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Stokes and Gevrey Phenomena in Relation to Index Theorems in the Theory of Meromorphic Linear Difference Equations

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Introduction

This paper deals with the meromorphic linear difference equation

$$a_m(z)f(z+m)+a_{m-1}(z)f(z+m-1)+\cdots+a_0(z)f(z)=g(z),$$

where the functions $a_m, \cdots, a_0$ and g are holomorphic in a full neighborhood of infinity. If is well known that the equation admits m linearly independent formal solutions (see for instance Birkhoff [3], Turrittin [13], Duval [5] or Praagman [10]).

Having determined a full set of formal solutions the question arises whether these solutions determine a holomorphic solution in a neighborhood of infinity, or a solution on a sector of $\boldsymbol{C}$ with prescribed asymptotic behavior, or perhaps reflect some properties of holomorphic solutions. The last two questions have been discussed thoroughly by Immink [7], although some problems remain open. The general opinion on the first question is that formal solutions almost never converge, although I have not found a rigorous proof in the litterature.

In § 1 this question is related to the existence of flat solutions of a homogeneous equation, that is solutions which tend to zero very rapidly if the argument tends to infinity in a given sector. In § 2 some of the results of Ramis [12] are extended to differential operators of infinite order, and in § 3 these results are applied to difference operators. This yields that formal power series solutions of linear difference equations satisfy a growth condition of Gevrey type. In § 4 a number of examples is treated, for which both the sheaf of flat solutions and the Stokes directions are determined. The main theorem on the Stokes directions follows in § 5. In § 6 the link with § 3 and the original problem is established in the determination of the cohomology group of the sheaf of flat solutions, and of the significant rôle of the Stokes directions. Finally in 7 the main theorem is proved.

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§ 1. Solvability of equations in a subspace

Indices. (For more properties and more details consult Ramis [12] ch. 0). Let M and N be linear spaces over $\boldsymbol{C}$, and L a $\boldsymbol{C}$-linear mapping from M to N, then L is called of *index*, if either $\dim_{\boldsymbol{C}} \ker L$ or $\dim_{\boldsymbol{C}} \operatorname{coker} L$ is finite, and of *finite index* if both are finite. If L is of index, then the index of L is defined by $\chi(L, M, N) = \chi_{MN}(L) = \dim_{\boldsymbol{C}} \ker L - \dim_{\boldsymbol{C}} \operatorname{coker} L \in [-\infty, \infty]$. If $M = N$ then I write $\chi_M(L)$.

Now assume $L\colon N \to N$, and $M \subset N$ with $LM \subset M$, and consider the equation.

$$Lx = y, \qquad y \in M.$$

If $x \in N$ solves this equation, the question arises whether $x \in M$ or not. Clearly the dimension of the quotient space $L^{-1}M/M$ gives a measure for the solvability of the equation in M compared with the solvability in N. Since $L^{-1}M/M = \ker(L, N/M)$, where the induced map on N/M is still denoted by L, define the *solvability index* of L with respect to M and N as $\iota_{MN}(L) = \dim_{\boldsymbol{C}} \ker(L, N/M)$. If L is surjective on N/M then $\iota_{MN}(L) = \chi_{N/M}(L) = \chi_N(L) - \chi_M(L)$. The following lemma gives a criterion for the surjectivity of L on N/M (for the definition of the topological vector spaces, see Grothendieck [6]):

Lemma 1. *Let N_1 and N_2 be two Banach spaces (resp. Fréchet spaces, resp. duals of Fréchet-Schwartz spaces) over $\boldsymbol{C}$, and M_1 and M_2, too (N_1 and M_1 (resp. N_2 and M_2) need not be of the same type, N_1 and N_2 (resp. M_1 and M_2) do). Let the following diagram of continuous $\boldsymbol{C}$-linear mappings commute:*

$$\begin{array}{ccc} M_1 & \xrightarrow{I_1} & N_1 \\ \downarrow{\scriptstyle L_1} & & \downarrow{\scriptstyle L_2} \\ M_2 & \xrightarrow{I_2} & N_2 \end{array}$$

and assume that I_1 and I_2 are injective, I_2M_2 is dense in N_2, and L_2 is of finite index. Then the induced map L_2 maps N_1/M_1 onto N_2/M_2.

Proof. Although the formulation is more general then in Ramis [12] Lemma 0.13, it is in fact proven there.

Formal power series and asymptotic expansions. Let O be the sheaf on $\boldsymbol{C}$ of holomorphic functions, and $\hat{\underline{O}}$ the completion of $\underline{O}$, the stalk in zero, with respect to the *valuation* v: $v(\sum_{j=n}^{\infty} a_j t^j) = n$, if $a_n \neq 0$; $\hat{\underline{O}} = \boldsymbol{C}[[t]]$ the ring of formal power series over $\boldsymbol{C}$. Let S denote the unit circle in $\boldsymbol{C}$, $S = \{t \in \boldsymbol{C} \mid |t| = 1\}$, equipped with the induced topology. For $\alpha, \beta, R \in \boldsymbol{R}$, $\beta - \alpha < 2\pi$, $R > 0$, $U_{\alpha\beta}$ will be the open subset of S defined by $U_{\alpha\beta} = \{t \in S \mid \alpha < \arg t < \beta\}$, and $V_{\alpha\beta R}$ the open subset of $\boldsymbol{C}$: $V_{\alpha\beta R} = \{t \in \boldsymbol{C} \mid |t| < R,\ t/|t| \in U_{\alpha\beta}\}$.

If $f \in O(V_{\alpha\beta R})$ and there exists an $\hat{f}=\sum_{j=0}^{\infty} f_j t^j$ such that for all $n \in N$ and all $\varepsilon>0$ there is an $R' \leq R$ such that for all $t \in V_{\alpha\beta R'}$

$$\left| t^{-n}\left(f(t)-\sum_{j=0}^{n} f_j t^j\right) \right| < \varepsilon,$$

then f is said to have the *uniform asymptotic expansion* $\hat{f}$ on $V_{\alpha\beta R}$, $f \sim \hat{f}$. It is well-known that if $f \sim \hat{f}$, then $\hat{f}$ is uniquely determined by f, and that for all $\hat{f} \in \underline{\hat{O}}$ there exists an $f \in O(V_{\alpha\beta R})$ such that $f \sim \hat{f}$ on $V_{\alpha\beta R}$.

Clearly the functions having a uniform asymptotic expansion form an injective system with respect to decreasing R, and a projective system with respect to increasing $\beta-\alpha$. Define

$$A(U_{\alpha\beta})=\varprojlim \varinjlim \{f \in O(V_{\gamma\delta R}) | \exists \hat{f} \in \underline{\hat{O}} \text{ such that } f \sim \hat{f} \text{ on } V_{\gamma\delta R}\}$$

where the direct limit is taken over all $R>0$, and the inverse limit over all γ, δ satisfying $\alpha<\gamma<\delta<\beta$. Let A be the sheaf on S generated by the $A(U_{\alpha\beta})$ and let A_0 be the subsheaf of *flat* functions:

$$A_0(U_{\alpha\beta})=\{f \in A(U_{\alpha\beta}) | \hat{f}=0\}.$$

The isomorphism of malgrange. Let $U=\{U_n\}$ be an open cover of S, and $f_n \in A(U_n)$ satisfying $\hat{f}_n=\hat{f}$ for all n. Then $f_{np} \in U_n \cap U_p$ defined by f_n-f_p is flat: $f_{np} \in A_0(U_n \cap U_p)$, so $\{f_{np}, U_n \cap U_p\}$ defines an element of $H^1(S, A_0)$. In fact there is a welldefined map $\mathscr{L}$: $\underline{\hat{O}} \to H^1(S, A_0)$ based on this procedure: Let $\hat{f} \in \underline{\hat{O}}$, choose a cover U of S and $f_n \in A(U_n)$ such that $\hat{f}_n=\hat{f}$, then the image in $H^1(S, A_0)$ is independent of the choice of the cover and the choice of the f_n, and the following holds:

Lemma 2. *The sequence* $0 \longrightarrow \underline{O} \longrightarrow \underline{\hat{O}} \xrightarrow{\mathscr{L}} H^1(S, A_0) \longrightarrow O$ *is exact.*

The proof of this lemma and the definition of $\mathscr{L}$ may be found in Malgrange [9] §1.

The base injection. Let L be a C-linear map on the sheaf A_0. Define K^L as the sheaf Ker L, I^L as the image of L and C^L as the cokernel of L. Applying the global section functor to the exact sequences of sheafs:

$$0 \longrightarrow K^L \longrightarrow A_0 \longrightarrow I^L \longrightarrow 0,$$

and

$$0 \longrightarrow I^L \longrightarrow A_0 \longrightarrow C^L \longrightarrow 0$$

yields the exact sequences (since $A_0(S)=0$, and $H^2(S, X)=0$ for any sheaf X):

$$0 \longrightarrow H^1(S, K^L) \longrightarrow H^1(S, A_0) \longrightarrow H^1(S, I^L) \longrightarrow 0,$$

and

$$0 \longrightarrow H^0(S, C^L) \longrightarrow H^1(S, I^L) \longrightarrow H^1(S, A_o) \longrightarrow H^1(S, C^L) \longrightarrow 0.$$

This yields $\operatorname{Ker}(L, H^1(S, A_o)) = H^1(S, K^L) \oplus H^0(S, C^L)$. So if L^* is the map on $\hat{\underline{O}}/\underline{O}$ induced by L and $\mathscr{L}$ in Lemma 2 one finds

Lemma 3. $\operatorname{Ker}(L^*, \hat{\underline{O}}/\underline{O}) \cong H^1(S, K^L) \oplus H^0(S, C^L)$.

Corollary. $\iota_{\underline{O}\hat{\underline{O}}}(L^*) \geq \dim_C H^1(S, K^L)$.

Gevrey classes. Let $\hat{\underline{O}}_s = \{f = \sum_{j=0}^{\infty} f_j t^j \in \hat{\underline{O}} \mid \sum_{j=0}^{\infty} f_j (j!)^{1-s} t^j \in \underline{O}\}$ and $\hat{\underline{O}}_{(s)} = \{f = \sum_{j=0}^{\infty} f_j t^j \in \hat{\underline{O}} \mid \sum_{j=0}^{\infty} f_j (j!)^{1-s} t^j \in \underline{O}(C)\}$. Formal power series satisfying this growth condition are called Gevrey of order s (resp. (s)). They are closely related to analytic functions satisfying a growth condition: for $s \geq 1$ let

$$A_s(U_{\alpha\beta}) = \varinjlim \varprojlim \{f \in O(V_{\gamma\delta R}) \mid \exists A, C \in \boldsymbol{R}: \forall t \in V_{\gamma\delta R}, p \in \boldsymbol{N} \cup \{0\}, |f^{(p)}(t)| \leq C(p!)^s A^p\}$$

where the direct limit is taken over all $R > 0$, and the inverse limit over all γ, δ satisfying $\alpha < \gamma < \delta < \beta$. Let A_s be the sheaf on S generated by the $A_s(U_{\alpha\beta})$ and let $A_{os} = A_o \cap A_s$. Then one has:

Lemma 4. a) *$A_s \subset A_t \subset A$ if $s \leq t$.*

b) *If $f \in A_s(U_{\alpha\beta})$ then $\hat{f} \in \hat{\underline{O}}_s$.*

c) *$0 \to A_{os} \to A_s \to \hat{\underline{O}}_s \to 0$ is an exact sequence of sheafs.*

d) *$0 \to \underline{O} \to \hat{\underline{O}}_s \to H^1(S, A_{os}) \to 0$ is an exact sequence of $\boldsymbol{C}$-linear spaces.*

e) *If L and L^* are as above, and moreover $LA_{os} \subset A_{os}$, K_s^L, I_s^L and C_s^L are the intersections of K^L, I^L and C^L with A_{os}, then* $\operatorname{Ker}(L^*, \hat{\underline{O}}_s/\underline{O}) \cong H^1(S, K_s^L) \oplus H^0(S, C_s^L)$.

Proof. See Ramis [11].

Remarks. 1) Note that Malgrange's definition of the irregularity of a differential operator (Malgrange [8], déf. 1.5) is a special case of the solvability index. I avoided the word irregularity, since there is already a notion of regularity for difference operators, which does not agree with the above definition.

2) It is also possible, and maybe even better, to define the indices as plus or minus a cardinal: the difference between the cardinality of a Hamel basis of the kernel and the cardinality of a Hamel basis of the cokernel if these two are distinct. The solvability index becomes plus of minus a cardinal, too in this way.

§ 2. Differential operators of infinite order

Ramis proved a number of index theorems for differential operators on rings of formal power series of a Gevrey class (Ramis [12], Th. 1.5.9 and 1.5.10). I shall generalize these results to differential operators of infinite order. Recall the following definitions ([12], § 1.3).

Definition 1. a) For $\kappa=(k, \rho, \mu) \in (0, \infty]\times(0, \infty)\times \boldsymbol{R}$ and $a=\sum_j a_j t^j \in \underline{\hat{O}}$, $\|a\|_\kappa=\sum_j |a_j|(j!)^{-1/k}\rho^j(1+j)^\mu$.

b) $l_\kappa=\{a \in \hat{O} \mid \|a\|_\kappa<\infty\}$.

Lemma 5. (Ramis [12] Lemmata 1.3.2 and 1.3.3, Immink [7] Lemma 16.1).

a) *l_κ is a Banach space over $\boldsymbol{C}$.*

b) *If $\kappa>\kappa'$ (in the lexicographic ordering) then $l_\kappa \hookrightarrow l_{\kappa'}$ and this injection is compact.*

c) *l_κ is a $\boldsymbol{C}$-algebra, and there exists a $C \in (0, \infty)$ depending only on ρ and μ such that for all $a, b \in l_\kappa$: $\|ab\|_\kappa<C\|a\|_\kappa\|b\|_\kappa$.*

From b) follows that the l_κ form an injective system for decreasing ρ, and a projective system for increasing ρ. *For the rest of this paper let k and s be connected as follows*: $k=(s-1)^{-1}$, $s=1+k^{-1}$, then these injective and projective systems define:

Definition 2. For $(s, A) \in [1, \infty)\times(0, \infty)$ let $\underline{\hat{O}}_{s,A-} = \varinjlim l_\kappa$, where the direct limit is taken over all κ with $k=(s-1)^{-1}$, $\rho>1/A$ and μ arbitrary but fixed. And let $\underline{\hat{O}}_{(s,A+)}=\varprojlim l_\kappa$, where the inverse limit is taken over the same values of κ, except that $\rho<1/A$.

Remarks. 1) Note that these definitions are independent of the value of μ.

2) Note that $\underline{\hat{O}}_s=\varinjlim l_\kappa$, with $\rho>0$, and $\underline{\hat{O}}_{(s)}=\varprojlim l_\kappa$ with $\rho<\infty$.

Definition 3. a) $\hat{D}=\underline{\hat{O}}[[t(d/dt)]]$ the skew formal power series ring of differential operators of infinite order with coefficients in $\underline{\hat{O}}$.

b) $B(l_{\kappa_1}, l_{\kappa_2})$ is the Banach space of bounded linear operators from l_{κ_1} to l_{κ_2}. The norm on $B(l_{\kappa_1}, l_{\kappa_2})$ is denoted by $\|\cdot\|_{\kappa_1\kappa_2}$.

c) $\hat{D}_{\kappa_1\kappa_2}=\hat{D}\cap B(l_{\kappa_1}, l_{\kappa_2})$.

Definition 4. Let $D=\sum_{h,j} a_{hj}t^j(t(d/dt))^h \in \hat{D}$.

a) The Newton polygon of D, $N(D)$, is the convex hull of $\{(x, y) \in \boldsymbol{R}^2 \mid \exists h, j \in \boldsymbol{N}_o^2$ with $a_{hj}\neq 0$ and $y=j$, $x\leq h\}$.

b) For all $k \in [0, \infty]$, $E_k(D)=\{(x, y) \in \boldsymbol{N}_o^2\cap N(D) \mid (t, y+k(t-x)) \notin \operatorname{Int}(N(D))$ for all $t \in \boldsymbol{R}\}$.

c) (h_k, j_k) is point of $E_k(D)$ with the smallest coördinates.

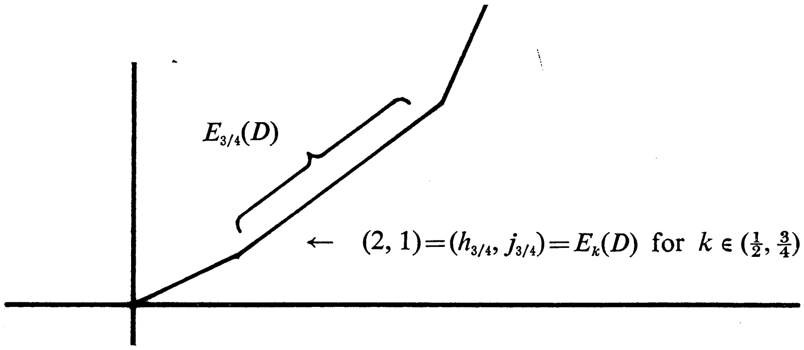
d) k is called characteristic for D if $E_k\neq(h_k, j_k)$.

e) D_k, the k-th principal part of D is defined by

$$D_k=\sum_{(h,j)\in E_k} a_{hj}t^{j_k}\left(t^{1+k}\frac{d}{dt}\right)^{h-h_k}\left(t\frac{d}{dt}\right)^{h_k}.$$

f) $F_{k,D} \in \boldsymbol{C}[[\xi]]$, the k-th characteristic function of D by

$$F_{k,D}(\xi)=\xi^{j_k}\sum_{(h,j)\in E_k} a_{hj}\xi^{(h-h_k)k}$$



Before I formulate and prove the main theorem of this section, let me introduce some notation: Let $D \in \hat{D}$, then $\chi_\infty(D)=\chi(D, \hat{\underline{O}})$. If $D \in \hat{D} \cap \mathrm{End}_C \hat{\underline{O}}_s$ then $\chi_s(D)=\chi(D, \hat{\underline{O}}_s)$; likewise $\chi_{s,A^-}(D)$, $\chi_{(s,A^+)}(D)$, $\chi_{(s)}(D)$ are defined while finally $\chi(D)=\chi(D, \underline{O})$, if $D \in \hat{D} \cap \mathrm{End}_C \underline{O}$.

Theorem 1. *Let* $D=\sum_{h,j} a_{hj} t^j (t(d/dt))^h \in D_{\kappa_1\kappa_2}$, $\kappa_1=(k, \rho, \mu)$, $\kappa_2=(k, \rho, \mu-h_k+j_k/k)$, *and assume that there exists a* $\kappa_3>\kappa_2$ *such that*

$$D-D_k \in \hat{D}_{\kappa_1\kappa_3}.$$

Assume moreover that $F_{k,D}$ *is holomorphic inside the disc* $|\xi|<\rho$, *then for* $A>1/\rho$:

a) $\chi_{s,A^-}(D)$ *equals minus the number of zeroes of* $F_{k,D}$, *counted with multiplicity, on the closed disc* $|\xi|\leq 1/A$.

b) $\chi_{(s,A^+)}(D)$ *equals minus the number of zeroes of* $F_{k,D}$, *counted with multiplicity, inside the open disc* $|\xi|<1/A$.

Proof. (Compare Ramis [12], § 1.5.) By Lemma 5. b $D-D_k$ is compact in $\hat{D}_{\kappa_1\kappa_2}$, so if D_k is of finite index, then $\chi(D, l_{\kappa_1}, l_{\kappa_2})=\chi(D_k, l_{\kappa_1}, l_{\kappa_2})$. Since $F_{k,D}$ is holomorphic inside $|\xi|<\rho$, it has only a finite number of zeroes on $|\xi|\leqslant 1/A$. Write $F_{k,D}(\xi)=\xi^{j_k}\prod_{n=1}^N (\xi^k-\xi_n) f(\xi^k)$, f having no zeroes on $|\xi|\leqslant 1/A$, then D_k may be written as

$$D_k=t^{j_k}\prod_{n=1}^{N}\left(t^{1+k}\frac{d}{dt}-\xi_n\right) f\left(t^{1+k}\frac{d}{dt}\right)\left(t\frac{d}{dt}\right)^{h_k}.$$

By direct calculation one verifies that $\chi_{s,A^-}(t)=-1$, that $\chi_{s,A^-}(t(d/dt))=0$, and that for $k \in \boldsymbol{N}$, $\chi_{s,A^-}(t^{1+k}(d/dt))-\xi_n)=-k$ if $|\xi_n|\leqslant A^{-k}$. Using a substitution $u=t^k$ one establishes that for $k \in \boldsymbol{Q}$, $\chi_{s,A^-}(\prod_{n=1}^N (t^{1+k}(d/dt)-\xi_n))=-kN$ if $|\xi_n|\leqslant A^{-k}$ for all n. So, using the additivity of the index $(\chi(MN)=\chi(M)+\chi(N))$, a follows if $\chi_{s,A^-}(f(t^{1+k}(d/dt))=0$.

Since $f(0)\neq 0$, $f(t^{1+k}(d/dt))$ is a bijection on $\hat{\underline{O}}$, its formal inverse being $1/f(t^{1+k}(d/dt))$. Since $1/f(\xi^k)$ is holomorphic on a small neighborhood of $|\xi|\leqslant 1/A$, $1/f(t^{1+k}(d/dt))=\sum_r c_r(t^{1+k}(d/dt))^r$, with $|c_r|\leqslant C(A-\varepsilon)^{kr}$. Further holds for all $\kappa=(k, \sigma, \nu)$ that $t^{1+k}(d/dt) \in \hat{D}_\kappa$, and $\|t^{1+k}(d/dt)\|_\kappa\leqslant(k+1)^\nu\sigma^k$ and hence $\|1/f(t^{1+k}(d/dt))\|_\kappa<\infty$, if $\sigma<1/(A-\varepsilon)$, implying that $f(t^{1+k}(d/dt))$ is a bijection of $\hat{\underline{O}}_{s,A^-}$.

A similar argument proves b.

§ 3. Linear difference operators

Let φ: $\underline{\hat{O}} \to \underline{\hat{O}}$ be the operator defined by $\varphi(\sum a_j t^j) = \sum a_j(t/(1+t))^j$, and Φ: $\underline{\hat{O}} \to \underline{\hat{O}}$ a difference operator defined by $\Phi = \sum_{h=0}^{m} b_h \varphi^h$, $b_h \in \underline{\hat{O}}$ and $b_0 b_m \neq 0$. φ is an automorphism of $\underline{\hat{O}}$, and its extension to the quotientfield of $\underline{\hat{O}}$ would yield $\varphi(1/t) = 1/t+1$; so the substitution $z = 1/t$ transforms Φ into a formal linear difference operator in the usual sense. The following lemma may be found in Immink [7] §16, but it is more convenient to see it as a special case of Lemma 9 below.

Lemma 6. *Let $s \geq 2$ and $A \in (0, \infty)$, then:*

a) *φ is an automorphism of $\underline{\hat{O}}_s$, $\underline{\hat{O}}_{s,A-}$, $\underline{\hat{O}}_{(s,A+)}$ and $\underline{\hat{O}}_{(s)}$.*

b) *If $b_h \in \underline{\hat{O}}_{s,A-}$ (resp. $\underline{\hat{O}}_{(s,A+)}$) then $\Phi(\underline{\hat{O}}_{s,A-}) \subset \underline{\hat{O}}_{s,A-}$ (resp. $\Phi(\underline{\hat{O}}_{(s,A+)}) \subset \underline{\hat{O}}_{(s,A+)}$).*

Due to the fact that $\varphi = \exp(-t^2(d/dt)) = \sum_{n=0}^{\infty} (-t^2(d/dt))^n/n!$ one may regard Φ as a differential operator of infinite order $\sum_{n=0}^{\infty} \sum_{h=0}^{m} (-h)^n b_h/n! (t^2(d/dt))^n$. To avoid possible confusion, I shall distinguish between the two expressions for Φ: $\Phi = \sum_{h=0}^{m} b_h \varphi^h$, $D = \sum_{n=0}^{\infty} \sum_{h=0}^{m} ((-h)^n b_h/n!) (t^2(d/dt))^n = \sum_{n=0}^{\infty} a_n (t^2(d/dt))^n$. To simplify the discussion I assume in the sequel that $v(b_n) = 0$ for some $h \in \{0, \cdots, m\}$. It is clear in which way this effects the indices that I am going to calculate.

Lemma 7. *For all $n \in N$, there exists an $h(n) \in \{0, \cdots, m-1\}$ such that $v(a_{n+h(n)}) = \min_{h \in \{1,\cdots,m\}} v(b_n)$.*

Proof. Let $b_h = \sum_{j=0}^{\infty} b_{hj} t^j$, $v = \min_{h \in \{1,\cdots,m\}} v(b_h)$ and $a_n = \sum_{j=0}^{\infty} a_{nj} t^j$. Then $a_{nj} = 0$ if $j < v$ and $a_{nv} = \sum_{h=1}^{m} (-h)^n b_{hv}/n!$. Let $\underline{a}_n = (a_{n,v}, \cdots, a_{n+m-1,v})^t$, $\underline{b} = (b_{1v}, \cdots, b_{mv})^t$ and $S_n = (s_{hj})_{j,h=1}^{m}, s_{hj} = (-j)^{n+h-1}/(n+h-1)!$, then $\underline{a}_n = S_n \underline{b}$. Now det S_n is a Vandermonde determinant and therefore not zero, so $\underline{b} = S_n^{-1} \underline{a}_n$. Since $b_{hv} \neq 0$ for some h, $\underline{a}_n \neq 0$, hence $v(a_{n+h(n)}) = v$ for some $h(n)$.

Corollary. $N(D)$ is bordered by at most m (or even $-\frac{1}{2}+\frac{3}{2}\sqrt{m}$) straight lines, that is there are at most m characteristic values for D, and the largest one equals 1.

Lemma 8. *Let κ be fixed, then*

a) *$t(d/dt) \in \hat{D}_{\kappa\kappa'}$, with $\kappa' = \kappa - (0, 0, 1)$ and $\|td/dt\|_{\kappa\kappa'} = 1$.*

b) *$t \in \hat{D}_{\kappa\kappa'}$, with $\kappa' = \kappa + (0, 0, 1/k)$ and $\|t\|_{\kappa\kappa'} = \rho \max(2^{\mu+1/k}, 1)$.*

c) *For all $\nu < 0$, the injection $l_\kappa \to l_{\kappa+(0,0,\nu)}$ has norm 1.*

d) *If $a \in l_\kappa$, and $v(a) \geqslant 1$, then $\|t^{-1}a\|_{\kappa-(0,0,1/k)} \leqslant \rho^{-1} \max(2^{-\mu}, 1)\|a\|_\kappa$.*

e) *For $h = 0, \cdots, m$ let $b_h \in l_\kappa$ and $\|b_h\|_\kappa \leqslant B$, then $\|\sum_{h=0}^{m} (-h)^n b_h\|_\kappa \leqslant (m+1)^{n+1} B$.*

This lemma may be verified by straightforward calculation. The next lemma verifies the last condition needed to apply Theorem 1 to D.

Lemma 9. a) *Let* $b_h \in \underline{\hat{O}}_{(s,A+)}$ *for* $h=0, \cdots, m$. *Let* $(s, A)>(2, 0)$ *and* $\kappa<(1/(s-1), 1/A, -\infty)$; $D=\sum_{n=0}^{\infty} a_n(t^2(d/dt))^n$, *with* $a_n=\sum_{h=0}^{m}((-h)^n/n!)b_h$. *Then* $D \in \hat{D}_{\kappa\kappa_1}$, *with* $\kappa_1=\kappa-(0, 0, h_k-j_k/k)$. *Moreover, there exists a* $\kappa_2>\kappa_1$ *such that* $D-D_k$ $\hat{D}_{\kappa\kappa_2}$.

b) *If* $b_h \in \underline{\hat{O}}_{s,A-}$, *then there exists a* $\rho_o>1/A$ *such that the same statement as in* a) *is true for* $\kappa<(1/(s-1), \rho_o, -\infty)$.

Proof. In case a) $b_h \in l_{\kappa'}$ for all $\kappa'<(1/(s-1), 1/A, -\infty)$, and in case b) there exists a ρ_o such that $b_h \in l_{\kappa'}$ for all $\kappa'<(1/(s-1), \rho_o, -\infty)$.

$(t^2(d/dt))^n$ maps l_κ into $l_{\kappa-(0,0,n-n/k)}$, which is contained in l_{κ_1} if $n \geqslant j_k$. So then by Lemma 8

$$\left\|\sum_{n=j_k}^{\infty} a_n\left(t^2\frac{d}{dt}\right)^n\right\|_{\kappa\kappa_1} \leqslant \sum_{n=j_k}^{\infty}\frac{(m+1)^{n+1}}{n!}B(\rho C)^n<\infty$$

where $B=\max\|b_h\|_{\kappa_1}$ and $C=\max(2^{\mu+1/k}, 1)$.

If $n<j_k$, then $v(a_n) \geqslant j_k+k(n-h_k)-n>0$. Let v_n be the smallest integer such that $v_n \geqslant j_k+k(n-h_k)-n$: Then $t^{-v_n}a_n \in l_{\kappa_1}$, and $t^{v_n}(t^2(d/dt))^n$ maps l_κ into $l_{\kappa-(0,0,n-(n+v_n)/k)}$. Here $n-(n+v_n)k \leqslant n-n/k-j_k/k-n+h_k+n/k=h_k-j_k/k$, so $a_n(t^2(d/dt))^n \in \hat{D}_{\kappa\kappa_1}$ for $n<j_k$ and therefore $D \in \hat{D}_{\kappa\kappa_1}$. A similar argument for $D-D_k$ yields that this operator maps l_κ into l_{κ_2} with $\kappa_2=\kappa-(0, 0, h'_k-j'_k/k)$ where $(h'_k-j'_k/k)<(h_k-j_k/k)$ and hence $\kappa_2>\kappa_1$.

Theorem 2. *Let* $\Phi=\sum_{h=0}^{m} b_h\varphi^h$ *be a difference operator and* $D=\sum_{n=0}^{\infty} a_n(t^2(d/dt))^n$ *the same operator written as a differential operator of infinite order. Assume* $b_h \in \underline{\hat{O}}_{s,A-}$ (*resp.* $\underline{\hat{O}}_{(s,A+)}$) *with* $(s, A)>(2, 0)$. *Then*

i) $\chi_{(s_1,A_1^+)}+(\Phi)>-\infty$ *for all* $(s_1, A_1)\geq(s, A)$ (*resp.* $>(s, A)$) *and equals minus the number of zeros of* $F_{k_1,D}$ *inside the open disc* $|\xi|<A_1^{-1}$.

ii) $\chi_{s_1,A_1^-}(\Phi)>-\infty$ *for all* $(s_1, A_1)>(s, A)$ *and equals minus the number of zeroes of* $F_{k_1,D}$ *on the closed disc* $|\xi|\leq A_1^{-1}$.

In particular one has $\chi_{s_1}(\Phi)=-j_{k_1}$ *and* $\chi_{(s_1)}=-j_{\hat{k}_1}$, $\hat{k}_1$ *the smallest characteristic value for D exceeding* k_1.

Proof. The theorem follows immediately from Theorem 1, Lemma 8 and 9 and the definition of $\underline{\hat{O}}_{s,A_1^-}$ resp. $\underline{\hat{O}}_{(s,A+)}$, $\underline{\hat{O}}_s$ and $\underline{\hat{O}}_{(s)}$.

The next theorem I prove will show that every formal solution of a difference equation is Gevrey for a clearly defined s and A, in accordance with the results for differential equations (Ramis [12], théorème 1.5.14). Therefore I need a couple of preparatory lemma's.

Lemma 10. *Let* $\Phi=\sum_{h=0}^{m} b_h\varphi^h$, $b_h \in \underline{\hat{O}}_{s,A-}$. *Assume* $\chi_{s,A-}(\Phi)$ *and* $\chi_{(s,A+)}(\Phi)$ *are finite. Then* Φ *is surjective on the following quotient spaces*:

$$
\begin{array}{ll}
\underline{\hat{O}}_{(s,A+)}/\underline{\hat{O}}_{\sigma,B-} & \textit{for all } (\sigma, B)<(s, A), \\
\underline{\hat{O}}_{(s,A+)}/\underline{\hat{O}}_{(\sigma,B+)} & \textit{for all } (\sigma, B)\leq(s, A), \\
\underline{\hat{O}}_{s,A-}/\underline{\hat{O}}_{\sigma,B-} & \textit{for all } (\sigma, B)\leq(s, A), \\
\underline{\hat{O}}_{s,A-}/\underline{\hat{O}}_{(\sigma,B+)} & \textit{for all } (\sigma, B)<(s, A).
\end{array}
$$

Proof. As an inverse limit (resp. direct limit) of Banach spaces $\underline{\hat{O}}_{(s,A+)}$ (resp. $\underline{\hat{O}}_{s,A-}$) is a Fréchet space (resp. the dual of a Fréchet-Schwartz space). So the lemma follows by Lemma 1.

Definition 5. Let $0<A<\infty$. If $\chi(\Phi, \underline{\hat{O}}_{(s,A+)}/\underline{\hat{O}}_{s,A-})\neq 0$, then A is called a k-exceptional value for Φ.

Lemma 11. *Let $\Phi \in \underline{\hat{O}}[\varphi]$. Then $\chi_\infty(\Phi)=-j_0$.*

Proof. Let D be as usual, then $v(a_i)=j_o-i$ for only a finite number of i. Hence the argument of Malgrange ([8], Prop. 1.3) for differential operators of finite order stays valid.

The next theorem states that any formal solution of a linear difference equation is of Gevrey type, the exact type depending on $N(D)$:

Theorem 3. *Let $\Phi=\sum_{h=0}^{m} b_h\varphi^h$, $b_h \in \underline{\hat{O}}_{(s,A+)}$ for all h. Let $f \in \underline{\hat{O}}$ satisfy $\Phi f \in \underline{\hat{O}}_{s,A-}$. Then either*

i) *$f \in \underline{\hat{O}}_{(\sigma,B+)}\backslash\underline{\hat{O}}_{\sigma,B-}$ with $(\sigma, B)>\max((s, A^+), (2, 0^+))$, $k'=1/(1-\sigma)$ a characteristic value for Φ and B a k'-exceptional value, or*

ii) *$f \in \underline{\hat{O}}_{(s,A+)} \cup \underline{\hat{O}}_{(2)}$.*

Proof. Note that for $\sigma>2$ there are only a finite number of k'-exceptional values, while for $\sigma=2$ there exists a maximal 1-exceptional value. So for all $(\sigma, B)>\max((2, 0), (s, A))$ there are precisely defined (s_1, A_1) and (s_2, A_2) with k_1 and k_2 characteristic, A_1 and A_2 k_1 resp. k_2 exceptional, and no such pairs in between such that $\max((2, 0), (s, A))\leq(s_1, A_1)<(\sigma, B)\leq(s_2, A_2)\leq\infty$ (with a small abuse of notation). Now assume $f \in \underline{\hat{O}}_{\sigma,B-}$, with possibly σ, $B^-=\infty$ and $\underline{\hat{O}}_\infty=\underline{\hat{O}}$, and consider the exact commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \underline{\hat{O}}_{(s_1,A_1^+)} \arrow[r] \arrow[d, "\Phi"] & \underline{\hat{O}}_{\sigma,B-} \arrow[r] \arrow[d, "\Phi"] & \underline{\hat{O}}_{\sigma,B-}/\underline{\hat{O}}_{(s_1,A_1^+)} \arrow[r] \arrow[d, "\Phi"] & 0 \\
0 \arrow[r] & \underline{\hat{O}}_{(s_1,A_1^+)} \arrow[r] & \underline{\hat{O}}_{\sigma,B-} \arrow[r] & \underline{\hat{O}}_{\sigma,B-}/\underline{\hat{O}}_{(s_1,A_1^+)} \arrow[r] & 0
\end{tikzcd}
$$

Then $\chi_{(s_1,A_1^+)}(\Phi)=\chi_{\sigma,B-}(\Phi)$ by Theorem 2 and Lemma 11. Further, by Lemma 10, one has that the third vertical arrow is onto and hence an isomorphism. This implies that the first two kernels are isomorphic, so $f \in \underline{\hat{O}}_{(s_1,A_1^+)}$. Repeating this procedure if $f \in \underline{\hat{O}}_{s_1,A_1^-}$ leads to the result.

The next theorem calculates $\chi_{(2)}(\Phi)$

Theorem 4. *Let $\Phi=\sum_{h=0}^{m} b_h\varphi^h$ be a difference operator, $b_h \in \underline{\hat{O}}_{(2)}$.*
a) *If $v(b_h)\neq 0$ for all $h\neq h'$, and $v(b'_h)=0$ then $\chi_{(2)}(\Phi)=\chi_\infty(\Phi)=0$.*
b) *Otherwise $\chi_{(2)}(\Phi)=-\infty$.*

Proof. a) Write again $D=\sum_{n=0}^{\infty} a_n(t^2(d/dt))^n$, $a_n=\sum_{h=0}^{m}(-h)^n b_h/h!$. Then $D_1=\sum_{n=0}^{\infty} a_{no}(t^2(d/dt))^n=b_{h'o}\varphi^{h'}$ is an automorphism of $l_{1,\rho,\mu}$ for all ρ and μ. Hence $\chi_{(2)}(\Phi)=\chi_{(2)}(D_1)=0$. Further $a_{oo}=b_{h'o}\neq 0$, so $j_o=0$, and thus $\chi_\infty(\Phi)=0$.

b) In case $v(b_h)=0$ for $h \in H$, where $|H|\geq 2$, then $D_1=\sum_{n=0}^{\infty}\sum_{h\in H}(-h)^n b_{ho}/n! \times(t^2(d/dt))^n$ and $F_{1,D}(\xi)=\sum_{h\in H} b_{ho}e^{-h\xi}=\prod_n(e^{-\xi}-\lambda_n)$. This function has infinitely many zeroes in the finite plane, so $\chi_{(2,A+)}(\Phi)$ is decreasing to $-\infty$ if $A\to 0$ and from the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\hat{O}}_{(2)} & \longrightarrow & \underline{\hat{O}}_{(2,A+)} & \longrightarrow & \underline{\hat{O}}_{(2,A+)}/\underline{\hat{O}}_{(2)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\hat{O}}_{(2)} & \longrightarrow & \underline{\hat{O}}_{(2,A+)} & \longrightarrow & \underline{\hat{O}}_{(2,A+)}/\underline{\hat{O}}_{(2)} & \longrightarrow & 0,
\end{array}$$

it follows that $\chi_{(2)}(\Phi)=\chi_{(2,A+)}(\Phi)-\chi(\Phi,\underline{\hat{O}}_{(2,A+)}/\underline{\hat{O}}_{(2)})$. By Lemma 10 the last factor is nonnegative, so $\chi_{(2)}(\Phi)\leq\chi_{(2,A+)}(\Phi)$ for all A, which proves the theorem.

Remarks. 1) Since Φ is injective if $v(a_o)=0$, as one easily verifies, Theorem 4 implies that in case $v(b_{h'})=0$ and $v(b_h)>0$ if $h\neq h'$, Φ is a bijection on $\underline{\hat{O}}_{s,A-}$-resp. $\underline{\hat{O}}_{(s,A+)}$ for all $(2,0)<(s,A)<\infty$.

2) In Praagman [10] I defined a different Newton polygon for Φ: if $\Phi=\sum_{h=0}^{m} b_h\varphi^h$ then $N(\Phi)$ is the convex hull of the $m+1$ straight vertical half-lines: $\{x=h,\ y\geq v(b_h)\}$. The condition in Theorem 4a may also be described by: $N(\Phi)$ has no edge of slope zero. In § 6 the significance of both Newton polygons will become evident.

3) Duval [5] works with the operator $\delta=\varphi-1$. It is not very difficult to verify the her formulation leads to the same Newton polygon $N(D)$, and that one obtains without any problem the characteristic functions in this way.

Denoting the solvability index for a difference operator $\Phi \in \underline{O}[\varphi]$, with respect to $\underline{O}$ and $\underline{\hat{O}}, \underline{\hat{O}}_s, \underline{\hat{O}}_{s,A-}, \underline{\hat{O}}_{(s,A+)}, \underline{\hat{O}}_{(s)}$ by resp. $\iota(\Phi)$, $\iota_s(\Phi)$, $\iota_{s,A-}(\Phi)$, $\iota_{(s,A+)}(\Phi)$ and $\iota_{(s)}(\Phi)$, one now has the following weak version of the main theorem of this paper:

Theorem 5. *Let $\Phi \in \underline{O}[\varphi]$. Then*
a) *If $N(\Phi)$ has an edge of slope zero, then $\iota(\Phi)=\infty$.*
b) *If $N(\Phi)$ has no edge of slope zero, then $\iota(\Phi)=\iota_{(2)}(\Phi)$.*

Proof. This follows from Lemma 10, Theorem 4, and Remark 2.

Since this approach clearly does not lead to a determination of $\iota(\Phi)$ in the most general case, I shall use the basic injections, as explained in § 1 to find a more complete answer.

§ 4. Examples; Three elementary operators

In this section I shall compute the sheafs K_s^φ (see § 1) for the operators $\Phi=\varphi-1$, $\Phi=\varphi-c$, $\Phi=\varphi-t$. Moreover I shall define Stokes directions and compute these for all three examples. But let me start with the following characterization of the sheafs A_{os}:

Lemma 12 (Ramis [11] Theorem 2.4 (iv)). *Let $f\in O(V_{\alpha\beta R})$ Then $f\in A_{os}(U_{\alpha\beta})$ if and only if for all γ, δ with $\alpha<\gamma<\delta<\beta$ there exist R', B, $C>0$ such that for all $t\in V_{\gamma\delta R'}$:*

$$|f(t)|\leq C\exp(-B|t|^{-k}).$$

Corollary. *For all s A_{os} is a sheaf of ideals in A.*

Example 1. Let me start with the most trivial example: $\Phi=\varphi-1$. The equation $\Phi f=0$ is solved by any φ-periodic function. First some notation.

Definition 6. For all $n\in \boldsymbol{Z}$, let $e_n(t)=\exp(2\pi in/t)$. Define an inner product by $\langle e_n, e_m\rangle=\int_x^{\varphi(x)}(e_n(t)e_m(-t)/t^2)dt=\delta_{mn}$, the Kronecker symbol. Note that the substitution $z=1/t$ relates these definitionf to the usual definitions concerning Fourier series. Conclude from that, that $\langle e_n, e_m\rangle$ is independent of the choice of x!. So if f is an arbitrary solution, then f is holomorphic on a region of the type $\{-\infty\leq R_1<\mathrm{Im}(1/t)<R_2\leq\infty\}$ and then $f=\sum_{n\in Z}\langle f, e_n\rangle e_n$, $|\langle f, e_n\rangle|<CR(n)^n$, $C\in \boldsymbol{R}$, $R(n)=R_2$ if $n<0$, $R(n)=R_1$ if $n\geq 0$, for an x in the region of definition of f. The question is to determine those f's which are flat in a certain region. Therefore the next lemma is fundamental.

Lemma 13. *Let $u=\sum_{n\in Z}u_ne_n\in \underline{O}(\{\mathrm{Im}(1/t)>R\})$. Then $U\in A_o(U_{o\pi})$ if and only if $u_n=0$ for $n\geq 0$. And then $u\in A_{o2}(U_{o\pi})$.*

Proof. Let $u\in A_o(U_{o\pi})$. Then

$$\begin{aligned}u_ne_n(t)&=\langle u, e_n\rangle e_n(t)=\int_x^{\varphi(x)}u(\tau)e^{2\pi in(t^{-1}-\tau^{-1})}/\tau^2d\tau\\&=\int_{(1/t)-(1/x)}^{(1/t)-(1/x+1)}u(t/(1-\delta t))e^{2\pi in\delta}d\delta.\end{aligned}$$

Since $u(t/1-\delta t)\in A_o(U_{o\pi})$ for all δ on the integration path, this implies that $u_n=0$ if $n\geq 0$.

On the other hand, let $u=\sum_{n<0} u_n e_n$, and let $|u_n|\leq R^n$ for some constant R. Then

$$|u(t)|\leq \sum_{n<0} (Re^{-2\pi n \mathrm{Im}(1/t)})^n = Re^{-2\pi \mathrm{Im}(1/t)}/(1-Re^{-2\pi \mathrm{Im}(1/t)})\leq Ce^{-B/|t|},$$

for $|t|$ small enough and arg t bounded away from 0 and π. Hence by Lemma 12 $u \in A_{02}(U_{o\pi})\subset A_o(U_{o\pi})$.

This lemma yields a complete description of K^Φ for $\Phi=\varphi-1$ (in the sequel I drop the Φ from the K if no confusion is possible). But before I state the theorem let me define *Stokes directions* for Φ. According to Wasow [14] § 15, a Stokes ray is a ray at which a solution changes its asymptotic behavior. A precise definition: let K_α be the stalk of K in $e^{i\alpha}$, and $K_\alpha^+=\varinjlim_{\beta>\alpha} K(U_{\mu\beta})$, $K_\alpha^-=\varinjlim_{\beta<\alpha} K(U_{\beta\alpha})$, then, identifying α with a direction in $\boldsymbol{C}$:

Definition 7. α is a positive (resp. negative) *Stokes direction* for Φ, $\alpha \in \Sigma^+(\Phi)$ (resp. $\alpha \in \Sigma^-(\Phi)$) if $K_\alpha\neq K_\alpha^+$ (resp. $K_\alpha\neq K_\alpha^-$). $\Sigma(\Phi)=\Sigma^+(\Phi)\cup\Sigma^-(\Phi)$.

The basic property of these Stokes direction is expressed in the following lemma, which is immediate from the definition (here again I drop the Φ from Σ):

Lemma 14. *Let $u \in K(U_{\gamma\delta})$, let $\alpha<\gamma<\delta<\beta$ and assume $(\alpha,\gamma]\cap\Sigma^+=[\delta,\beta)\cap\Sigma^-=\emptyset$. Then $u \in K(U_{\alpha\beta})$.*

Theorem 6. *Let $\Phi=\varphi-1$. Then $\Sigma^+=\Sigma^-=\pi\boldsymbol{Z}$, and $K=K_2$ is described by*

$$K(U_{\alpha\beta})=\begin{cases}\{\sum_{n<0} u_n e_n \mid \sum_{n<0} u_n t^{-n} \in \underline{O}\} & \text{if } 2m\pi<\alpha<\beta<(2m+1)\pi,\\ 0 & \text{if } (\alpha,\beta)\cap\Sigma\neq\emptyset,\\ \{\sum_{n>0} u_n e_n \mid \sum_{n>0} u_n t^{n} \in \underline{O}\} & \text{if } (2m-1)\pi<\alpha<\beta<2m\pi.\end{cases}$$

Proof. Let $0<\alpha<\beta<\pi$, and $u \in K(U_{\alpha\beta})$, then the proof of Lemma 13 implies immediately that $u \in K(U_{o\pi})$ and hence the Theorem is a direct consequence of Lemma 13 and its companions on $U_{\gamma\delta}$ where $\gamma=m\pi$, and $\delta=(m+1)\pi$.

Remark. Note that $K_s=0$ for $s<2$.

Example 2. The next example is slightly more complicated, although it is again of order 1 and has constant coefficients, too: Let $\Phi=\varphi-c$, $c \in \boldsymbol{C}^*$. For simplicity write $c=e^{2\pi(a+bi)}$, $a \in \boldsymbol{R}$, $b \in [0, 1)$. Solutions of $\Phi f=0$ have the form $u(t)=\sum_{n\in\boldsymbol{Z}} u_n e_n(t)c^{1/t}$, with a growth condition on the u_n. Since $u_n=\langle u(t)c^{-1/t}, e_n\rangle$ the following lemma may be proved analogously to the proof of Lemma 13:

Lemma 15. *Let $u \in \underline{O}(\{\mathrm{Im}(1/t)>R\})$ be defined by $u(t)=\sum_{n\in\boldsymbol{Z}} u_n e_n(t) e^{2\pi(a+bi)/t}$*

and let $0<\alpha<\beta<\pi$. *Then* $u \in A_0(U_{\alpha\beta})$ *if and only if* $u_n=0$ *for all* $n \geq (-b+a \operatorname{cotg} \alpha)$.

From this Lemma and similar lemma's for $m\pi<\alpha<\beta<(m+1)\pi$ follows immediately:

Theorem 7. *Let* $\Phi=\varphi-e^{2\pi(a+bi)}$, $a \in \boldsymbol{R}$, $b \in [0, 1)$, $(a, b) \neq (0, 0)$.

i) *If* $a=0$, *then* $\Sigma^+=\Sigma=\pi\boldsymbol{Z}$, *and* $K=K_2$ *is described by*:

$$K(U_{\alpha\beta})=\begin{cases}\{e^{2\pi bi/t}\sum_{n<0} u_n e_n \mid \sum_{n<0} u_n t^{-n} \in \underline{O}\}, & \text{if } 2m\pi<\alpha<\beta<(2m+1)\pi, \\ 0, & \text{if } (\alpha, \beta) \cap \Sigma \neq \emptyset, \\ \{e^{2\pi bi/t}\sum_{n\geq 0} u_n e_n \mid \sum_{n\geq 0} u_n t^{n} \in \underline{O}\}, & \text{if } (2m-1)\pi<\alpha<\beta<2m\pi.\end{cases}$$

ii) *If* $a<0$, *then* $\Sigma=\{\alpha \in \boldsymbol{R} \mid b+a \operatorname{cotg} \alpha \in \boldsymbol{Z} \cup \{\pm\infty\}\}$, $\Sigma^+=\Sigma \cap ([0, \pi]+2\pi\boldsymbol{Z})$, $\Sigma^-=\Sigma \cap ([-\pi, 0]+2\pi\boldsymbol{Z})$, *and if* $\nu(\alpha)=b+a \operatorname{cotg} \alpha$ *then* $K=K_2$ *is described by*:

$$K(U_{\alpha\beta})=\begin{cases}K_\alpha^+=\{\sum_{n<0} u_n e_n f \mid \sum_{n<0} u_n t^{-n} \in \underline{O}, f=\exp(2\pi(a+(b-\operatorname{ent}[\nu(\alpha)])i)/t)\}, & \\ \qquad \text{if } 2m\pi<\alpha<\beta<(2m+1)\pi, & \\ 0, \qquad \text{if } (\alpha, \beta) \cap 2\pi\boldsymbol{Z} \neq \emptyset, & \\ K_\beta^-=\{\sum_{n>0} u_n e_n f \mid \sum_{n>0} u_n t^{n} \in \underline{O}, f=\exp(2\pi(a+(b+\operatorname{ent}[-\nu(\beta)])i)/t)\}, & \\ \qquad \text{if } (2m-1)\pi<\alpha<\beta<2m\pi, & \\ K_\alpha^+ \cap K_\beta^- \qquad \text{if } 2m\pi<\alpha<(2m+1)\pi<\beta<(2m+2)\pi. & \end{cases}$$

iii) *If* $a>0$, *then* $\Sigma=\{\alpha \in \boldsymbol{R} \mid -b+a \operatorname{cotg} \alpha \in \boldsymbol{Z} \cup \{\pm\infty\}\}$, $\Sigma^+=\Sigma \cap ([-\pi, 0]+2\pi\boldsymbol{Z})$, $\Sigma^-=\Sigma \cap ([0, \pi]+2\pi\boldsymbol{Z})$, *and* $K=K_2$ *is described by* (ν *as above*):

$$K(U_{\alpha\beta})=\begin{cases}K_\beta^-=\{\sum_{n<0} u_n e_n f \mid \sum_{n<0} u_n t^{-n} \in \underline{O}, f=\exp(2\pi(a+(b-\operatorname{ent}[\nu(\beta)])i)/t)\}, & \\ \qquad \text{if } 2m\pi<\alpha<\beta<(2m+1)\pi, & \\ 0, \qquad \text{if } (\alpha, \beta) \cap \pi+2\pi\boldsymbol{Z} \neq \emptyset, & \\ K_\alpha^+=\{\sum_{n>0} u_n e_n f \mid \sum u_n t^{n} \in \underline{O}, f=\exp(2\pi(a+b+\operatorname{ent}[-\nu(\alpha)])i)/t\}, & \\ \qquad \text{if } (2m-1)\pi<\alpha<\beta<2m\pi, & \\ K_\alpha^+ \cap K_\beta^- \qquad \text{if } (2m-1)\pi<\alpha<2m<\beta<(2m+1). & \end{cases}$$

Remarks. 1) Note that again $K_s=0$ if $s<2$.

2) In case $a=0$, Σ is finite, but if $a \neq 0$ Σ is a countable set, and $\pi\boldsymbol{Z}$ is the set of accumulation points for Σ.

Example 3. The last example is the operator $\Phi=\varphi-t$, which has the Γ-function as solution: if $\Phi u=0$, then $u(t)=\sum_{n \in Z} u_n e_n(t)/\Gamma(1/t)$, with some growth condition on the u_n. In this case it is difficult to determine the sheafs K_s completely, but the

Stokes directions may be calculated.

Theorem 8. *Let $\Phi=\varphi-t$. Then $\Sigma^{+}=[-\pi/2, 0]+2\pi\boldsymbol{Z}$, $\Sigma^{-}=[0, \pi/2]+2\pi\boldsymbol{Z}$, and $K_\alpha=0$ if $\alpha\in[\pi/2, 3\pi/2]+2\pi\boldsymbol{Z}$.*

Proof. Let $\alpha\in(\pi/2, \pi)$, then $\Gamma(1/t)\in A_{o,\alpha}$, so if $u\in K_\alpha$ then $u(t)\Gamma(1/t)\in A_{o,\alpha}$. So by Lemma 13 $u(t)=\sum_n u_n e_n(t)/\Gamma(1/t)$. Assume $u_n\neq 0$ for $n>h$, $u_h\neq 0$. Then $u(t)=(u_h e_h(t)/\Gamma(1/t))(1+v(t))$, with $v(t)\to 0$ if $t\to 0$. Hence $u\notin A_{o,\alpha}$, and thus $u_n=0$ for all n. The same reasoning for $\alpha\in(\pi, 3\pi/2)$ yields that $K_\alpha=0$ if $\alpha\in[\pi/2, 3\pi/2]$. Thus $[\pi/2, 3\pi/2]+2\pi\boldsymbol{Z}\cap\Sigma=\emptyset$.

To prove the statements on Σ, one has to prove that for all $\alpha\in(-\pi/2, \pi/2)$ there exists a φ-periodic function which has the same growth rate as $\Gamma(1/t)$ if $t\to 0$ along the ray $\arg t=\alpha$. Or equivalently: construct a function v which grows as $(\log|t|)^{\log|t|}$.

Define:

$$v(t)=\prod_{n=0}^{\infty}(1+a_n t),\quad a_n=\exp(-\exp(\lambda n)),\quad \lambda>0.$$

This infinite product converges: if $t\notin\{-a_0^{-1}, -a_1^{-1}, \cdots, -a_n^{-1}, \cdots\}$ then

$$\log|v(t)|=\sum_{n=0}^{\infty}\log|1+a_n t|\leq\sum_{n=0}^{\infty}a_n|t|<C|t|.$$

So v is an entire function of order zero, and v takes its maximum modulus values on the positive real axis: $\max_{|t|=r}|v(t)|=v(r)$. Let $r>e$, and $N=\operatorname{ent}[\lambda^{-1}\log\log r]$.

Define:

$$v_1(r)=\prod_{n=0}^{N}(1+a_n r),$$

$$v_2(r)=\prod_{n>N}(1+a_n r),$$

Then:

$$a_o\cdots a_N r^{N+1}<v_1(r)<a_o\cdots a_N(2r)^{N+1},$$
$$1<v_2(r)<\exp(1/(1-e^{-\lambda})),$$

and:

$$a_o\cdots a_N=\sum_{n=0}^{N}\exp(-\exp(\lambda n))=\exp\left(-\sum_{n=0}^{N}\exp\lambda n\right)$$
$$=\exp(-(\exp\lambda(N+1)-1)/(e^\lambda-1))=\exp(1/(e^\lambda-1))\times(\exp\exp\lambda N)^{e^\lambda/(1-e^\lambda)}.$$

So there exist positive constants c_1, c_2, c_3, c_4 depending on λ, but not on r and N, such that:

$$c_1 R^{N+1-c_2} < v(r) < c_3(2r)^{N-c_4},$$

which yields, after substituting $N = \text{ent}\,[\lambda^{-1}\log\log r]$, the estimate:

$$c_1 r^{\lambda^{-1}\log\log r - c_2} < v(r) < c_3(2r)^{\lambda^{-1}\log\log r - c_4}.$$

Let $\alpha \in (0, \pi/2) + 2\pi \boldsymbol{Z}$, and consider $u(t) = v(e^{2\pi i/t})/\Gamma(1/t)$ in $V_{o(\pi/2)R}$

$$\begin{aligned}|u(t)| &\leq c_3 \exp((\log 2 + 2\pi \operatorname{Im} t/|t|^2)(\lambda^{-1}\log(2\pi \operatorname{Im} t/|t|^2) - c_4)) \\ &\quad \times K \exp(((\operatorname{Re} t - \tfrac{1}{2})\log|t| - \operatorname{Im} t \arg t + \operatorname{Re} t)/|t|^2) \\ &\leq K' \exp(2(-2(\pi/\lambda)\sin\arg t + \cos\arg t)|t|^{-1}\log|t|) \\ &\leq K'' \exp(-L/|t|),\end{aligned}$$

if $t \to 0$, in a sector where $(-2(\pi/\lambda)\sin\arg t + \cos\arg t) > 0$. So $u \in K_{2,\alpha}$ if $\operatorname{tg}\alpha < \lambda/2\pi$. On the other hand, if $e^{2\pi i/t} > 0$ one has:

$$|u(t)| \geq K_1 \exp(\tfrac{1}{2}(\cos\arg t - 2(\pi/\lambda)\ \sin\arg t)|t|^{-1}\log|t|),$$

hence $u \notin K_\alpha$ if $\operatorname{tg}\alpha > \lambda/2\pi$. This proves that $\alpha \in \Sigma^-$ if $\operatorname{tg}\alpha = \lambda/2\pi$. So if λ varies between zero and infinity α ranges through the set $(0, \pi/2) + 2\pi\boldsymbol{Z}$. Similarly one proves that $(-\pi/2, 0) + 2\pi\boldsymbol{Z} \subset \Sigma^+$. That $2\pi\boldsymbol{Z} \subset \Sigma^+ \cap \Sigma^-$ is a consequence of the fact that periodic functions are in general only defined on half planes while a holomorphic periodic function defined in a small sector containing the positive or negative real axis is entire. Finally, $\pi/2$ and $-\pi/2$ have to be included since the Γ function changes its asymptotic behavior at the imaginary axis.

The last thing to show is that $\alpha \notin \Sigma^+$ if $\alpha \in (0, \pi/2)$: Let $u \in K_\alpha$, and γ, δ such that $0 < \gamma < \delta < \alpha$, and β, η such that $\beta < \alpha < \eta$ and $u \in K(U_{\beta\eta})$. Let R be so small that $\varphi(t) \in V_{\beta\eta R}$ if $\arg t = \eta$ and $t < R$. If R' is small enough then there exists for every $t \in V_{\gamma\delta R'}$ an integer $h(t) > 0$ such that $t' = \varphi^{-h(t)}(t) \in V_{\beta\eta R}$, and $h(t) \leq H/|t|$, where H is a constant depending only on β, η, γ and δ. Using the equation for u: $\Phi u = 0$, one gets

$$u(t) = u(\varphi^{h(t)}(t')) = \varphi^{h(t)-1}(t') \cdots t' u(t').$$

Hence $|t|^{-N}|u(t)| \leqslant (1+h)^N |t'|^{-N}|u(t)| \leqslant (1+H)^N |t'|^{-N}|u(t')|$, and therefore $u \in K(U_{\gamma\delta})$.

Remarks. 1. Note that $K_s = 0$ if $s < 2$: If $u(t)/\Gamma(1/t) \in K_s$, then $|u(t)|/|\Gamma(1/t)| \leqslant C\exp(-B|t|^{-k})$, hence $|u(t)| \leqslant C'\exp(-B|t|^{-k} - B'|t|^{-1}) = C'\exp(-B|t|^{-k}(1 + (B'/B)|t|^{-1+k})) \leqslant C'\exp -B''|t|^{-k}$ if $k > 1$. Hence $u(t) \in A_{0,s}$, and hence by Lemma 13 $u = 0$.

2. I have been unable to conclude whether $K = K_2$ here. The problem is that there may exist a φ-periodic function growing fast enough if $\operatorname{Im} t \to 0$ to disturb the growth rate of the Γ-function without disturbing its flatness. It turns out, however, that this knowledge is not essential to prove the main theorem in § 7.

3. Essentially, nothing changes in Example 3 if one takes $\Phi=t^m\varphi-t^n$. Then main solution becomes $(\Gamma(1/t))^{m-n}$, leading to the same Stokes directions if $m<n$, and to its reflection in the imaginary axis if $m>n$. This evidently still holds if m and n are rational.

§ 5. Stokes directions

In the remainder of the paper let $\Phi=\sum_{h=0}^{m} b_h\varphi^h \in \underline{O}[\varphi]$. It is well known that the equation $\Phi f=0$ admits m formal solutions, linearly independent over $C[[e^{2\pi i/t}]]$ (Birkhoff [3], Turrittin [13], Duval [5], Praagman [10]). These solutions have the form:

$$u_r(t)=(\Gamma(1/t))^{d_r}(\exp P_r(t^{-1/p}))t^{c_r}\hat{Q}(\log t),$$

where $p \in N$, dividing l.c.m $(1, \cdots, m)$, c_r and $d_r \in (1/p)Z$, $P_r \in C[X]$, degr $P_r \leq p$, $P_r(0)=0$, and $\hat{Q}_r \in \underline{O}[X]$. Let $q_r=$ degr P_r, and let μ_r be the leading coefficient of P_r. One may assume that Im $\mu_r \in [0, 2\pi)$ if $q_r=p$.

Let $\alpha \in R$, and assume there exist β, γ with $\beta<\alpha<\gamma$ and $Q_r \in A(U_{\beta\gamma})[X]$ such that $Q_r \sim \hat{Q}_r$ on $U_{\beta\gamma}$, and $v_r(t)=(\Gamma(1/t))^{d_r}\exp(P_r(t^{-1/p}))t^{c_r}Q_r(\log t)$ satisfies $\Phi v_r=0$. Then the v_r form a fundamental set of solutions on $U_{\beta\gamma}$, and

Theorem 9. **a.** *$\alpha \in \Sigma^+(\Phi)$ if and only if at least one of the following conditions is satisfied*:

i) $\alpha \in \pi Z$.

ii) $d_r<0$ *for some* r, *and* $\alpha \in [-\pi/2, 0]+2\pi Z$.

iii) $d_r>0$ *for some* r, *and* $\alpha \in [\pi/2, \pi]+2\pi Z$.

iv) $d_r=0$, $q_r=p$ *and* Re $\mu_r<0$ *for some* r, $\alpha \in (0, \pi)+2\pi Z$ *and* Im μ_r+Re μ_r cotgα $\in 2\pi Z$.

v) $d_r=0$, $q_r=p$ *and* Re $\mu_r>0$ *for some* r, $\alpha \in (-\pi, 0)+2\pi Z$ *and* Im μ_r+Re μ_r cotgα $\in 2\pi Z$.

vi) $d_r=0$, $q_r<p$, *and* $\alpha \in (p/q_r)(\arg \mu_r+3\pi/2)+(2\pi p/q_r)Z$.

b. *$\alpha \in \Sigma^-(\Phi)$ if and only if at least one of the following conditions is satisfied*:

i) $\alpha \in \pi Z$.

ii) $d_r<0$ *for some* r, *and* $\alpha \in [0, \pi/2]+2\pi Z$.

iii) $d_r>0$ *for some* r, *and* $\alpha \in [\pi, 3\pi/2]+2\pi Z$.

iv) $d_r=0$, $q_r=p$ *and* Re $\mu_r<0$ *for some* r, $\alpha \in (-\pi, 0)+2\pi Z$ *and* Im μ_r+Re μ_r cotgα $\in 2\pi Z$.

v) $d_r=0$, $q_r=p$ *and* Re $\mu_r>0$ *for some* r, $\alpha \in 0, \pi)+2Z$ *and* Im μ_r+Re α_r cotg $\alpha \in 2\pi Z$.

vi) $d_r=0$, $q_r<p$, *and* $\alpha \in (p/q_r)(\arg \mu_r+\pi/2)+(2\pi p/q_r)Z$.

Proof. Clearly $\pi Z \in \Sigma^+ \cap \Sigma^-$ because of the regions of holomorphy and flatness of φ-periodic functions. The function v constructed in example 3 yields for a suitable choice of λ the Stokes directions mentioned in ii) and iii) a and b, since P_r and Q_r do not play any significant rôle in the estimates involved. For the same reason the asymptotic behavior of v_r, with $d_r=0$ is completely determined by the leading coeffifficient of P_r. Analogously to example 2 this yields iv) and v) of both a and b. If $q_r<p$ then $\exp P_r(t^{-1/p}) \in A_{o\alpha}$ if and only if $\mathrm{Re}(\mu_r t^{-q_r/p})<0$ for $\arg t=\alpha$. So Stokes directions are formed if $\mathrm{Re}(\mu_r t^{-q_r/p})=0$, leading to $\arg \mu_r-(q_r/p) \arg t \in \pi/2+\pi \boldsymbol{Z}$. And hence $\alpha \in \Sigma^+$ if $\alpha \in (p/q_r)(\arg \mu_r+3\pi/2)+(2p\pi/q_r)\boldsymbol{Z}$, and $\alpha \in \Sigma^-$ if $\alpha \in (p/q_r)(\arg \mu_r+\pi/2)+(2p\pi/q_r)\boldsymbol{Z}$. This proves the "if" part of the theorem.

Now assume $u \in K_\alpha$, then $u(t)=\sum_{r=1}^m u_r(t)v_r(t)$ with u_r φ-periodic and v_r as above. Consider the matrix

$$((\varphi^{j-1}v_r(t))/v_r(t))_{j,r=1}^m.$$

Since the v_r form a fundamental set of solutions, this matrix is invertible (its determinant is a Casorati determinant). Further the expression given above for v_r yields

$$\varphi^{j-1}v_r(t)=\left(\frac{1}{t}+j-2\right)\cdots\frac{1}{t}\exp(P_r((t/(jt+1))^{-1/p})-P_r(t^{-1/p}))(1+jt)^{-c_r}$$
$$\times Q_r(\log(t/(jt+1)))/Q_r(\log t)v_r(t).$$

And hence the entries of the matrix above have a non zero asymptotic expansion in $U_{\beta\gamma}$, and the same holds for its inverse (in both cases modulo an unimportant positive or negative fractional power of t). Solving the equations

$$\varphi^{j-1}u(t)=\sum_{r=1}^m((\varphi^{j-}v_r(t))/v_r(t))u_r(t)v_r(t)$$

for $u_r(t)v_r(t)$ yields that $u_r v_r \in K_\alpha$. Hence the Stokes directions are completely determined by the behavior of the separate fundamental solutions; which proves the only if part of the theorem.

Remarks. 1) Apart from the phenomenon mentioned in Remark 2 following Example 3 in the preceding section, the only flat solutions which are not in K_2 are the solutions with $d_r=0$, $q_r<p$. This type of solution is only finite in number, and belongs to K_s, $s=1+p/q_r$ as is easily verified.

2) In the sequel it will be convenient to distinguish two subclasses of Σ. For $k \in (0,1)$ let $\Sigma_k=\bigcup k^{-1}(\arg \mu_r+(\pi/2)+\pi\boldsymbol{Z})$, the union taken over all r such that $q_r/p=k$. Probably this corrresponds exactly to the Stokes directions associated with the solutions in K_s, $s=1+1/k$, which are not Gevrey for smaller s.

3) Because of the special rôle played by the real axis, and the unknown rôle of the imaginary axis, let $\Sigma_o=(\pi/2)\boldsymbol{Z}$.

4) The proof of the only if part of the theorem yields immediately the following lemma:

Lemma 16. *Let $\alpha \in (\beta, \gamma)$ and let $v_1, \cdots, v_m$ form a fundamental set of solutions of $\Phi u=0$ on $U_{\beta\gamma}$. Then $K_{s,\alpha}$ is the direct sum of $V_1, \cdots, V_m$ where each V_r is a cyclic module over a suitable subring of the periodic functions, generated by v_r. Moreover $V_r=0$ if $v_r \notin K_{s,\alpha}$.*

The subrings mentioned in the lemma depend on d_r, μ_r and α.

5) For $\alpha \notin \Sigma_o$ one may split the stalks K_α in a direct sum: for $\alpha \in (0, \pi)\backslash\pi/2$ (resp. $(-\pi, 0)\backslash-\pi/2$), let K_α^p (resp. K_α^n)$=\oplus V_r$, the sum ranging over those r for which $d_r>0$, or $d_r=0$, $p=q_r$ and $\operatorname{Re}\mu_r\leq 0$; K_α^n (resp. K_α^p)$=\oplus V_r$, for those r with $d_r<0$ or $d_r=0$, $p=q_r$, and $\operatorname{Re}\mu_r>0$; and finally $K_\alpha^o=\oplus V_r$ with $d_r=0$, $p>q_r$. Then by Lemma 16 $K_\alpha=K_\alpha^p\oplus K_\alpha^n\oplus K_\alpha^o$. Let K^p, K^n, K^o denote the associated sheafs on $S\backslash\exp i\Sigma_o$. The reason for these definitions lies in the pattern of the Stokes directions for these subsheafs: on its domains $S\backslash\exp i\Sigma_o$ the sheaf K^p has only positive Stokes directions, while K^n has only negative ones and K^o has only the Stokes directions Σ_k, $k \in [0, 1)$. K_s^p, K_s^n, K_s^o are defined in the obvious way.

6) Note that $K_2^o=0$.

§ 6. Cohomology

Using the Stokes directions determined in the preceding section, and the decomposition of K defined in Remark 5 of that section it is possible to obtain some information about the cohomology groups $H^1(S, K_s)$. First recall (Immink [7] Th. 18.13) that for $\beta \in \pi/2+\pi\boldsymbol{Z}$ and $\gamma=\beta+\pi$, there exists a fundamental set of solutions $v_1, \cdots, v_m$ defined on $U_{\beta\gamma}$ of the form described directly above Theorem 9. The implies that $K_s(U_{\alpha\delta})=K_s^p(U_{\alpha\delta})\oplus K_s^n(U_{\alpha\delta})\oplus K_s^o(U_{\alpha\delta})$ if $\beta<\alpha<\delta<\gamma$ and $(\alpha, \delta)\cap\pi\boldsymbol{Z}=0$.

Lemma 17. *Let $(\alpha, \beta)\cap\Sigma_o=\emptyset$ and assume (α, β) contains at most one Stokes direction of type $\Sigma_{k'}$, $k' \in (k, 1)$. Then $\check{H}^1(W, K_s)=0$ for all finite covers W of $U_{\alpha\beta}$.*

Proof. Without loss of generality assume that $W=\{W_h\}$, $W_h\cap W_j=\emptyset$ if $h \notin \{j-1, j, j+1\}$, $\alpha \in \overline{W}_1$, $\beta \in \overline{W}_n$, and let $W_h'=W_h\cap W_{h+1}$, if $h<n$. Let $u_h \in K_s(W_h')$ be given for $h=1, \cdots, n-1$. Write $u_h=u_h^p+u_h^n+u_h^o$, with $u_h^{\cdot} \in K_s^{\cdot}(W_h')$. Depending on the position of α and β and the nature of the Stokes direction of type Σ_k, several cases should be distinguished. Since all cases may be treated similarly, assume $(\alpha, \beta)\subset(0, \pi/2)$, and the Stokes direction of type Σ_k is positive. Then:

$$K_s^p(W_h')=K_s^p(\bigcup_{j>h} W_j),\quad K_s^n(W_h')=K_s^n(\bigcup_{j\leq h} W_j),\quad K_s^o(W_h')=K_s^o(\bigcup_{j>h} W_j),$$

by Lemma 14. Defining $\tilde{u}_h=\sum_{j<h} u_j^p-\sum_{j\geq h} u_j^n+\sum_{j<h} u_j^o \in K_s(W_h)$ yields

$$u_{h+1}-u_h=u_h^p+u_h^n+u_h^o=\tilde{u}_h.$$

Consequently one may restrict oneself to covers of the following type: For $\alpha \in \Sigma_0 \cup \Sigma_{K'}$, $k' \in (k, 1)$, let U_α be a small sector containing $e^{i\alpha}$, such that no pair of such sectors intersect each other, and let $U = S \setminus \cup_\alpha \exp i\alpha$. Then one has to consider covers of the type $\{U, U_\alpha\}$.

Distinguish three cases: $s<2$, $s=2$ and $s>2$. From Lemma 12 and the form of the v_γ one immediately derives $K_s=0$ if $s<2$. So $H^1(S, K_s)=0$ for $s<2$.

If $s>2$, let $\hat{K}_s = K_s/K_2$. $\hat{K}_s$ is a sheaf of $\boldsymbol{C}$-linear spaces, probably finite dimensional. Anyway $H^1(S, \hat{K}_s)$ may be calculated using the method of Bertrand [1], § 2, counting the jumps at the Σ_h, $h \in (k, 1)$. This yields $H^1(S, K_s) \cong \boldsymbol{C}^{n(s)}$, where $n(s)$ is the number defined by

$$n(s) = \sum_{h \in (k,1)} \#\{\lambda \in \boldsymbol{C}^*,\ F_{h,D}(\lambda)=0\}, \quad \text{counted with multiplicity.}$$

Since $0 \to K_2(S) \to K_s(S) \to \hat{K}_s(S) \to 0$ is exact, $H^1(S, K_s) \cong H^1(S, K_2) \oplus \boldsymbol{C}^{n(s)}$. Let me concentrate on K_2.

In this case the covers $U = S \setminus \exp(i\pi/2)\boldsymbol{Z}$, U_α, $\alpha \in (\pi/2)\boldsymbol{Z}$ suffice. Let the $\boldsymbol{C}$-linear map

$$\beta\colon \oplus K_{2,\alpha} \longrightarrow \oplus K_{2,\alpha}^{n+} \oplus K_{2,\alpha}^{p-},$$

where K^{n+} and K^{p-} are defined analogously to K^+ and K^-, be defined by

$$\beta(u_1, u_2, u_3, u_4) = (v_1^+, v_1^-, v_2^+, v_2^-, v_3^+, v_3^-, v_4^+, v_4^-)$$

where u_j, $v_j^\pm$ are defined in $\alpha = j\pi/2$, and the $v_j^\pm$ are as follows: $u_j \in K_{j\pi/2}^-$ and $u_j = u_j^{p-} + u_j^{n-}$, with u_j^{p-} and u_j^{n-} in $K_{j\pi/2}^{p-}$ resp. $K_{j\pi/2}^{n-}$. Also $u_j \in K_{j\pi/2}^+$ and hence $u_j = u_j^{p+} + u_j^{n-}$ (Note that u_j^{p+} does not have to equal u_j^{p-}). But the nature of K^{p+}, u_j^{p+} extends to a flat solution on $U_{\gamma\delta}$, $\gamma = j\pi/2$, $\delta = (j+1)\pi/2$, and hence $u_j^{p+} \in K_{(j+1)\pi/2}^{p-}$. With these notations:

$$v_j^- = u_j^{p-} - u_{j-1}^{p+},\ v_j^+ = u_j^{n+} - u_{j+1}^{n-}.$$

Theorem 10. $H^1(S, K_2) \cong \operatorname{coker} \beta$.

Proof. Let U, U_α be a cover as defined above. Let $U_\alpha^+ = U_\alpha \cap U_{\alpha\gamma}$ and $U_\alpha^- = U_\alpha \cap U_{\delta\alpha}$ with $U_\alpha \subset U_{\delta\gamma}$. Consider the following exact commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_2(U) & \longrightarrow & K_2(U) \oplus_\alpha K_2(U_\alpha) & \longrightarrow & \oplus K_2(U_\alpha) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \oplus_\alpha K_2^p(U_\alpha^+) \oplus K_2^n(U_\alpha^-) & \longrightarrow & \oplus_\alpha K_2(U_\alpha^+) \oplus K_2(U_\alpha^-) & \longrightarrow & \oplus_\alpha K_2^n(U_\alpha^+) \oplus K_2^p(U_\alpha^-) & \longrightarrow & 0
\end{array}$$

Since the first column is an isomorphism by Lemma 14, the theorem follows by taking the direct limit over all covers of the described type.

Let me examine the examples of § 5 to get an impression about coker β.

Example 1. $\Phi=\varphi-1$. Then $K_{2,\pi/2}=K^{p-}_{2,\pi/2}\oplus K^{n+}_{2,\pi/2}$, and $K^{n-}_{2,\pi}\subset K^{n+}_{2,0}$, so β may be restricted to 0 and π. Since $K_{2,0}=K_{2,\pi}=0$, and $K^{n+}_{2,0}=K^{p+}_{2,0}=0$:

$$H^1(S, K_2)\cong K^{p-}_{2\pi}\oplus K^{n+}_{2\pi}\cong \underline{O}\oplus\underline{O}.$$

Example 2. $\Phi=\varphi-a$, $|a|>1$. Again β may be restricted to 0 and π. Since $K_{2,0}=0$, $K_{2,\pi}\cong \boldsymbol{C}[t, t^{-1}]$, $K^{n+}_{2,\pi}=K^{p-}_{2,0}=0$, $K^{n+}_{2,\pi}\cong K^{p-}_{2,\pi}\cong\underline{O}[t^{-1}]$, and β embeds $\boldsymbol{C}[t, t^{-1}]$ onto the diagonal of $\underline{O}[t^{-1}]\oplus\underline{O}[t^{-1}]$ by $\beta(p(t))=p(t)\oplus p(t^{-1})$:

$$H^1(S, K_2)\cong\underline{O}\oplus\underline{O}.$$

Example 3. $\Phi=\varphi-t$. Again restriction to 0 and π. Since $K_{2,0}=0$, and $K_{2,\pi}$ is isomorphic to a subset of $O(\boldsymbol{C}^*)$, characterized by a growth condition for $t\to 0$ and for $t\to\infty$, while $K^{n+}_{2,\pi}$ and $K^{p-}_{2,\pi}$ form a subset of $O(\boldsymbol{C}^*)$, characterized by just one growth condition for $t\to 0$ (resp. $t\to\infty$), here again $H^1(S, K_2)$ has infinite dimension, having an uncountable basis.

In general a combination of these phenomena occur. Each v_r gives rise to one of these three phenomena. So:

Theorem 11. *For $s<2$, $K_s=H^1(S, K_s)=0$, while for $s\geq 2$, $H^1(S, K_s)$ has an uncountable Hamel basis over $\boldsymbol{C}$. The injection $H^1(S, K_2)\to H^1(S, K_s)$ has an finite dimensional cokernel.*

Remarks. 1) In fact $K_s=0$ for $s<2$ does not provide a lot information. One might even rightfully ask about the definition of K_s, since φ does not map $A_{o,s}$ into itself for $s<2$.

2) It is possible to give a finer filtration on A_o to clarify the rôle of the zeroes of the characteristic functions and of the Γ factors involved. This leads, however, to very tedious technical computations.

3) The shape of the Newton polygons $N(\Phi)$ and $N(D)$ determine in a certain sense $H^1(S, K_s)$: every nonzero slope of $N(\Phi)$ gives rise to factor involving growth condition as in Example 3 while zero slopes yield factors $\underline{O}\oplus\underline{O}$. Whether these arise directly, or by an embedding on the diagonal as in Example 2 may be seen from $N(D)$.

§ 7. The main theorem

As a direct consequence of Lemma 3 and Theorem 11 one has now:

Theorem 12. *The solvability index of a difference operator $\Phi\in\underline{O}[\varphi]$ in $\hat{\underline{O}}_s$, $i_s(\Phi)=\infty$ for $s\geq 2$.*

References

[1] Bertrand, D., Travaux récents sur les points singuliers des équations différentielles linéaires, Sém. Bourbaki 78/79 exp. n° 538. L.N.M. 770, Springer, Berlin, 1980.
[2] Birkhoff, G. D., General theory of linear difference equations, Trans. Amer. Math. Soc., **12** (1911), 243–284.
[3] ——, Formal theory of irregular linear difference equations, Acta Math., **54** (1930), 205–246.
[4] Birkhoff, G. D. and Trjitzinski, W. J., Analytic theory of singular difference equations, Acta Math., **60** (1933), 1–89.
[5] Duval, A., Lemmes de Hensel et factorisation formelle pour les opérateurs aux différences, Funkcial. Ekvac., **26** (1983), 349–368.
[6] Grothendieck, A., Sur certain espaces de fonctions holomorphes I, J. Reine Angew. Math., **192** (1955) 35–64.
[7] Immink, G. K., Asymptotics of analytic difference equations, L.N.M. 1082, Springer, Berlin, 1984.
[8] Malgrange, B., Sur les points singuliers des équations différentielles, Enseignement Math., **20**(1–2) (1974), 147–176.
[9] ——, Remarques sur le équations différentielles à points singuliers irréguliers, L.N.M. 712, 77–86, Springer, Berlin, 1979.
[10] Praagman, C., The formal classification of linear difference equations, Proc. Kon Ned. Ak. Wet. Ser. A, **88** (= Ind. Math., **45**) (1983), 249–261.
[11] Ramis, J. P., Les séries k-sommables et leurs applications, L.N.P. 126, Springer, Berlin, 1980.
[12] ——, Théorème d'indices Gevrey pour les équations différentielles ordinaires, Mem. Amer. Math. Soc., **296** (1984), 95 pp.
[13] Turrittin, H. L., The formal theory of systems of irregular homogeneous linear difference and differential equations, Bol. Soc. Math. Mexicana, (2) **5** (1960), 255–264.
[14] Wasow, W., Asymptotic expansions for ordinary differential equations, Interscience, New York, 1965.

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