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On Some Nonlinear Evolution Equation

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§ 0. Introduction

This paper is concerned with the initial valued problem of the following nonlinear evolution equation

$$\partial\varphi^{t}(u'(t))+Au(t)\ni f(t),\qquad 0\leqq t\leqq T, \tag{0.1}$$

$$u(0)=u_{0} \tag{0.2}$$

in a real Hilbert space H. Here $\partial\varphi^{t}$ is the subdifferential of a lower semicontinuous proper convex function φ^{t} from H into $[0, +\infty]$ for each $t\in[0, T]$, and A is a positive definite self-adjoint operator in H. Under certain hypothesis on φ^{t}, A, $f(t)$ and u_{0} we show the existence and uniqueness of a strong solution of (0.1), (0.2).

The equation (0.1) of the form (0.1) was investigated by H. Brezis [4], V. Barbu [2] and T. Arai [1]. In these results it was assumed that $\varphi^{t}\equiv\varphi$ is independent of t; however, in [1] and [2] A may be a nonlinear subdifferential $\partial\psi$. It was required that φ satisfies some compactness condition in [2] and φ is positively homogeneous of order $p>1$ in [1].

We apply our result to the following initial boundary value problem:

$$\begin{aligned}
&0\leqq(\partial u/\partial t)(t, x)\leqq h(t, x) && \text{for a.e. } (t, x)\in(0, T)\times\Omega,\\
&\nu(|\partial u/\partial t|^{p-2}\partial u/\partial t)-\Delta u=f && \text{on } \{0<\partial u/\partial t<h\},\\
&\nu(|\partial u/\partial t|^{p-2}\partial u/\partial t)-\Delta u\geqq f && \text{on } \{\partial u/\partial t=0\},\\
&\nu(|\partial u/\partial t|^{p-2}\partial u/\partial t)-\Delta u\leqq f && \text{on } \{\partial u/\partial t=h\},\\
&u(t, x)=0 && \text{on } (0, T)\times\Omega,\\
&u(0, x)=0 && \text{for a.e. } x\in\Omega,
\end{aligned}$$

where Ω is a bounded domain in $\boldsymbol{R}^{n}$ with smooth boundary, h is in $W^{1,2}([0, T]; L^{2}(\Omega))\cap L^{2}(0, T; H^{2}(\Omega))$, and $\nu\geqq 0$, $p>1$ are constants. Here h may be dependent on t, while the results of [1], [3] and [4] can be applied only when h is independent of t.

The contents of this paper are the following.

The section 1 contains a brief review of the basic properties of subdifferentials of lower semicontinuous convex functions and positive self-adjoint operators. In the section 2 we state our main result. In the section 3 we consider the approximate problem. And, in the section 4 we consider the convergence of approximate solution and the uniqueness of the strong solution of (0.1) with $u(0)=u_0$. In the section 5 we illustrate our result by applying it to the initial boundary value problem for a certain differential inequality.

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Notation. We use the following notation throughout this paper. H denotes a real Hilbert space with the inner product $(\ ,\ )$ and the norm $|\ |$. T is a fixed positive number and $C([0, T]; H)$ denotes the space of strongly continuous functions u: $[0, T]\to H$ with the norm $|u|_{C([0,T];H)}=\sup_{s\in[0,T]}|u(s)|$. Moreover for $1\leqq p\leqq +\infty$, $L^p(0, T; H)$ denotes the space of strongly measurable functions u: $[0, T]\to H$ such that $|u|_{L^p(0,T;H)}<+\infty$, where

$$|u|_{L^p(0,T;H)}=\begin{cases}\left(\int_0^T|u(s)|^p ds\right)^{1/p} & \text{if } 1\leqq p<+\infty,\\ \operatorname*{ess.\,sup}_{s\in[0,T]}|u(s)| & \text{if } p=+\infty.\end{cases}$$

$W^{1,2}([0, T]; H)$ denotes the space of strongly absolutely continuous function u; $[0, T]\to H$ such that du/dt is in $L^2(0, T; H)$ with the norm

$$|u|_{W^{1,2}([0,T];H)}=|u|_{L^2(0,T;H)}+|du/dt|_{L^2(0,T;H)}.$$

§ 1. Preliminaries

In this section we collect some well-known results on the subdifferential of convex functions as well as self-adjoint operators. For the proofs we refer to the books of H. Brezis [6] and K. Yosida [10].

Let φ be a proper lower semicontinuous convex function from H into $(-\infty, +\infty]$. The effective domain $D(\varphi)$ of φ is defined by

$$D(\varphi)=\{u\in D(\varphi);\ \varphi(u)<+\infty\}.$$

For each $u\in D(\varphi)$ the set

$$\partial\varphi(u)=\{f\in H;\ \varphi(v)-\varphi(u)\geqq(f, v-u)\text{ for all } v\in H\}$$

is called the subdifferential of φ at u and the subdifferential operator $\partial\varphi$ of φ is defined as an operator which assigns to each u the set $\partial\varphi(u)$. The operator $\partial\varphi$ is a (possibly multi-valued) maximal monotone operator with the domain

$$D(\partial\varphi)=\{u\in D(\varphi);\ \partial\varphi(u)\text{ is not empty}\}.$$

Hence, for each $\lambda>0$, we can define a single-valued operator $J_\lambda=(1+\lambda\partial\varphi)^{-1}$ and the Yosida approximation $(\partial\varphi)_\lambda=\lambda^{-1}(1-J_\lambda)$ of $\partial\varphi$.

Now, for each $\lambda>0$ and each $u \in H$, we define

$$\varphi_\lambda(u)=\inf_{v\in H}\{\varphi(v)+(2\lambda)^{-1}|u-v|^2\}. \tag{1.1}$$

Then it is seen that

$$\varphi_\lambda(u)=\varphi(J_\lambda u)+(2\lambda)^{-1}|u-J_\lambda u|^2. \tag{1.2}$$

Moreover we have

$$\varphi(J_\lambda u)\leqq\varphi_\lambda(u)\leqq\varphi(u) \qquad \text{for } u \in H, \tag{1.3}$$

$$\varphi_\lambda(u)\uparrow\varphi(u) \quad \text{as } \lambda\downarrow 0 \quad \text{for } u \in H, \tag{1.4}$$

and

$$\partial\varphi_\lambda=(\partial\varphi)_\lambda. \tag{1.5}$$

For a closed convex subset K of H let I_K be the indicator function of K:

$$I_K u=0 \quad \text{if } u \in K, \qquad I_K u=\infty \quad \text{if } u \in K.$$

Then it is known that

$$P_K=(1+\lambda\partial I_K)^{-1} \qquad \text{for all } \lambda>0 \tag{1.6}$$

is the projection operator onto K.

Next, let A be a self-adjoint operator in H with the domain $D(A)$. The operator A is called positive if

$$(Av, v)\geqq 0 \qquad \text{for all } v \in D(A). \tag{1.7}$$

In this case A is a maximal monotone operator, and for each $\lambda>0$, $A_\lambda=A(1+\lambda A)^{-1}$ is also self-adjoint operator, and is equal to the Yosida approximation of A.

§ 2. Statement of the main result

We state our main result on the initial value problem for the equation (0.1). We make the following assumptions.

(A.1) $\{\varphi^t: 0\leqq t\leqq T\}$ is a family of lower semicontinuous convex functions from H into $[0, +\infty]$ with nonempty effective domain and there is a function $g \in W^{1,2}(0, T)$ having the following property (*);

(*) For each $t_0 \in [0, T]$, $v_0 \in D(\varphi^{t_0})$ and $t \in [0, T]$ there is an element $v(t) \in D(\varphi^t)$ such that

$$|v(t)-v_0|\leqq|g(t)-g(t_0)|(\varphi^{t_0}(v_0)+1)^{1/2}$$

and

$$\varphi^t(v(t))\leqq\varphi^{t_0}(v_0)+|g(t)-g(t_0)|(\varphi^{t_0}(v_0)+1).$$

(A.2) A is a self-adjoint operator and

$$(Av, v)\geqq C_0|v|^2 \qquad \text{for any } v \in D(A)$$

where C_0 is a positive constant.

(A.3) There exists a function $a \in L^2(0, T; H)$ such that

$$\varphi^t((1+\varepsilon A)^{-1}(v+\varepsilon a(t)))\leqq\varphi^t(v) \quad \text{for any } v \in D(\varphi^t), \quad t \in [0, T] \quad \text{and } \varepsilon>0.$$

Next we define the strong solution of (0.1) on $[0, T]$.

Definition 2.1. A function $u \in C([0, T]; H)$ is called a strong solution of (0.1) on $[0, T]$ if the following conditions (i), (ii) and (iii) hold;

(i) u is absolutely continuous on any closed subinterval of $(0, T)$.
(ii) For a.e. $t \in [0, T]$, $u(t)$ is in $D(A)$ and $(du/dt)(t)$ is in $D(\varphi^t)$.
(iii) u satisfies (0.1) for a.e. $t \in [0, T]$.

Our main result is the following,

Theorem 2.2. *Assume that* (A.1), (A.2) *and* (A.3) *be satisfied. Then for every* $f \in W^{1.2}([0, T]; H)$ *and* $u_0 \in A^{-1}(f(0)-\overline{R(\partial\varphi^0)})$, (0. 1), (0.2) *has one and only one strong solution* u *such that*

(i) $Au \in L^\infty(0, T; H)$,
(ii) $u, A^{1/2}u \in C([0, T]; H)$,
(iii) $\sqrt{t}\,u' \in L^2(0, T; H)$

and

(iv) $\varphi^{\cdot}(u') \in L^1(0, T)$.

Moreover, if $u_0 \in A^{-1}(f(0)-R(\partial\varphi^0))$, *then*

(iii′) $u' \in L^2(0, T; H)$

and

(iv′) $\varphi^{\cdot}(u') \in L^2(0, T)$.

Remark 2.3. The assumption (A.1) is a modification of the assumption of N. Kenmochi [7] and Y. Yamada [9] in the t-dependence of φ^t, and it is stronger than that of Y. Yamada [9]. Further, in the case $\varphi^t=I_{K(t)}$, the assumption (A.3) is the same of H. Brezis [5].

Remark 2.4. In the case $\varphi^t=I_{K(t)}$ we show that the conclusion of our theorem holds under the following assumption (A.1′).

(A.1′) $\varphi^t = I_{K(t)}$ for each $t \in [0, T]$, where $\{K(t);\ 0 \leqq t \leqq T\}$ is a family of nonempty closed convex subsets of H. Further, let $P(t)$ be the projection onto $K(t)$ for every $t \in [0, T)$. There exist a real number $\alpha \in (1, \infty)$ and a nonnegative constant L such that for each $\xi \in (0, T/2)$ and $v \in C([0, T]; H)$,

$$\int_0^{T-\xi} |P(t+\xi)v(t) - P(T)v(t)|^\alpha dt \leqq (L\xi)^\alpha.$$

§ 3. Approximate problems

Let u_0 be in $D(A)$. In order to construct a strong solution of (0.1) with the initial condition $u(0) = u_0$, we consider the existence problem of the following approximate equations.

$$\varepsilon u_\varepsilon'(t) + \partial\varphi^t(u_\varepsilon'(t)) + A_\varepsilon u_\varepsilon(t) \ni f(t), \qquad 0 \leqq t \leqq T, \tag{3.1}$$

$$u_\varepsilon(0) = u_{0\varepsilon}\ (= (1+\varepsilon A)u_0), \tag{3.2}$$

for $0 < \varepsilon \leqq 1$.

We consider the continuity of $(\varepsilon + \partial\varphi^t)^{-1}$ with respect to t. The following lemma is due to Y. Yamada [9, Proposition 3.1] (see Remark 2.3).

Lemma 3.1. *Let* $\{\varphi^t;\ 0 \leqq t \leqq T\}$ *satisfy* (A.1). *Then, for each* $0 < \varepsilon \leqq 1$ *and* $v \in H$, $(\varepsilon + \partial\varphi^t)^{-1} v$ *is strongly continuous in* $0 \leqq t \leqq T$.

The following lemma is easy to show.

Lemma 3.2. *Under the assumption* (A.2), *for each* $v \in H$ *and* $\varepsilon > 0$.

$$|A_\varepsilon v| \geqq C_0(1+\varepsilon C_0^2)^{-1/2}|v|, \qquad |A_\varepsilon^{1/2} v| \geqq C_0^{1/2}(1+\varepsilon C_0^2)^{-1/4}|v|.$$

From now on we write $C_{0\varepsilon}$ instead of $C_0(1+\varepsilon C_0^2)^{-1/2}$.

Proposition 3.3. *Let* $f \in W^{1,2}([0, T]; H)$ *and* $u_0 \in A^{-1}(f(0) - R(\partial\varphi^0))$. *Then* (3.1) *and* (3.2) *have a unique solution* u_ε *such that* u_ε' *is in* $C([0, T]; H)$.

Proof. We can rewrite the equation (3.1) as follows.

$$u_\varepsilon'(t) = (\varepsilon + \partial\varphi^t)^{-1}(f(t) - A_\varepsilon u_\varepsilon(t)), \qquad 0 \leqq t \leqq T. \tag{3.3}$$

For each $\varepsilon > 0$ and $t \in [0, T]$, $(\varepsilon + \partial\varphi^t)^{-1}$ and A_ε are Lipschitz continuous operators on H. Further, by Lemma 3.1, we can prove that there exists a unique solution u_ε such that u_ε' is in $C([0, T]; H)$.

To prove the convergence of $\{u_\varepsilon;\ 0 < \varepsilon\}$ we need the following a priori estimates.

A priori estimates [I].

In this subsection we consider some estimates for $A_\varepsilon u_\varepsilon$, $\sqrt{\varepsilon}\, A_\varepsilon^{1/2} u_\varepsilon'$ and $u_\varepsilon'(0)$.

Proposition 3.4. *There is a positive constant* $R_1 = R_1(|a|_{L^2(0,T;H)}, |f|_{W^{1,2}([0,T];H)}, |Au_0|)$ *independent of* $\varepsilon \in (0, 1]$, *such that*

$$|A_\varepsilon u_\varepsilon(t)| \leqq R_1, \quad \varepsilon\int_0^T |A_\varepsilon^{1/2} u'_\varepsilon(s)|^2 ds \leqq R_1 \quad \textit{for any } t \in [0, T] \quad \textit{and } \varepsilon \in (0, 1].$$

Proof. From condition (A.3) it follows that

$$\begin{aligned}(3.4)\qquad &\frac{\varepsilon}{2}|A_\varepsilon^{1/2}u'_\varepsilon(t)|^2 + \frac{d}{dt}\left\{\frac{1}{2}|A_\varepsilon u_\varepsilon(t)|^2 - (f(t), A_\varepsilon u_\varepsilon(t))\right\} \\ &\leqq \left\{\frac{\varepsilon}{2}|A_\varepsilon^{-1/2}(1+\varepsilon A)^{-1}a(t)|^2 + (f(t), (1+\varepsilon A)^{-1}a(t))\right\} \\ &\quad -(f'(t) + (1+\varepsilon A)^{-1}a(t), A_\varepsilon u_\varepsilon(t))\end{aligned}$$

holds for a.e. $t \in [0, T]$.

Further, integrating (3.4) leads to

$$(3.5)\qquad \frac{\varepsilon}{2}\int_0^t |A_\varepsilon^{1/2}u'_\varepsilon(s)|^2 ds + \frac{1}{4}|A_\varepsilon u_\varepsilon(t)|^2 \leqq C_1 + \int_0^t k_1(s)|A_\varepsilon u_\varepsilon(s)|ds,$$

for $0 < \varepsilon \leqq 1$ and $0 \leqq t \leqq T$, where

$$C_1 = \frac{1}{2}|Au_0|^2 + |f(0)||Au_0| + |f|^2_{L^\infty(0,T;H)} + (1+C_0)C_0^{-1}|a|_{L^2(0,T)} + |f|_{L^2(0,T;H)}|a|_{L^2(0,T;H)}$$

and $k_1(t) = |f'(t)| + |a(t)| \in L^2(0, T)$. Therefore,

$$\frac{1}{2}|A_\varepsilon u_\varepsilon(t)| \leqq \sqrt{C_1} + \int_0^t k_1(s)ds,$$

$$(3.6)\qquad \frac{\varepsilon}{2}\int_0^t |A_\varepsilon^{1/2}u_\varepsilon(t)|^2 ds + \frac{1}{4}|A_\varepsilon u_\varepsilon(t)|^2 \leqq \left(\sqrt{C_1} + \int_0^t k_1(s)ds\right)^2.$$

Then the estimates of Proposition 3.4 follow from (3.6).

Lemma 3.5. $|u'_\varepsilon(0)| \leqq |w_0|$ *for any* $\varepsilon > 0$, *where* w_0 *is the element in H with* $f(0) - Au_0 \in \partial\varphi^0(w_0)$.

Proof. Since u_0 is in $A^{-1}(f(0) - R(\partial\varphi^0))$ there exists an element $w_0 \in D(\partial\varphi^0)$ satisfying $f(0) - Au_0 \in \partial\varphi^0(w_0)$. Hence we have

$$(\{f(0) - Au_0\} - \{f(0) - A_\varepsilon u_{0\varepsilon} - \varepsilon u'_\varepsilon(0)\}, w_0 - u'_\varepsilon(0)) \geqq 0,$$

which implies

$$|u'_\varepsilon(0)| \leqq |w_0|, \qquad \text{for any } \varepsilon > 0.$$

A priori estimates [II].

Next we consider the estimate of $A_\varepsilon^{1/2}u_\varepsilon'$ and $\varphi^{\cdot}(u_\varepsilon')$.

Proposition 3.6. *There is a positive constant*

$$R_2=R_2(|a|_{L^2(0,T;H)}, |f|_{W^{1,2}([0,T];H)}, |g|_{L^2(0,T)}, |Au_0|, |w_0|),$$

independent of $\varepsilon \in (0, 1]$, *such that*

$$|A_\varepsilon^{1/2}(u_\varepsilon'(\cdot))|_{L^2(0,T;H)}\leqq R_2, \qquad |\varphi^{\cdot}(u_\varepsilon'(\cdot))|_{L^2(0,T)}\leqq R_2.$$

Proof. Let $0<t\leqq T$, N an integer and $\delta=t/N$. Put $t_k=\delta k$, $k=0, 1, \cdots, N$. Put

$$v_\varepsilon(s)\equiv f(s)-\varepsilon u_\varepsilon'(s)-A_\varepsilon u_\varepsilon(s) \in \partial\varphi^s(u_\varepsilon'(s)) \qquad \text{for } s \in [0, T].$$

We have for $k=0, 1, \cdots, N-1$,

$$\begin{aligned}(3.7)\qquad &\varepsilon(u_\varepsilon'(t_{k+1})-u_\varepsilon'(t_k), u_\varepsilon'(t_{k+1}))+(v_\varepsilon(t_{k+1})-v_\varepsilon(t_k), u_\varepsilon'(t_{k+1}))\\ &+(A_\varepsilon(u_\varepsilon(t_{k+1})-u_\varepsilon(t_K)), u_\varepsilon'(t_{k+1}))\\ &=(f(t_{k+1})-f(t_k), u_\varepsilon'(t_{k+1})).\end{aligned}$$

Observe that

$$\begin{aligned}(v_\varepsilon(t_{k+1})-v_\varepsilon(t_k), u_\varepsilon'(t_{k+1}))&=(v_\varepsilon(t_{k+1}), u_\varepsilon'(t_{k+1}))-(v_\varepsilon(t_k), u_\varepsilon'(t_k))\\ &\quad+(v_\varepsilon(t_k), u_\varepsilon'(t_k)-z)+(v_\varepsilon(t_k), z-u_\varepsilon'(t_{k+1}))\end{aligned}$$

for any $z \in H$. Now, take $z \in D(\varphi^{t_k})$ so that

$$|z-u_\varepsilon'(t_{k+1})|\leqq|g(t_{k+1})-g(t_k)|(\varphi^{t_{k+1}}(u_\varepsilon'(t_{k+1}))+1)^{1/2}$$

and

$$\varphi^{t_k}(z)-\varphi^{t_{k+1}}(u_\varepsilon'(t_{k+1}))\leqq|g(t_{k+1})-g(t_k)|(\varphi^{t_{k+1}}(u_\varepsilon'(t_{k+1}))+1).$$

Then,

$$\begin{aligned}(3.8)\qquad &(v_\varepsilon(t_{k+1})-v_\varepsilon(t_k), u_\varepsilon'(t_{k+1}))\\ &\geqq(v_\varepsilon(t_{k+1}), u_\varepsilon'(t_{k+1}))-(v_\varepsilon(t_k), u_\varepsilon'(t_k))+\varphi^{t_k}(u_\varepsilon'(t_k))-\varphi^{t_{k+1}}(u_\varepsilon'(t_{k+1}))\\ &\quad-|g(t_{k+1})-g(t_k)|(\varphi^{t_{k+1}}(u_\varepsilon'(t_{k+1}))+1)\\ &\quad-|g(t_{k+1})-g(t_k)|(\varphi^{t_{k+1}}(u_\varepsilon'(t_{k+1}))+1)^{1/2}|v_\varepsilon(t_k)|.\end{aligned}$$

Hence, we infer from (3.7), (3.8) that

$$\begin{aligned}(3.9)\qquad &(\varepsilon/2)|u_\varepsilon'(t_{k+1})|^2+(v_\varepsilon(t_{k+1}), u_\varepsilon'(t_{k+1}))-\varphi^{t_{k+1}}(u_\varepsilon'(t_{k+1}))+\delta|A_\varepsilon^{1/2}u_\varepsilon'(t_{k+1})|^2\\ &\leqq(\varepsilon/2)|u_\varepsilon'(t_k)|^2+(v_\varepsilon(t_k), u_\varepsilon'(t_k))-\varphi^{t_k}(u_\varepsilon'(t_k))\end{aligned}$$

$$\begin{aligned}
&+\delta|A_\varepsilon u'_\varepsilon(t_{k+1})||u'_\varepsilon(t_{k+1})-(u_\varepsilon(t_{k+1})-u_\varepsilon(t_k))/\delta| \\
&+\delta((f(t_{k+1})-f(t_k))/\delta,\ u'_\varepsilon(t_{k+1})) \\
&+\delta|(g(t_{k+1})-g(t_k))/\delta|(\varphi^{t_{k+1}}(u'_\varepsilon(t_{k+1}))+1) \\
&+\delta|(g(t_{k+1})-g(t_k))/\delta|(\varphi^{t_{k+1}}(u'_\varepsilon(t_{k+1}))+1)^{1/2}|v_\varepsilon(t_k)|.
\end{aligned}$$

Adding there inequalities from $k=0$ up to $N-1$, we get

$$\begin{aligned}
(3.10)\qquad &(\varepsilon/2)|u'_\varepsilon(t)|^2+(v_\varepsilon(t),\ u'_\varepsilon(t))-\varphi^t(u'_\varepsilon(t))+\delta\sum_{k=0}^{N-1}|A_\varepsilon^{1/2}u'_\varepsilon(t_{k+1})|^2 \\
&\leqq(\varepsilon/2)|u'_\varepsilon(0)|^2+(v_\varepsilon(0),\ u'_\varepsilon(0))-\varphi^0(u'_\varepsilon(0)) \\
&\quad+\delta\sum_{k=0}^{N-1}|A_\varepsilon u'_\varepsilon(t_{k+1})||u'_\varepsilon(t_{k+1})-(u_\varepsilon(t_{k+1})-u_\varepsilon(t_k))/\delta| \\
&\quad+\delta\sum_{k=0}^{N-1}|(f(t_{k+1})-f(t_k))/\delta||u'_\varepsilon(t_{k+1})| \\
&\quad+\frac{3}{2}\delta\sum_{k=0}^{N-1}|(g(t_{k+1})-g(t_k))/\delta|(\varphi^{t_{k+1}}(u'_\varepsilon(t_{k+1}))+1) \\
&\quad+\frac{1}{2}\delta\sum_{k=0}^{N-1}|(g(t_{k+1})-g(t_k))/\delta||v_\varepsilon(t_k)|^2.
\end{aligned}$$

Moreover we note that

$$(v_\varepsilon(t),\ u'_\varepsilon(t))-\varphi^t(u'_\varepsilon(t))\geqq(v_\varepsilon(t),\ q(t))-\varphi^t(q(t)),$$

where $q\in W^{1,2}([0,\ T];\ H)$ with $\sup_{0\leqq t\leqq T}\varphi^t(q(t))<\infty$; and

$$\varphi^s(u'_\varepsilon(s))\leqq|v_\varepsilon(s)||q(s)-u'_\varepsilon(s)|+\varphi^s(q(s))\qquad\text{for all } s\in[0,\ T].$$

The existence of such a function q is shown in [10; Theorem 1]. Taking these inequalities into account, we deduce from (3.10) by letting $N\to\infty$ that

$$\begin{aligned}
(3.11)\quad &(\varepsilon/2)|u'_\varepsilon(t)|^2+\int_0^t|A_\varepsilon^{1/2}u'_\varepsilon(s)|^2ds \\
&\leqq(\varepsilon/2)|w_0|^2+|v_\varepsilon(0)||w_0|+|v_\varepsilon(t)||q(t)|+\varphi^t(q(t))+\int_0^t|f'(s)||u'_\varepsilon(s)|ds \\
&\quad+\int_0^t(3/2)|g'(s)|\{|v_\varepsilon(s)|(|u'_\varepsilon(s)|+|q(s)|)+\varphi^s(q(s))+1\}ds \\
&\quad+\int_0^t(1/2)|g'(s)||v_\varepsilon(s)|^2ds \\
&\leqq(\varepsilon/2)|w_0|^2+(\varepsilon|u'_\varepsilon(0)|+|Au_0|+|f(0)|)|w_0| \\
&\quad+(\varepsilon|u'_\varepsilon(t)|+|A_\varepsilon u_\varepsilon(t)|+|f(t)|)|q(t)|+\varphi^t(q(t))+\int_0^t|f'(s)||u'_\varepsilon(s)|ds
\end{aligned}$$

$$+\frac{3}{2}\int_0^t |g'(s)|\{(\varepsilon|u'_\varepsilon(s)|+|A_\varepsilon u_\varepsilon(s)|+|f(s)|)(|u'_\varepsilon(s)|+|q(s)|)+\varphi^s(q(s))+1\}ds$$

$$+(3/2)\int_0^t |g'(s)|\{\varepsilon^2|u'_\varepsilon(s)|^2+|A_\varepsilon u_\varepsilon(s)|^2+|f(s)|^2\}ds$$

hold for $t \in [0, T]$. Rearranging this inequality we see that

$$(\varepsilon/4)|u'_\varepsilon(t)|^2+\int_0^t |A_\varepsilon^{1/2}u'_\varepsilon(s)|^2 ds$$

$$\leqq k_{2\varepsilon}(t)+\int_0^t k_{3\varepsilon}(s)|u'_\varepsilon(s)|ds+\int_0^t |g'(s)|\left\{\left(\frac{3}{2}\varepsilon+\frac{3}{2}\varepsilon^2\right)|u'(s)|^2\right\}ds$$

hold for any $t \in [0, T]$ and $\varepsilon \in (0, 1]$. Here, for each $\varepsilon \in (0, 1]$ and $t \in [0, T]$, where

$$\begin{aligned} k_{2\varepsilon}(t)=&(\varepsilon/2)|w_0|^2+(\varepsilon|u'_\varepsilon(0)|+|Au_0|+|f(0)|)|w_0| \\ &+\varepsilon|q(t)|^2+(|A_\varepsilon u_\varepsilon(t)|+|f(t)|)|q(t)|+\varphi^t(q(t)) \\ &+(3/2)\int_0^t |g'(s)|(\varphi^s(q(s))+1)ds \\ &+\int_0^t |g'(s)|(|A_\varepsilon u_\varepsilon(s)|^2+|f(s)|^2)ds \\ &+(3/2)\int_0^t |g'(s)|(|A_\varepsilon u_\varepsilon(s)|+|f(s)|)|q(s)|ds \\ k_{3\varepsilon}(t)=&\{|f'(t)|+(3/2)|g'(t)|(|A_\varepsilon u_\varepsilon(t)|+|f(t)|)\}+(3/2)\varepsilon|g'(t)||q(t)|. \end{aligned}$$

By Proposition 3.4 and Lemma 3.5 there is a constant

$$C_2=C_2(|a|_{L^2(0,T;H)}, |f|_{W^{1,2}([0,T];H)}, |Au_0|, |w_0|)$$

such that

$$\sup_{\varepsilon\in(0,1]}|k_{2\varepsilon}|_{L^\infty(0,T)}\leqq C_2$$
$$\sup_{\varepsilon\in(0,1]}|k_{3\varepsilon}|_{L^2(0,T)}\leqq C_2.$$

Further, we note that $\varphi^t(u'_\varepsilon(t))$ satisfies

$$\varphi^t(u'_\varepsilon(t))\leqq(f(t)-\varepsilon u'_\varepsilon(t)-A_\varepsilon u_\varepsilon(t),\ q(t)-u'_\varepsilon(t))+\varphi^t(q(t))$$

for each $\varepsilon \in (0, 1]$ and $t \in [0, T]$. Then we have the estimates of Proposition 3.6.

§ 4. Convergence of the approximate solutions and uniqueness

By Proposition 3.4 and 3.6 there exist a sequence $\{\varepsilon(k); k\geqq 1\}\subset(0, 1]$ converging to 0 and a function $u \in W^{1,2}([0, T]; H)$ such that

$$(4.1)\qquad w\text{-}\lim_{k\to\infty} u_{\varepsilon(k)}(t)=u(t) \quad \text{in } H, \quad \text{for } 0\leqq t\leqq T,$$

$$(4.2)\qquad w\text{-}\lim_{k\to\infty} A^{1/2}(1+\varepsilon(k)A)^{-1}u_{\varepsilon(k)}(t)=A^{1/2}u(t) \quad \text{in } H,$$

for $0\leqq t\leqq T$,

$$(4.3)\qquad w^*\text{-}\lim_{k\to\infty} A_{\varepsilon(k)}u_{\varepsilon(k)}=Au \qquad \text{in } L^\infty(0, T; H),$$

$$(4.4)\qquad w\text{-}\lim_{k\to\infty} u'_{\varepsilon(k)}=u' \qquad \text{in } L^2(0, T; H),$$

$$(4.5)\qquad w\text{-}\lim_{k\to\infty} A^{1/2}(1+\varepsilon(k)A)^{-1}u'_{\varepsilon(k)}=A^{1/2}u' \qquad \text{in } L^2(0, T; H).$$

Now we set $D(\Phi)=\{v \in L^2(0, T; H);\ \varphi^s(v) \in L^1(0, T; H)\}$ and for each $v \in D(\Phi)$

$$\Phi(v)=\int_0^T \varphi^s(v(s))ds.$$

Then, we note that for $\varepsilon \in (0, 1]$ u_ε satisfies

$$(4.6)\qquad \Phi(v)-\Phi(u'_\varepsilon)\geqq\int_0^T (f(s)-A_\varepsilon u_\varepsilon(s)-\varepsilon u'_\varepsilon(s),\ v(s)-u'_\varepsilon(s))ds$$

for any $v \in D(\Phi)$.

We can show that Φ is a proper lower semicontinuous convex function from $L^2(0, T; H)$ into $[0, +\infty]$. Therefore,

$$(4.7)\qquad \liminf_{k\to\infty}\int_0^T \varphi^s(u'_{\varepsilon(k)}(s))ds\geqq\int_0^T \varphi^s(u'(s))ds.$$

Using there sequences, (4.6) can be rewritten as

$$\begin{aligned}
&\Phi(v)-\Phi(u'_{\varepsilon(k)})\\
&\quad\geqq\int_0^T (f(t),\ v(t)\quad u'_{\varepsilon(k)}(t))dt-\int_0^T (A_{\varepsilon(k)}u_{\varepsilon(k)}(t),\ v(t))dt\\
&\qquad+\int_0^T (A_{\varepsilon(k)}u_{\varepsilon(k)}(t),\ u'_{\varepsilon(k)}(t))dt+\int_0^T (\varepsilon(k)u'_{\varepsilon(k)}(t),\ v(t)-u'_{\varepsilon(k)}(t))dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T (A_{\varepsilon(k)}u_{\varepsilon(k)}(t),\ u'_{\varepsilon(k)}(t))dt\\
&\quad=(1/2)\int_0^T (d/dt)|A^{1/2}_{\varepsilon(k)}u_{\varepsilon(k)}(t)|^2dt\\
&\quad=(1/2)|A^{1/2}_{\varepsilon(k)}u_{\varepsilon(k)}(T)|^2-(1/2)(Au_0,\ (1+\varepsilon(k)A)u_0).
\end{aligned}$$

Moreover, by (4.2),

$$\begin{aligned}\liminf_{k\to\infty}|A_{\varepsilon(k)}^{1/2}u_{\varepsilon(k)}(T)| &\geqq \liminf_{k\to\infty}|A^{1/2}(1+\varepsilon(k)A)^{-1}u(T)|\\ &\geqq |A^{1/2}u(T)|.\end{aligned}$$

Therefore, by (4.3), (4.4), (4.5) and (4.7), we have

$$\int_0^T \varphi^s(v(s))ds-\int_0^T \varphi^s(u'(s))ds \geqq \int_0^T (f(s)-Au(s),\ v(s)-u'(s))ds, \qquad \text{for } v\in D(\Phi). \tag{4.8}$$

By (4.1)–(4.5) and (4.8) we can show u is a strong solution of (0.1) on $[0, T]$ provided $u_0 \in A^{-1}(f(0)-R(\partial\varphi^0))$, and that u satisfies (i), (ii), (iii′) and (iv′).

Next we shall prove Theorem 2.1 for the case in which $u_0 \in A^{-1}(f(0)-\overline{R(\partial\varphi^0)})$.

Multiplying both sides of (3.9) by t_{k+1}, adding the resultant ralations from $j=0$ up to $N-1$ and then letting $N\to\infty$, we have the following inequality in a way similar to the derivation of (3.10):

$$\begin{aligned}
&(\varepsilon t/2)|u_\varepsilon'(t)|^2+\int_0^t s|A_\varepsilon^{1/2}u_\varepsilon'(s)|^2ds+t\{(v_\varepsilon(t),\ u_\varepsilon'(t))-\varphi^t(u_\varepsilon'(t))\}\\
&\leqq (\varepsilon/2)\int_0^t |u_\varepsilon'(s)|^2ds+\int_0^t\{(v_\varepsilon(s),\ u_\varepsilon'(s))-\varphi^s(u_\varepsilon'(s))\}ds\\
&\quad+\int_0^t s(f'(s),\ u_\varepsilon'(s))ds+(3/2)\int_0^t s|g'(s)|(\varphi^s(u_\varepsilon'(s))+1)ds\\
&\quad+(1/2)\int_0^t s|g'(s)||v_\varepsilon'(s)|^2ds\\
&\leqq (\varepsilon/2)\int_0^t |u_\varepsilon'(s)|^2ds+\Big\{-\varepsilon\int_0^t|u_\varepsilon'(s)|^2ds-\frac{1}{2}|A_\varepsilon^{1/2}u_\varepsilon(t)|^2\\
&\quad+\frac{1}{2}|A_\varepsilon^{1/2}u_{0\varepsilon}|^2+(f(t),\ u_\varepsilon(t))-(f(0),\ u_{0\varepsilon})-\int_0^t(f'(s),\ u_\varepsilon(s))ds\Big\}\\
&\quad+\int_0^t s|f'(s)||u_\varepsilon'(s)|ds+(3/2)\int_0^t s|g'(s)|\\
&\quad\times\{\varphi^s(q(s))+(\varepsilon|u_\varepsilon'(s)|+|A_\varepsilon u_\varepsilon(s)|+|f(s)|)(|u_\varepsilon'(s)|+|q(s)|)+1\}ds\\
&\quad+(3/2)\int_0^t s|g'(s)|(\varepsilon^2|u_\varepsilon'(s)|^2+|A_\varepsilon u_\varepsilon(s)|^2+|f(s)|^2)ds,
\end{aligned} \tag{4.9}$$

for $0\leqq t\leqq T$. On the other hand

$$\begin{aligned}
&\text{the first member of (4.9)}\\
&\geqq(\varepsilon t/2)|u_\varepsilon'(t)|^2+t(v_\varepsilon(t),\ q(t))+\int_0^t s|A_\varepsilon^{1/2}u_\varepsilon'(s)|^2ds-t\varphi^t(q(t))\\
&\geqq(\varepsilon t/4)|u_\varepsilon'(t)|^2-t\{|q(t)|^2+(|A_\varepsilon u_\varepsilon(t)|+|f(t)|)|q(t)|\}\\
&\quad+\int_0^t s|A_\varepsilon^{1/2}u_\varepsilon'(s)|^2ds-t\varphi^t(q(t))
\end{aligned} \tag{4.10}$$

for $0 \leqq t \leqq T$. Rearrenging this inequality we see that

$$(4.11) \qquad (\varepsilon t/4)|u_\varepsilon'(t)|^2+\int_0^t s|A_\varepsilon^{1/2}u_\varepsilon'(s)|^2 ds$$

$$\leqq k_{4\varepsilon}(t)+\int_0^t s|g'(s)|\left(\frac{3}{2}+\frac{\varepsilon}{2}+\frac{3}{2}|q(s)|\right)\varepsilon|u_\varepsilon'(s)|^2 ds$$

$$+\int_0^t s\left\{\frac{3}{2}|g'(s)|(|A_\varepsilon u_\varepsilon(s)|+|f(s)|)+|f'(s)|\right\}|u_\varepsilon'(s)|ds$$

hold for any $t \in [0, T]$ and $\varepsilon \in (0, 1]$. Here, for each $\varepsilon \in (0, 1]$ and $t \in [0, T]$, where

$$k_{4\varepsilon}(t)=t\varphi^t(q(t))+t\{|q(t)|^2+(|A_\varepsilon u_\varepsilon(t)|+|f(t)|)|q(t)|\}$$

$$+|f(t)||u_\varepsilon(t)|+|f(0)||u_{0\varepsilon}|+\int_0^t|f'(s)||u_\varepsilon(s)|ds+\frac{1}{2}|A_\varepsilon^{1/2}u_{0\varepsilon}|^2$$

$$+(3/2)\int_0^t s|g'(s)|\{\varphi^s(q(s))+(|A_\varepsilon u_\varepsilon(s)|+|f(s)|)|q(s)|+1\}ds$$

$$+(1/2)\int_0^t s|g'(s)|(|A_\varepsilon u_\varepsilon(s)|^2+|f(s)|^2)ds.$$

By Proposition 3.4 there exists a positive constant

$$C_3=C_3(|a|_{L^2(0,T;H)}, |f|_{W^{1,2}(0,T;H)}, |g'|_{L^2(0,T)}, |Au_0|)$$

such that

$$\sup_{\varepsilon\in(0,1]}|k_{4\varepsilon}|_{L^\infty(0,T)}\leqq C_3.$$

Therefore we have a positive

$$R_3=R_3(|a|_{L^2(0,T;H)}, |f|_{W^{1,2}(0,T;H)}, |g'|_{L^2(0,T)}, |Au_0|)$$

such that

$$(4.12) \qquad \int_0^T t|A^{1/2}u'(t)|^2 dt\leqq R_3.$$

Let u_i $(i=1, 2)$ denote the strong solution of (0.1) on $[0, T]$ with $u_i(0)=u_{0i} \in A^{-1}(f(0)-R(\partial\varphi^0))$ $(i=1, 2)$, and satisfy (i), (ii), (iii) and (iv). Then we have that

$$(4.13) \qquad |A^{1/2}(u_1(t)-u_2(t))|\leqq|A^{1/2}(u_{01}-u_{02})|, \qquad \text{for } 0\leqq t\leqq T.$$

If u_0 is in $A^{-1}(f(0)-\overline{R(\partial\varphi^0)})$, then there exists a sequence $\{u_{0n}; n\geqq 1\}\subset A^{-1}(f(0)-R(\partial\varphi^0))$ satisfying $\lim_{n\to\infty} Au_{0n}=Au_0$ in H. Let u_n be the strong solution with $u_n(0)=u_{0n}$. By (4.11) there exists a function $u \in C([0, T]; H)$ satisfying

$$\lim_{n\to\infty} A^{1/2}u_n = A^{1/2}u \qquad \text{in } C([0, T]; H).$$

Since R_3 can be chosen so as to be bounded when $|Au_0|$ is bounded, there exists a positive constant R_4 depending upon $\sup_{n\geqq 1}|Au_{0n}|$ such that

$$\sup_{n\to\infty} \int_0^T t|A^{1/2}u_n'(t)|^2 dt \leqq R_4.$$

Consequently, we have

$$w\text{–}\lim_{n\to\infty} A^{1/2}u_n' = A^{1/2}u' \quad \text{in } L^2(\delta, T; H), \quad \text{for each } \delta \in (0, T).$$

Hence,

$$\int_0^T t|A^{1/2}u'(t)|^2 dt \leqq R_4.$$

Thus we can show by an argument analogous to that for the case $u_0 \in A^{-1}(f(0) - R(\partial\varphi^0))$ that u is a strong solution which satisfies (i), (ii), (iii) and (iv).

Finally we can prove the uniqueness of Theorem 2.1 by using the following inequality.

$$|A^{1/2}(u_1(t) - u_2(t))| \leqq |A^{1/2}(u_1(s) - u_2(s))|, \qquad \text{for } 0 \leqq s \leqq t \leqq T,$$

where u_i $(i=1, 2)$ denote the strong solution of (0.1) on $[0, T]$ with $u_i(0) = u_{0i} \in A^{-1}(f(0) - R(\partial\varphi^0))$ $(i=1, 2)$ and satisfies (i), (ii), (iii′) and (iv′). Q.E.D.

§ 5. Application

In the section we apply Theorem 2.1 to the initial boundary value problem for the following differential equations.

Let Ω be a bounded domain in $\boldsymbol{R}^n$ having a sufficiently smooth boundary. We put $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Omega$. Consider the following initial boundary value problem;

$$\text{(Ex)}\begin{cases} 0 \leqq u_t \leqq h & \text{on } Q, \\ \nu|u_t|^{p-2}u_t - \Delta u = f & \text{on } \{0 < u_t < h\}, \\ \nu|u_t|^{p-2}u_t - \Delta u \leqq f & \text{on } \{u_t = h\}, \\ \nu|u_t|^{p-2}u_t - \Delta u \geqq f & \text{on } \{u_t = 0\}, \\ u = 0 & \text{on } \Sigma, \\ u(0, \cdot) = u_0 & \text{on } \Omega, \end{cases}$$

where $\nu \geqq 0$, $p > 1$ are constants and $u_0(x)$, $h(t, x)$, $f(t, x)$ are given functions.

Then we have the following result by applying Theorem 2.1;

Corollary 5.1. *Let u_0, f and h be given functions satisfying*

$$h \in W^{1/2}([0, T]; L^p(\Omega)) \cap L^2(0, T; H^2(\Omega))$$
$$\Delta h(t, x) \leqq 0 \quad \text{for a.e. } x \in \Omega \quad \text{and a.e. } t \in [0, T].$$

Then there exists a unique solution u of (Ex) with the following properties;

(i) $u \in C([0, T]; H_0^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$,

(ii) $u_t \in L^2(Q)$.

Proof. We take $H = L^2(\Omega)$, $Au = -\Delta u$ for $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and for each $t \in [0, T]$

$$\varphi^t(v) = \begin{cases} \int_\Omega |v(x)|^{p+1} dx & \text{if } v \in K(t), \\ +\infty & \text{if } v \in L^2(\Omega) \backslash K(t), \end{cases}$$

where $K(t)$ is the closed convex subset of $L^2(\Omega)$ defined by

$$\{u \in L^p(\Omega);\ 0 \leqq u(x) \leqq h(t, x) \text{ for a.e. } x \in \Omega\}.$$

Then we see that A is a positive self-adjoint operator satisfying the assumption (A.2) and that for $0 \leqq t \leqq T$, φ^t is a proper lower semicontinuous convex function on $L^2(\Omega)$.

Next we show that $\{\varphi^t; 0 \leqq t \leqq T\}$ satisfies the assumption (A.1). We see that for each $0 \leqq t \leqq T$, φ^t is a proper lower semicontinuous convex function on $L^2(\Omega)$ and that the effective domain $D(\varphi^t)$ is the convex closed subset $K(t)$. Let $t_0 \in [0, T]$ and take $v_0 \in D(\varphi^{t_0})$. Then, setting for each $x \in \Omega$ and $t \in [0, T]$

$$v(t, x) = \begin{cases} h(t, x) & \text{if } v_0(x) > h(t, x), \\ v_0(x) & \text{if } 0 \leqq v_0(x) \leqq h(t, x). \end{cases}$$

We can show by the assumption (A.1) that $v(t, \cdot)$ is in $D(\varphi^t)$ and that it satisfies

$$\int_\Omega |v(t, x) - v_0(x)|^2 dx \leqq \int_\Omega |h(t, x) - h(t_0, x)|^2 dx \qquad \text{for } 0 \leqq t \leqq T$$

and

$$\varphi^t(v(t, \cdot)) \leqq \varphi^{t_0}(v_0) \qquad \text{for } 0 \leqq t \leqq T.$$

Thus (A.2) is verified.

In order to apply Theorem 2.1 to (Ex), we have only to verify that A and $\{\varphi^t; 0 \leqq t \leqq T\}$ satisfy the assumption (A.3). The assumption easily follows from the maximal principle, taking $a(t) = -\Delta h(t, \cdot)$. Q.E.D.

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