) in Proposition 1 we have

$$\sup_{t\leqq s\leqq t+1} E(u(s)) \leqq C_2\{D(t)^{2p/(p+1)} \sup_{t\leqq s\leqq t+1} E(u(s))^{(1-\theta_1)/(p+1)}+D(t)^2+\delta_0(t)^2\}.$$

Recalling the definition of $D(t)$ and using the boundedness of $D(t)$ we arrive at the inequality

$$\sup_{t\leqq s\leqq t+1} E(u(s))^{1+\theta_1/p} \leqq C_2\{E(u(t))-E(u(t+1))+\delta_0(t)^2+\delta_0(t)^{2(p+\theta_1)/p}\}. \tag{22}$$

Now, applying Lemma 2 to (22) we obtain

$$E(u(t)) \leqq C_2(1+t)^{-4mp/N}$$

if $\delta_0(t)^2=o(t^{-1-4mp/N})$ as $t\to\infty$, which completes the proof of Theorem 1 for the case $k=2$.

In order to get further estimates we must establish the boundedness of $E(D_t^ju(t))$, $j=3, 4,\cdots, k$. For this we utilize j-times differentiated (w.r.t. t) equations:

$$U_{tt} + LU + a(x)U_t + \frac{\partial^j}{\partial t^j}f(x, u(t)) = D_t^jg \tag{23}$$

where $U\equiv D_t^ju$, $j=3, 4,\cdots, k+1$. Note that $U|_{\partial\Omega}=0$ for $t\in \boldsymbol{R}^+$ (by the compatibility condition) and $\partial^j/\partial t^j f(x, u(t))$ is caluculated as follows:

$$\frac{\partial^j}{\partial t^j} f(x, u(t)) = \sum_{i=1}^{j} \sum_\sigma C_\sigma \frac{\partial^i}{\partial t^i} f(x, u) u_t^{\sigma_1} \cdots (D_t^j u)^{\sigma_j} \tag{24}$$

with certain integers C_σ, $\sigma=(\sigma_1,\cdots,\sigma_j)$, where the summation $\sum_\sigma$ is taken over the set

$$\{\sigma=(\sigma_1,\cdots,\sigma_j) \in N^j \mid \sigma_1+\cdots+\sigma_j=i \text{ and } \sigma_1+2\sigma_2+\cdots+j\sigma_j=j\}.$$

Proposition 4. *Let $k \geqq 2$ be an integer. In addition to* A1, A2, *suppose that*

$$\sum_{i=0}^{k-1} \sup_{t\in \boldsymbol{R}^+} \left\| \frac{\partial^i}{\partial t^i} g(t) \right\|_2 + \sum_{i=0}^{k} \int_0^\infty \int_\Omega a^{-1}(x) \left| \frac{\partial^{i+1}}{\partial t^{i+1}} g(x, t) \right|^2 dxdt \quad < \infty. \tag{25}$$

Then we have

$$E(D_t^j u(t)) \leqq C_j < \infty, \quad j = 1, 2, \cdots, k, \tag{26}$$

and

$$\begin{aligned} &\sum_{j=1}^{k+1} \int_0^\infty \int_\Omega a(x) |D_t^j u(t)|^2 dxdt + \sum_{j=0}^{k-1} \sup_{t\in \boldsymbol{R}^+} \|D_t^j u(t)\|_{H_{(k+1-j)m}} \\ &\leqq C_k < \infty \end{aligned} \tag{27}$$

where C_k denotes a constant depending on $\|u_0\|_{H_{(k+1)m}}$ and $\|u_1\|_{H_{km}}$.

Proof. The proof is given by induction. The case $k=2$ is the contents of propositions 2 and 3. Suppose that the estimates (26), (27) are valid for arbitrarily fixed k ($\geqq 2$).

Multiplying the equation (23) with $j=k+1$ by $U_t \equiv D_t^{k+2} u$ and integrating we have

$$\begin{aligned} &E(U(t)) + \frac{1}{2} \int_0^t \int_\Omega a(x) |U_t|^2 dxds \\ &\leqq E(U(0)) + C \sum \int_0^t \int_\Omega a(x) \left| \frac{\partial^i}{\partial u^i} f(x, u) \right| |u_t|^{\sigma_1} \cdots |D_t^{k+1} u|^{\sigma_{k+1}} |U_t| dxds \\ &\quad + \frac{1}{2} \int_0^t \int_\Omega a^{-1}(x) |D_t^{k+1} g|^2 dxds \\ &\leqq E(U(0)) + C_k \sum \int_0^t \int_\Omega a(x) |u_t|^{\sigma_1} \cdots |D_t^{k+1} u|^{\sigma_{k+1}} |U_t| dxds \\ &\quad + \frac{1}{2} \int_0^t \int_\Omega a^{-1}(x) |D_t^{k+1} g|^2 dxds \\ &\leqq E(U(0)) + C_k \sum_{i=1}^{k+1} \int_0^t \int_\Omega a(x) |D_t^i u| |U_t| dxds \\ &\quad + \frac{1}{2} \int_0^t \int_\Omega a^{-1}(x) |D_t^{k+1} g|^2 dxds \end{aligned}$$

where we have used the facts $\|D_t^i u\|_\infty \leqq C_k$ if $0 \leqq i \leqq k-1$, $0 \leqq \sigma_k \leqq 1$ and $0 \leqq \sigma_{k+1} \leqq 1$ for $k \geqq 2$. Therefore we conclude

$$E(U(t)) + \frac{1}{4}\int_0^t \int_\Omega a(x)|U_t|^2 dxds \leqq E(U(0))$$
$$+ C_k \sum_{i=1}^{k+1} \int_0^\infty \int_\Omega a(x)|D_t^i u|^2 dxds + C\int_0^\infty \int_\Omega a^{-1}(x)|D_t^{k+1} g|^2 dxds$$
$$\leqslant C(E(U(0)), \|u_0\|_{H_{(k+1)m}}, \|u_1\|_{H_{km}}) < \infty.$$

Now, it is not difficult to see

$$E(U(0)) \leqq C(\|u_0\|_{H_{(k+2)m}}, \|u_1\|_{H_{(k+1)m}}).$$

Therefore,

$$\sup_{t \in \boldsymbol{R}^+} E(D_t^{k+1} u(t)) + \int_0^\infty \int_\Omega a(x)|D_t^{k+2} u|^2 dxds \leqq C_{k+1} < \infty.$$

From this and the boundedness of $\|D_t^{k+1} g(t)\|_2$ we can obtain also, in virtue of the elliptic regularity theory,

$$\sum_{i=0}^k \|D_t^i u(t)\|_{H_{(k+2-i)m}} \leqq C_{k+1} < \infty.$$

Thus the required estimates for $k+1$ are established.

Now, we are in a position to prove our Theorem 1 easily. In fact, since

$$\|u_t(t)\|_\infty \leqq C\|u_t(t)\|_{H_{km}}^{\theta_k} \|u_t\|_2^{1-\theta_k}$$
$$\leqq C_k E(u(t))^{(1-\theta_k)/2}$$

with $\theta_k = N/km$, we can obtain from (8) (see (22))

$$\sup_{t \leqq s \leqq t+1} E(u(s))^{1+\theta_k/p} \leqq C_k\{E(u(t)) - E(u(t)) + C\delta_0(t)^2\}$$

which yields by Lemma 2 the final estimate

$$E(u(t)) \leqq C_k(1+t)^{-2kmp/N}$$

if $\delta_0(t)^2 = o(t^{-1-2kmp/N})$ as $t \to \infty$.

References

[1] Inoue, A., Sur $\Box u + u^3 = f$ dans un domaine noncylindrique, J. Math. Anal. Appl., **46** (1974), 771–819.

[2] Nakao, M., Convergence of solutions of the wave equation with nonlinear dissipative term to the steady state, Mem. Fac. Sci. Kyushu Univ., **30** (1976), 257–265.

[3] Nakao, M., Decay of classical solutions of a semilinear wave equation, Math. Rep.

College Gen. Ed. Kyushu Univ., **11** (1977), 39–45.
[4] Nakao, M., Bounded, periodic and almost periodic classical solutions of the wave equation with a dissipative term, J. Math. Soc. Japan **30** (1978), 375–394.
[5] Nakao, M., Energy decay for the wave equation with a degenerate dissipative term, Proc. Royal Soc. Edinburgh, **100A** (1985), 19–27.
[6] Nakao, M., Energy decay for the nonlinear wave equations with degenerate dissipative terms, Patterns and waves —Qualitative Anal. Nonlinear P.D.E.— (1986), 583–596.
[7] Russell, D. L., Decay rates for weakly damped systems in Hilbert space, J. Differential Equations, **44** (1975), 334–370.
[8] Sather, J., The existence of global classical solutions of the initial-boundary value problem for $\Box u+u^3=f$. Arch. Rat. Mech. Anal., **22** (1966), 292–307.
[9] Strauss, W. A., Energy Method for Nonlinear Partial Differential Equations, Notas Math. Brazil, L969.
[10] von Wahl, W., Klassische Lösungen nichtlinearer Wellengleichungen im Grossen. Math. Z., **112** (1969), 241–279.
[11] von Wahl, W., Regular solutions of initial-boundary value problem for linear and nonlinear wave equations 11, Math. Z. **142** (1975), 121–130.

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