Funkcialaj Ekvacioj, 29 (1986) 219-236

Asymptotic Behavior of Periodic Systems Generated by Time-Dependent Subdifferential Operators

By

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Dedicated to Professor Isao Miyadera on the occasion of his 60th birthday

Introduction.

This paper is concerned with an abstract nonlinear evolution equation of the following form in a real Hilbert space H:

$$(0.1)\qquad u_t(t)+\partial\phi^t(u(t)) \ni f(t), \qquad t\in \boldsymbol{R},$$

where $\partial\phi^t$ is the subdifferential of a time-dependent lower semicontinuous convex function ϕ^t from H into $(-\infty, +\infty]$ with $\phi^t \not\equiv +\infty$; f is a given function in $L^2_{\mathrm{loc}}(\boldsymbol{R}; H)$; u is the unknown function from $\boldsymbol{R}$ into H and $u_t(t)$ is the strong derivative of $u(t)$ in H with respect to t. Many results on Cauchy or periodic problems for (0.1) were established so far. See e.g. [2, 4 11, 12, 14, 17, 18, 21, 22].

The purpose of this work is to investigate the asymptotic behavior of the system governed by (0.1). When ϕ^t and f have the same period $T>0$, i.e. $\phi^{t+T}=\phi^t$ and $f(t+T)=f(t)$ for all $t\in \boldsymbol{R}$, we are especially interested in the following property (*):

(*) (Asymptotical T-periodicity) For each solution u of (0.1) on $[0, +\infty)$ with relatively compact trajectory, there exists a T-periodic solution ω of (0.1) on $\boldsymbol{R}$ such that $u(t)-\omega(t)\to 0$ strongly in H as $t\to+\infty$.

In Dafermos [7, 8], the notion of compact and uniform process were introduced, and the ω-limitting sets of orbits and limiting processes were discussed. The results established there offer many suggestions to the study of the system governed by (0.1). However, these results are not directly applicable to the investigation of the asymptotic behavior of solutions of (0.1), because every solution u of (0.1) is constrained by $D(\phi^t)=\{z\in H; \phi^t(z)<+\infty\}$, i.e. $u(t)$ must belongs to $D(\phi^t)$ for all $t\in \boldsymbol{R}$, and the closure of $D(\phi^t)$ in H is allowed to vary with time t.

In the case where ϕ^t is independent of t, i.e. $\phi^t\equiv\phi$ and f is T-periodic, the asymptotic stability of solutions to

$$(0.2)\qquad u_t(t)+\partial\phi(u(t)) \ni f(t), \qquad t\in \boldsymbol{R},$$

was studied in detail by Baillon and Haraux [3]; for example, it was proved that the difference of any two T-periodic solutions of (0.2) is a constant vector, and that the property (*) corresponding to (0.2) is always satisfied. However, in our time-dependent case of ϕ^t, the asymptotic behavior of solutions is expected to be more complicated; in fact, as was exemplified in three dimensional space by Kenmochi and Ôtani [13], there exists a system governed by (0.1) such that the difference of two T-periodic solutions is not constant and property (*) does not hold, more precisely, there exist solutions with compact trajectories which are almost perodic but do not approach to any periodic solutions. From these reasons, it seems to be important and interesting to investigate the structure of the class of all T-periodic solutions in connection with property (*) as well as the classes of all almost periodic solutions and of all solutions with relatively compact trajectories. We shall show (in Theorem 1) that the property (*) with "T-periodic" replaced by "almost periodic" always holds. Also, we give a characterization of property (*) (in Theorem 2), and several sufficient conditions for property (*) from some practical points of view (in Theorem 3).

For the basic definitions used in this paper, such as convex functions, their effective domains and subdifferentials, we refer to Brézis [5]. Also, for almost periodic functions and their fundamental properties, we refer to Amerio and Prouse [1].

§ 1. Main results

Let H be a real Hilbert space with norm $|\cdot|$ and inner product $(\cdot, \cdot)$. Let $\{\phi^t\} = \{\phi^t; t \in \boldsymbol{R}\}$ be a family of lower semicontinuous convex functions ϕ^t from H into $(-\infty, +\infty]$ with $\phi^t \not\equiv +\infty$, and put $D(\phi^t) = \{z \in H; \phi^t(z) < +\infty\}$. Now, consider the evolution equation

$$E(\phi^t;f) \qquad u_t(t) + \partial\phi^t(u(t)) \ni f(t), \quad t \in J,$$

where J is an interval in $\boldsymbol{R}$, $\partial\phi^t$ is the subdifferential of ϕ^t and f is a given function in $L^2_{\mathrm{loc}}(\boldsymbol{R}; H)$.

By a *solution* of $E(\phi^t;f)$ on a compact interval $[t_0, t_1]$, we mean an H-valued function u in $C([t_0, t_1]; H) \cap W^{1,2}_{\mathrm{loc}}((t_0, t_1]; H)$ such that the relation $f(t) - u_t(t) \in \partial\phi^t(u(t))$ holds for a.e. $t \in [t_0, t_1]$. Also, for an interval J in $\boldsymbol{R}$, an H-valued function u on J is called a solution of $E(\phi^t;f)$ on J, if it is a solution of $E(\phi^t;f)$ on every compact subinterval of J in the above sense. In this work we shall use the terminology "*c-solution*" of $E(\phi^t;f)$ on J to mean a solution u such that the trajectory $\{u(t); t \in J\}$ is relatively compact in H.

We now introduce three classes $\mathscr{P}$, $\mathscr{AP}$, $\mathscr{RC}$ of solutions of $E(\phi^t;f)$ on $\boldsymbol{R}$.

$\mathscr{P} = \{\omega; \omega$ is a T-periodic solution of $E(\phi^t;f)$ on $\boldsymbol{R}\}$,

$\mathscr{AP} = \{w; w$ is an almost periodic solution of $E(\phi^t;f)$ on $\boldsymbol{R}\}$,

$\mathscr{RC} = \{u; u$ is a c-solution of $E(\phi^t;f)$ on $\boldsymbol{R}\}$.

Here $u \in C(R; H)$ is called T-periodic, if $u(t+T)=u(t)$ for all $t \in \boldsymbol{R}$, and is called almost periodic, if it is almost periodic as a function from $\boldsymbol{R}$ into H.

It is well known that among these classes the following inclusion holds. (See e.g. [1].)

$$\mathscr{P} \subset \mathscr{AP} \subset \mathscr{RC}. \tag{1.1}$$

In what follows, let T be a fixed positive number and suppose that the following hypotheses (H.1) and (H.2) are satisfied.

(H.1) (i) ϕ^t is T-periodic, i.e. $\phi^{t+T}=\phi^t$ for any $t \in \boldsymbol{R}$,

(ii) $f \in L^2_{\mathrm{loc}}(R; H)$ and $f(t+T)=f(t)$ for a.e. $t \in \boldsymbol{R}$.

(H2) There exists a monotone increasing function $M(r, s)$ of two variables r and s from $[0, +\infty)\times[0, +\infty)$ into $[0, +\infty)$ such that for any $t_0 \in \boldsymbol{R}$, $u_0 \in \overline{D(\phi^{t_0})}$ and $f \in L^2(t_0, t_0+2; H)$, $E(\phi^t; f)$ has a unique solution u on $[t_0, t_0+2]$ with $u(t_0)=u_0$ satisfying

$$\int_{t_0+1}^{t_0+2} |u_t(t)|^2 dt \leqq M(|u_0|, |f|_{L^2(t_0, t_0+2; H)}), \tag{1.2}$$

$$\phi^t(u(t)) \text{ is continuous on } (t_0, t_0+2]. \tag{1.3}$$

Remark 1. (1) Several sufficient conditions for (H.2) are known so far. For example, assume the following condition (C.1).

(C.1) There exist an exponent $\alpha \in [0, 1]$ and three families $\{K_r; r \geqq 0\} \subset [0, +\infty)$, $\{a_r(\cdot); r \geqq 0\} \subset W^{1,\beta}(0, T)$ with $\beta = \max\{2, 1/(1-\alpha)\}$ (resp. $\beta=+\infty$) if $0 \leqq \alpha < 1$ (resp. $\alpha=1$), and $\{b_r(\cdot); r \geqq 0\} \subset W^{1,1}(0, T)$ satisfying the following properties (h.1) and (h.2):

(h.1) $\phi^t(z)+K_r \geqq 0$ for any $t \in [0, T]$, $r \geqq 0$ and $z \in H$ with $|z| \leqq r$,

(h.2) For each $s, t \in [0, T]$, $r \geqq 0$ and $z \in D(\phi^s)$ with $|z| \leqq r$, there is $z' \in D(\phi^t)$ such that

$$\begin{cases} |z'-z| \leqq |a_r(t)-a_r(s)|(\phi^s(z)+K_r)^\alpha, \\ \phi^t(z')-\phi^s(z) \leqq |b_r(t)-b_r(s)|(\phi^s(z)+K_r). \end{cases} \tag{1.4}$$

Then (H.2) is satisfied and (1.3) is replaced by

$$(1.3)' \qquad \phi^t(u(t)) \text{ is absolutely continuous on } (t_0, t_0+2] \text{ and}$$
$$\sup\{(t-t_0)\phi^t(u(t));\ t_0 < t \leqq t_0+2\} \leqq M(|u_0|, |f|_{L^2(t_0, t_0+2; H)}).$$

(See, e.g. [11, 12, 17, 21, 22].)

(2) There is another important class $\mathscr{B}$ given by

$$\mathscr{B}=\{u;\ u \text{ is a bounded solution of } E(\phi^t; f) \text{ on } \boldsymbol{R}\},$$

where "bounded" means that $\sup_{t \in \boldsymbol{R}} |u(t)| < +\infty$. Clearly $\mathscr{RC} \subset \mathscr{B}$ holds. In this paper, we leave $\mathscr{B}$ out of consideration. However, note that $\mathscr{B}$ coincides with $\mathscr{RC}$ provided that (C.1) (see (1.3)′) and the following (C.2) are satisfied.

(C.2) For each $t \in [0, T]$ and $r \geqq 0$, the set $\{u; |u| \leqq r, \phi^t(u) \leqq r\}$ is compact in H.

The first main result is concerned with the almost periodicity of solutions of $E(\phi^t; f)$.

Theorem 1. *Suppose that* (H.1) *and* (H.2) *hold, and that* $\mathscr{RC} \neq \phi$. *Then* $\mathscr{AP} = \mathscr{RC}$, *and for each c-solution u of* $E(\phi^t; f)$ *on* $[t_0, +\infty)$, $t_0 \in \boldsymbol{R}$, *there exists* $\omega \in \mathscr{AP}$ *such that* $u(t) - \omega(t) \to 0$ *strongly in* $\boldsymbol{H}$ *as* $t \to +\infty$.

A result of the same kind as Theorem 1 was obtained by Haraux [10] in a more general framework, where the result is stated in terms of "periodic contractive process" on a fixed metric space. However, this result does not formally cover Theorem 1, since in our setting the periodic contractive process $U(t, \tau)$ generated by $\partial\phi^t$ is the operator from $\overline{D(\phi^t)}$ into $\overline{D(\phi^{t+\tau})}$, and $\overline{D(\phi^t)}$ may vary with the time t. Although our proof of Theorem 1 is a modification of that of Haraux's result, we shall present it in Section 2 for the sake of completeness.

The second main result is concerned with the T-periodicity of solutions of $E(\phi^t; f)$.

Theorem 2. *Suppose that* (H.1) *and* (H.2) *hold, and that* $\mathscr{RC} \neq \phi$. *Then the following statements* (a) *and* (b) *are equivalent to each other*:

(a) *For each c-solution u of* $E(\phi^t; f)$ *on* $[t_0, +\infty)$, $t_0 \in \boldsymbol{R}$, *there exists a solution* ω *in* $\mathscr{P}$ *such that* $u(t) - \omega(t) \to 0$ *strongly in H as* $t \to +\infty$.

(b) $\mathscr{P} = \mathscr{RC}$, *and hence* $\mathscr{P} = \mathscr{AP} = \mathscr{RC}$.

As to the system governed by the evolution equation, involving the time-independent subdifferential $\partial\phi$:

$$u_t(t) + \partial\phi(u(t)) \ni f(t), \qquad t \in \boldsymbol{R}, \tag{1.5}$$

Baillon and Haraux [3] proved under (ii) of (H.1) that for each solution u of (1.5) on $[t_0, +\infty)$, there is a T-periodic solution ω of (1.5) on $\boldsymbol{R}$ satisfying $u(t) - \omega(t) \to 0$ weakly in H as $t \to +\infty$. Therefore property (a) of Theorem 2 always holds. The proof of this result relies on the fact that the difference of any two T-periodic solutions is a constant vector. However, for the system governed by $E(\phi^t; f)$, such nice properties are no longer expected; in fact, there exists a system (c.f. [13]) which is generated by the subdifferential of the indicator function of a moving convex set in $\boldsymbol{R}^3$, and for which (H.1) and (H.2) hold, but property (a) of Theorem 2 does not hold and the difference of two T-periodic solutions is not constant. Hence it would be worth while to investigate sufficient conditions for property (a) of Theorem 2 from some practical points of view.

Theorem 3. *Suppose that* (H.1) *and* (H.2) *hold, and that* $\mathscr{RC} \neq \phi$. *Then any one of the following conditions* (i)–(iv) *is sufficient for property* (a) *of Theorem* 2:

(i) ϕ^t *is strictly convex on* $D(\phi^t)$ *for some* $t=t_0 \in [0, T]$.

(ii) *For every c-solution u of* $E(\phi^t; f)$ *on* $[0, T]$, $-u_t(t)$ *coincides with the minimal section* $(\partial\phi^t(u(t))-f(t))^0$ *of* $\partial\phi^t(u(t))-f(t)$ *for a.e.* $t \in [0, T]$.

(iii) *There exists an H-valued T-periodic continuous function* $\alpha_0(t)$ *such that if* $z_i^* \in \partial\phi^t(z_i)$; $i=1, 2$, $t \in \boldsymbol{R}$, *and* $(z_1^*-z_2^*, z_1-z_2)=0$, *then* z_1-z_2 *belongs to the subspace* $\{C\alpha_0(t);\ C \in \boldsymbol{R}\}$.

(iv) *For each* $z \in D(\phi^0)$, *there exists an element* z_0 *of* $\mathscr{P}(0) \equiv \{\omega(0);\ \omega \in \mathscr{P}\}$ *and a number* ε_0 *in* (0, 1) *such that* $z_0+\varepsilon_0(z-z_0) \in \mathscr{P}(0)$.

As will be seen from Lemma 1 in Section 2, the system governed by $E(\phi^t; f)$ is contractive. Therefore, property (a) of Theorem 2 is closely related to the study of the fixed points of contractive mappings on closed convex sets (Cf. [6, 7, 15, 16, 19, 20]). Especially the sufficient condition (iv) in Theorem 3 is suggested from such a point of view. We note that condition (iv) is satisfied if $\mathscr{P}(0)$ has at least one internal point of $D(\phi^0)$.

Remark 2. As will be seen in Section 3, property (1.3) of (H.2) is needed only for the proof of (i) of Theorem 3. Even if (1.3) is not satisfied, the following condition (i)′ is also sufficient for property (a).

(i)′ ϕ^t is strictly convex on $D(\phi^t)$ for t in a subset of $[0, T]$ with positive linear measure.

Then in order to assure all the assumptions of (H.2) except (1.3), condition (C.1) can be weakened slightly as follows: the property (h.2) can be replaced by the following (h.2)′ (see [12]):

(h.2)′ For each s, $t \in [0, T]$ with $s \leqq t$, $r \geqq 0$ and $z \in D(\phi^s)$ with $|z| \leqq r$, there is $z' \in D(\phi^t)$ satisfying (1.4).

§ 2. Some lemmas

We prepare several lemmas for the proofs of the theorems. Throughout this section, we always assume that (H.1) and (H.2) hold and $\mathscr{RC} \neq \phi$.

Lemma 1. (1) *Let u and v be any solutions of* $E(\phi^t; f)$ *on* $[t_0, +\infty)$. *Then*

$$|u(t)-v(t)| \leqq |u(s)-v(s)| \qquad \text{for any } t \geqq s \geqq t_0, \tag{2.1}$$

and hence $\lim_{t\to+\infty}|u(t)-v(t)|$ *exists.*

(2) *Let u be any solution of* $E(\phi^t; f)$ *on* $[t_0, +\infty)$. *Then u is bounded on* $[t_0, +\infty)$ *and the function* $t \mapsto \phi^t(u(t))$ *is continuous on* $[t_0+1, +\infty)$. *Moreover* $\sup_{t\geqq t_0+1}|u_t|_{L^2(t,t+1;H)} < +\infty$.

Proof. The first assertion is a direct consequence of the monotonicity of $\partial\phi^t$. Let u be any solution of $E(\phi^t; f)$ on $[t_0, +\infty)$, and take a bounded solution v in $\mathscr{RC}$. Then, by (2.1), $|u(t)|\leqq|v(t)|+|u(t_0)-v(t_0)|$ for all $t\geqq t_0$. Hence u is bounded on $[t_0, +\infty)$. Thus the second assertion follows from (1.2) and (1.3) of (H.2). Q.E.D.

Lemma 2. *Let u be any c-solution of $E(\phi^t; f)$ on $[t_0, +\infty)$. Then there exists a sequence $\{n_k\}$ of integers with $n_k\to+\infty$ as $k\to+\infty$ such taht for each $m=0, \pm1, \pm2, \cdots$, $u(n_kT-mT)$ converges strongly in H as $k\to+\infty$.*

Proof. Let m be any integer. Since the set $\{u(nT-mT);\ n\geqq m+t_0/T, n\in \boldsymbol{Z}\}$ is relatively compact in H, we can choose a sequence $\{n_{k(m)}\}$ with $n_{k(m)}\to+\infty$ as $k(m)\to+\infty$ such that $u(n_{k(m)}T-mT)$ converges strongly in H as $k(m)\to+\infty$. Hence, using the standard diagonal argument, we can obtain a sequence $\{n_k\}$ satisfying the desired properties. Q.E.D.

Lemma 3. (1) *Let u and v be in $\mathscr{B}$, the set of all bounded solutions of $E(\phi^t; f)$ on $\boldsymbol{R}$. Then $\lim_{t\to-\infty}|u(t)-v(t)|$ exists.*

(2) *Let u be in $\mathscr{B}$. Then $t\mapsto\phi^t(u(t))$ is continuous on $\boldsymbol{R}$, and*

$$\sup_{t\in \boldsymbol{R}}|u_t(t)|_{L^2(t,t+1;H)}<+\infty.$$

Proof. Let $u, v\in\mathscr{B}$. Then, by (1) of Lemma 1, the function $|u(t)-v(t)|$ of t is bounded and monotone non-increasing on $\boldsymbol{R}$, whence follows (1). The second assertion is assured by (1.2) and (1.3) of (H.2). Q.E.D.

Lemma 4. *Let u be in $\mathscr{RC}$. Then there exists a sequence $\{n_k\}$ of integers with $n_k\to+\infty$ as $k\to+\infty$ such that for each $m=0, \pm1, \pm2, \cdots$, $u(-n_kT-mT)$ converges strongly in H as $k\to+\infty$.*

Proof. It suffices to repeat the same argument as in the proof of Lemma 2. Q.E.D.

Lemma 5. *Let u be any c-solution of $E(\phi^t; f)$ on $[t_0, +\infty)$. Then there exists a sequence $\{n_k\}$ with $n_k\to\infty$ as $k\to+\infty$ such that $u_k(t)\equiv u(t+n_kT)$, $t\in[-n_kT+t_0, +\infty)$, converges strongly in H and uniformly in $t\in[-mT, +\infty)$ as $k\to+\infty$ for each $m=1, 2, \cdots$. Moreover, the limit function $\hat{u}$ belongs to $\mathscr{RC}$.*

Proof. First take a sequence $\{n_k\}$ as in Lemma 2. Then u_k is clearly a c-solution of $E(\phi^t; f)$ on $[-n_kT+t_0, +\infty)$, and by (2.1)

$$|u_k(t)-u_j(t)|\leqq|u_k(-mT)-u_j(-mT)|=|u(n_kT-mT)-u(n_jT-mT)|$$
$$\text{for } t\geqq-mT \text{ and sufficiently large } k, j.$$

Since $u(n_kT-mT)$ converges strongly in H as $k\to+\infty$, it follows from the above inequality that u_k converges strongly in H and uniformly on $[-mT, +\infty]$ as $k\to+\infty$ for each $m=1, 2, \cdots$. Hence the limit function $\hat{u}$ of u_k is well defined on $\boldsymbol{R}$. Now, let s be any number in $\boldsymbol{R}$ and v be the solution of $E(\phi^t; f)$ on $[s, +\infty)$ with $v(s)=\hat{u}(s)$. Then, by (2.1) again,

$$\begin{aligned}|v(t)-\hat{u}(t)| &\leqq |v(t)-u_k(t)|+|u_k(t)-\hat{u}(t)|\\ &\leqq |\hat{u}(s)-u_k(s)|+|u_k(t)-\hat{u}(t)|\end{aligned}$$

holds for all $t\geqq s$ and sufficiently large k. Letting $k\to+\infty$ in these inequalities, we find that $v(t)=\hat{u}(t)$ for all $t\geqq s$. Consequently $\hat{u}$ turns out to be a solution of $E(\phi^t; f)$ on $\boldsymbol{R}$. Furthermore, since $\{\hat{u}(t); t\in\boldsymbol{R}\}\subset\overline{\{u(t); t\geqq t_0\}}$, the trajectory of $\hat{u}$ is relatively compact. Thus $\hat{u}\in\mathscr{RC}$. Q.E.D.

Now, we introduce an auxiliary class, denoted by $\mathscr{RC}_0$, of solutions of $E(\phi^t; f)$ on $\boldsymbol{R}$:

$$\mathscr{RC}_0=\{u\in\mathscr{RC};\ |u(t)-\omega(t)|=|u(0)-\omega(0)| \text{ for any } t\in\boldsymbol{R} \text{ and any } \omega\in\mathscr{P}\}.$$

Then we have

$$\mathscr{AP}\subset\mathscr{RC}_0. \tag{2.2}$$

In fact, for each $u, v\in\mathscr{AP}$, $|u(t)-v(t)|$ becomes a real valued almost periodic function and $\lim_{t\to-\infty}|u(t)-v(t)|$ exists by (1) of Lemma 1. Hence $|u(t)-v(t)|$ must be constant for all $t\in\boldsymbol{R}$ (see [1] and [9]).

Lemma 6. *Let u be in $\mathscr{RC}$. Then there exists a sequence $\{n_k\}$ with $n_k\to+\infty$ as $k\to+\infty$ such that $u_{-k}(t)\equiv u(t-n_kT)$, $t\in\boldsymbol{R}$, converges strongly in H and uniformly in $t\in[-mT, +\infty)$ as $k\to+\infty$ for each $m=1, 2, \cdots$. Moreover the limit function $\hat{u}$ belongs to $\mathscr{RC}_0$.*

Proof. Let $u\in\mathscr{RC}$. Then, with the help of Lemma 4, we can see by the same argument as in the proof of Lemma 5 that $u_{-k}(t)$ converges strongly in H and uniformly in $t\in[-mT, +\infty)$ as $k\to+\infty$ for each $m=1, 2, \cdots$, and that its limit function $\hat{u}$ belongs to $\mathscr{RC}$. Next, let ω be any element in $\mathscr{P}$. Then, it follows from (1) of Lemma 3 that $d=\lim_{t\to-\infty}|u(t)-\omega(t)|$ exists. Hence

$$\begin{aligned}d&=\lim_{k\to+\infty}|u(-n_kT+t)-\omega(-n_kT+t)|\\ &=\lim_{k\to+\infty}|u_{-k}(t)-\omega(t)|=|\hat{u}(t)-\omega(t)| \qquad \text{for all } t\in\boldsymbol{R},\end{aligned}$$

so that $\hat{u}\in\mathscr{RC}_0$. Q.E.D.

Lemma 7. *Let u and v be solutions of $E(\phi^t; f)$ on $[t_0, +\infty)$ such that $|u(t)-v(t)|\equiv d$ for all $t\geqq t_0$, where d is a non-negative constant. Then, for each $\lambda\in(0, 1)$, $\lambda u+(1-\lambda)v$ becomes a solution of $E(\phi^t; f)$ on $[t_0, +\infty)$.*

Proof. Let $0<\lambda<1$, and let w be the solution of $E(\phi^t;f)$ on $[t_0, +\infty)$ with $w(t_0)=\lambda u(t_0)+(1-\lambda)v(t_0)$. Then, by (2.1),

$$|u(t)-w(t)|\leqq|u(t_0)-w(t_0)|=(1-\lambda)|u(t_0)-v(t_0)| \qquad \text{for } t\geqq t_0,$$

and similarly

$$|v(t)-w(t)|\leqq\lambda|u(t_0)-v(t_0)| \qquad \text{for } t\geqq t_0.$$

Hence we have

$$\begin{aligned} d=|u(t)-v(t)|&\leqq|u(t)-w(t)|+|w(t)-v(t)| \\ &\leqq(1-\lambda)|u(t_0)-v(t_0)|+\lambda|u(t_0)-v(t_0)|=d \qquad \text{for } t\geqq t_0. \end{aligned}$$

Therefore, $|u(t)-w(t)|=(1-\lambda)|u(t)-v(t)|$ and $|v(t)-w(t)|=\lambda|u(t)-v(t)|$ for $t\geqq t_0$, which implies that $w(t)=\lambda u(t)+(1-\lambda)v(t)$ for $t\geqq t_0$. Q.E.D.

Lemma 8. *Let u be in $\mathscr{RC}$, and let $\{\lambda_n\}$ be a sequence in $\boldsymbol{Z}$. Then there exist an element x_0 in H, a number $\varepsilon_0 \in (0, 1)$, a sequence $\{\nu_k\}$ in N with $\nu_k\to+\infty$ as $k\to+\infty$ and a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ such that*

$$|u(-\nu_l T+\lambda_{n_k}T)-x_0|\leqq\varepsilon_0^k \qquad \textit{for all } l, k \quad \textit{with} \quad l\geqq k.$$

Proof. By Lemma 4, there exists a sequence $\{\mu_l\}$ with $\mu_l\to+\infty$ as $l\to+\infty$ such that $u(-\mu_l T+\lambda_n T)$ converges strongly in H to some x_n for each $n \in N$. Since $x_n \in \overline{\{u(t);\ t \in \boldsymbol{R}\}}$ for all $n \in N$, $\{x_n\}_{n\in N}$ forms a relatively compact subset of H. Hence we can choose a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k}\to x_0$ strongly in H as $k\to+\infty$. For simplicity, let us denote $\{\lambda_{n_k}\}$ by $\{\lambda_n\}$ again. Now, given ε_0 in (0, 1), we take $n_1 \in N$ so that $|x_{n_1}-x_0|<\varepsilon_0/2$, and next take $l_1 \in N$ so that $|u(-\mu_l T+\lambda_{n_1}T)-x_{n_1}|<\varepsilon_0/2$ for all $l\geqq l_1$. We then have

$$|u(-\mu_l T+\lambda_{n_1}T)-x_0|<\varepsilon_0 \qquad \text{for all } l\geqq l_1.$$

We repeat this procedure by replacing ε_0 by ε_0^k, $k=2, 3, \cdots$, to get subsequences $\{\mu_{l_k}\}$ and $\{\lambda_{n_k}\}$ such that

$$|u(-\mu_l T+\lambda_{n_k}T)-x_0|<\varepsilon_0^k \qquad \text{for all } l\geqq l_k,\ k=1, 2, \cdots.$$

Thus $\{\nu_k\}=\{\mu_{l_k}\}$ and $\{\lambda_{n_k}\}$ satisfy the desired property. Q.E.D.

§ 3. Proofs of main theorem

We begin with the proof of Theorem 1.

Proof of Theorem 1. First we prove $\mathscr{AP}=\mathscr{RC}$. Since $\mathscr{AP}\subset\mathscr{RC}$, it suffices to verify $\mathscr{AP}\supset\mathscr{RC}$. In showing this, we use Bochner's criterion (cf. [1]) for the almost periodicity which is stated as follows: "A function $v \in C(\boldsymbol{R}; H)$ is almost periodic on $\boldsymbol{R}$ if and only if for any sequence $\{c_n\}$ in $\boldsymbol{R}$, there exists a subsequence $\{c_{n_k}\}$ such that $v(\cdot + c_{n_k})$ converges strongly in H and uniformly on $\boldsymbol{R}$ as $k \to +\infty$." Now, let $u \in \mathscr{RC}$ and let $\{c_n\}$ be any sequence in $\boldsymbol{R}$. Put

$$c_n = \lambda_n T + a_n, \quad \lambda_n \in \boldsymbol{Z}, \quad 0 \leqq a_n < T.$$

Then, by virtue of Lemma 8, there exist $x_0 \in H$, $\varepsilon_0 \in (0, 1)$, a sequence $\{\nu_l\}$ in $\boldsymbol{N}$ and a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\text{(3.1)} \qquad a_{n_k} \longrightarrow a_0 \in [0, T] \qquad \text{as } k \longrightarrow +\infty,$$

and

$$\text{(3.2)} \qquad |u(-\nu_l T + \lambda_{n_k} T) - x_0| \leqq \varepsilon_0^k \qquad \text{for all } l, k \text{ with } l \geqq k.$$

For each $t \in \boldsymbol{R}$ and $c_{n_p}, c_{n_q} \in \{c_{n_k}\}$, choose $l \in \boldsymbol{N}$ so that $l \geqq \max\{p, q\}$ and $-\nu_l T < t$. Then it follows from (3.2), (2.1) and (2) of Lemma 3 that

$$\begin{aligned}
|u(t+c_{n_p}) - u(t+c_{n_q})| &= |u(t+a_{n_p}+\lambda_{n_p}T) - u(t+a_{n_q}+\lambda_{n_q}T)| \\
&\leqq |u(t+a_{n_p}+\lambda_{n_p}T) - u(t+a_{n_p}+\lambda_{n_q}T)| \\
&\quad + |u(t+a_{n_p}+\lambda_{n_q}T) - u(t+a_{n_q}+\lambda_{n_q}T)| \\
&\leqq |u(-\nu_l T+\lambda_{n_p}T) - u(-\nu_l T+\lambda_{n_q}T)| + |a_{n_p} - a_{n_q}|^{1/2} L \\
&\leqq |u(-\nu_l T+\lambda_{n_p}T) - x_0| + |u(-\nu_l T+\lambda_{n_q}T) - x_0| \\
&\quad + |a_{n_p} - a_{n_q}|^{1/2} L \\
&\leqq \varepsilon_0^p + \varepsilon_0^q + |a_{n_p} - a_{n_q}|^{1/2} L, \qquad \text{for all } t \in \boldsymbol{R},
\end{aligned}$$

where $L = \sup_{s \in \boldsymbol{R}} |u_t|_{L^2(s, s+T; H)} < +\infty$. Hence, by (3.1), $u(\cdot + c_{n_k})$ converges strongly in H and uniformly on $\boldsymbol{R}$, whence $u \in \mathscr{AP}$.

Next, we prove the second assertion. Without loss of generality, we may assume that $t_0 = 0$ and u is a c-solution of $E(\phi^t; f)$ on $[0, +\infty)$. Then from Lemma 5 it follows that there exists a sequence $\{n_k\}$ in $\boldsymbol{N}$ such that $u(t + n_k T)$ converges to some $\hat{u}(t) \in \mathscr{RC}$ strongly in H and uniformly in $t \in [-mT, +\infty)$ for each $m = 1, 2, \cdots$, as $k \to +\infty$. Since $\hat{u} \in \mathscr{RC} = \mathscr{AP}$ as was seen above, there exists a subsequence $\{k_j\}$ of $\{k\}$ and an element $\omega \in \mathscr{AP}$

$$\hat{u}(t - n_{k_j} T) - \omega(t) \longrightarrow 0 \quad \text{strongly in } H \text{ and uniformly in } t \in \boldsymbol{R} \quad \text{as } j \to +\infty.$$

Let ε be any positive number, and choose $j_\varepsilon \in \boldsymbol{N}$ such that

$$|u(t + n_{k_j} T) - \hat{u}(t)| < \varepsilon/2 \qquad \text{for all } t \geqq 0 \text{ and } j \geqq j_\varepsilon,$$

or equivalently

$$|u(t) - \hat{u}(t - n_{k_j} T)| < \varepsilon/2 \qquad \text{for all } t \geqq n_{k_j} T \text{ and } j \geqq j_\varepsilon,$$

and such that

$$|\hat{u}(t-u_{k_j}T)-\omega(t)|<\varepsilon/2 \qquad \text{for all } t\in \boldsymbol{R} \text{ and } j\geqq j_\varepsilon.$$

Then we have

$$|u(t)-\omega(t)|<\varepsilon \qquad \text{for all } t\geqq n_{k_j}T \text{ and } j\geqq j_\varepsilon,$$

which implies that $u(t)-\omega(t)\to 0$ strongly in H as $t\to+\infty$. Q.E.D.

Now, recall the class $\mathscr{RC}_0$, which is introduced before the statement of Lemma 6 in Section 2. Then Theorem 1 and relation (2.2) give

$$\mathscr{AP}=\mathscr{RC}_0=\mathscr{RC}. \tag{3.3}$$

Proof of Theorem 2. (*Proof of* (a)→(b)) Let u be any element of $\mathscr{RC}=\mathscr{RC}_0$. Then, by (a), there exists $\omega\in\mathscr{P}$ such that $\lim_{t\to+\infty}|u(t)-\omega(t)|=0$. Since $|u(t)-\omega(t)|=|u(0)-\omega(0)|$ for all $t\in\boldsymbol{R}$, it follows that $u=\omega\in\mathscr{P}$.

(*Proof of* (b)→(a)). Let u be any c-solution of $E(\phi^t;f)$ on $[t_0,+\infty)$. Then, by virtue of Lemma 5, there exists a sequence $\{n_k\}$ such that $u_k(t)\equiv u(t+n_kT)$, $t\in[-n_kT+t_0,+\infty)$, converges to some $\hat{u}\in\mathscr{RC}=\mathscr{P}$ strongly in H and uniformly in $t\in[-mT,+\infty)$ for each $m=1,2,\cdots$. Furthermore, we see from (1) of Lemma 1 that $d=\lim_{t\to+\infty}|u(t)-\hat{u}(t)|$ exists and

$$d=\lim_{k\to+\infty}|u(n_kT)-\hat{u}(n_kT)|=\lim_{k\to+\infty}|u_k(0)-\hat{u}(0)|=0.$$

Thus (a) is derived. Q.E.D.

By virtue of (3.3) and Theorem 2, property (a) in Theorem 2 is equivalent to the relation $\mathscr{P}=\mathscr{RC}_0$. Therefore, in order to prove Theorem 3, it suffices to derive the relation $\mathscr{P}=\mathscr{RC}_0$ instead of (a) from each of the conditions (i)–(iv).

Proof of Theorem 3. (*Proof of* (i)→"$\mathscr{P}=\mathscr{RC}_0$") Let $u\in\mathscr{RC}_0$ and $\omega\in\mathscr{P}$. Then, from the definition of subdifferential it follows that

$$(f(t)-\omega_t(t),\ u(t)-\omega(t))\leqq\phi^t(u(t))-\phi^t(\omega(t)) \qquad \text{for a.e. } t\in\boldsymbol{R},$$

and

$$(f(t)-u_t(t),\ \omega(t)-u(t))\leqq\phi^t(\omega(t))-\phi^t(u(t)) \qquad \text{for a.e. } t\in\boldsymbol{R}.$$

Adding these inequalities and noting that $|u(t)-\omega(t)|=\text{Const.}$ on $\boldsymbol{R}$, we see that

$$\begin{aligned}0&=(u_t(t)-\omega_t(t),\ u(t)-\omega(t))\\&=(f(t)-\omega_t(t),\ u(t)-\omega(t))+(f(t)-u_t(t),\ \omega(t)-u(t))\\&\leqq\{\phi^t(u(t))-\phi^t(\omega(t))\}+\{\phi^t(\omega(t))-\phi^t(u(t))\}=0 \qquad \text{for a.e. } t\in\boldsymbol{R}.\end{aligned}$$

Therefore,

$$(3.4)\qquad \begin{aligned}(f(t)-u_t(t),\ \omega(t)-u(t))&=(f(t)-\omega_t(t),\ \omega(t)-u(t))\\&=\phi^t(\omega(t))-\phi^t(u(t))\qquad \text{for a.e. } t\in \boldsymbol{R}.\end{aligned}$$

Moreover, for any $\lambda\in(0,1)$, we infer from (3.4) that

$$\begin{aligned}\lambda\{\phi^t(u(t))-\phi^t(\omega(t))\}&=\lambda(f(t)-\omega_t(t),\ u(t)-\omega(t))\\&=(f(t)-\omega_t(t),\ \lambda u(t)+(1-\lambda)\omega(t)-\omega(t))\\&\leqq\phi^t(\lambda u(t)+(1-\lambda)\omega(t))-\phi^t(\omega(t))\\&\leqq\lambda\phi^t(u(t))+(1-\lambda)\phi^t(\omega(t))-\phi^t(\omega(t))\\&\leqq\lambda\{\phi^t(u(t))-\phi^t(\omega(t))\}\qquad \text{for a.e. } t\in\boldsymbol{R},\end{aligned}$$

so that

$$(3.5)\qquad \phi^t(\lambda u(t)+(1-\lambda)\omega(t))=\lambda\phi^t(u(t))+(1-\lambda)\phi^t(\omega(t))\qquad \text{for a.e. } t\in\boldsymbol{R}.$$

Furthermore, since $|u(t)-\omega(t)|=|u(0)-\omega(0)|$ for all $t\in\boldsymbol{R}$, $\lambda u+(1-\lambda)\omega$ is also a c-solution of $E(\phi^t;f)$ on $\boldsymbol{R}$ by Lemma 7. Hence, by (1.3) of (H.2), $\phi^t(u(t))$, $\phi^t(\omega(t))$ and $\phi^t(\lambda u(t)+(1-\lambda)\omega(t))$ are continuous on $\boldsymbol{R}$. This implies that (3.5) is valid for all $t\in\boldsymbol{R}$, in particular, for $t=t_0-nT$, $n=1,2,\cdots$. Accordingly, the strict convexity of ϕ^{t_0} yields that $u(t_0-nT)=\omega(t_0-nT)$ for all $n=1,2,\cdots$, so that $u=\omega\in\mathscr{P}$.

(Proof of (ii)→"$\mathscr{P}=\mathscr{RC}_0$"). Let $u\in\mathscr{RC}_0$ and $\omega\in\mathscr{P}$. Then, since $|u(t)-\omega(t)|$ $=$Const. on $\boldsymbol{R}$, we have

$$(3.6)\qquad (u^*(t)-\omega^*(t),\ u(t)-\omega(t))=0\qquad \text{for a.e. } t\in\boldsymbol{R},$$

where $u^*(t)=f(t)-u_t(t)\in\partial\phi^t(u(t))$ and $\omega^*(t)=f(t)-\omega_t(t)\in\partial\phi^t(\omega(t))$. Now, let us recall that $\partial\phi^t(\cdot)$ is cyclically monotone (cf. Brézis [5]). Then, for any $z_i^*\in\partial\phi^t(z_i)$, $i=1,2,3$, we have

$$(z_1^*,\ z_1-z_2)+(z_2^*,\ z_2-z_3)+(z_3^*,\ z_3-z_1)\geqq 0$$

which is equivalent to

$$(3.7)\qquad (z_1^*-z_2^*,\ z_1-z_2)+(z_2^*-z_3^*,\ z_1-z_3)\geqq 0.$$

Taking $z_1=\omega(t)$, $z_1^*=\omega^*(t)$, $z_2=u(t)$ and $z_2^*=u^*(t)$ in (3.7), we obtain

$$(\omega^*(t)-u^*(t),\ \omega(t)-u(t))+(u^*(t)-z^*,\ \omega(t)-z)\geqq 0\qquad \text{for any } z^*\in\partial\phi^t(z).$$

Hence, by (3.6),

$$(u^*(t)-z^*,\ \omega(t)-z)\geqq 0\qquad \text{for any } z^*\in\partial\phi^t(z),$$

so it follows from the maximal monotonicity of $\partial\phi^t$ that $u^*(t)=f(t)-u_t(t)\in\partial\phi^t(\omega(t))$, i.e., (3.8) $$-u_t(t) \in \partial\phi^t(\omega(t))-f(t) \qquad \text{for a.e } t \in \boldsymbol{R}.$$

Similarly we get

(3.9) $$-\omega_t(t) \in \partial\phi^t(u(t))-f(t) \qquad \text{for a.e. } t \in \boldsymbol{R}.$$

Also, by assumption, $-u_t(t)=(\partial\phi^t(u(t))-f(t))^0$ and $-\omega_t(t)=(\partial\phi^t(\omega(t))-f(t))^0$ for a.e. $t \in \boldsymbol{R}$. Since, for any closed convex set A in H, the minimal section A^0 denotes a unique element with the least norm in A, the above equalities together with (3.8) and (3.9) give $|u_t(t)|=|\omega_t(t)|$ for a.e. $t \in \boldsymbol{R}$, so that $u_t(t)=\omega_t(t)$ for a.e. $t \in \boldsymbol{R}$. Thus $u=\omega+\text{Const.}$ on $\boldsymbol{R}$, so $u \in \mathscr{P}$.

(*Proof of* (iii)→"$\mathscr{P}=\mathscr{RC}_0$") Let $u \in \mathscr{RC}_0$ and $\omega \in \mathscr{P}$. Then we have (3.6) just as in the above proof. Hence, by assumption,

(3.10) $$u(t)-\omega(t)=c(t)\alpha_0(t),\ c(t) \in \boldsymbol{R}, \qquad \text{for a.e. } t \in \boldsymbol{R},$$

and hence

(3.11) $$|u(t)-\omega(t)|=|c(t)||\alpha_0(t)|\equiv d,\ d \in \boldsymbol{R}, \qquad \text{for all } t \in \boldsymbol{R}.$$

If $|\alpha_0(t)|=0$ for some $t_0 \in \boldsymbol{R}$, then (3.11) implies that $u=\omega \in \mathscr{P}$. Therefore we have only to derive $u \in \mathscr{P}$ for the case that $|\alpha_0(t)|\neq 0$ for all $t \in \boldsymbol{R}$. In such a case, multiplying (3.10) by $\alpha_0(t)$, we get

$$c(t)=(u(t)-\omega(t),\ \alpha_0(t))/|\alpha_0(t)|^2 \qquad \text{for all } t \in \boldsymbol{R}.$$

Clearly $c(\cdot)$ is continuous on $\boldsymbol{R}$. Hence, by (3.11), $c(t)|\alpha_0(t)|=d$ or $-d$ for all $t \in \boldsymbol{R}$. Then we deduce, by (3.10),

$$u(t)=\omega(t)+d\alpha(t) \ \text{ or } \ \omega(t)-d\alpha(t) \qquad \text{for all } t \in \boldsymbol{R},$$

where $\alpha(t)=\alpha_0(t)/|\alpha_0(t)|$. Since $\alpha(\cdot)$ is T-periodic, we get $u \in \mathscr{P}$.

(*Proof of* (iv)→"$\mathscr{P}=\mathscr{RC}_0$") Let $u \in \mathscr{RC}_0$. Then,

(3.12) $$|u(-(n+1)T)-z|=|u(-nT)-z| \qquad \text{for all } z \in \mathscr{P}(0),\ n \in \boldsymbol{N}.$$

Since $x_{n+1}\equiv u(-(n+1)T) \in D(\phi^{-(n+1)T})=D(\phi^0)$, it follows from our assumption that there exist $z_0 \in \mathscr{P}(0)$ and $\varepsilon_0 \in (0, 1)$ such that

$$z_0+\varepsilon_0(x_{n+1}-z_0) \in \mathscr{P}(0).$$

Putting $z=z_0+\varepsilon_0(x_{n+1}-z_0)$ in (3.12), we get

$$\begin{aligned}(1-\varepsilon_0)^2|x_{n+1}-z_0|^2&=|x_n-z_0-\varepsilon_0(x_{n+1}-z_0)|^2\\&=|x_n-z_0|^2+\varepsilon_0^2|x_{n+1}-z_0|^2-2\varepsilon_0(x_n-z_0,\ x_{n+1}-z_0).\end{aligned}$$

Since $|x_{n+1}-z_0|=|x_n-z_0|$ by (3.12), it follows from the above equalities that

$$|x_{n+1}-z_0|^2=(x_n-z_0,\ x_{n+1}-z_0),$$

whence follows $x_n=x_{n+1}$, namely, $u(-nT)=u(-(n+1)T)$ for all $n \in \boldsymbol{N}$. Thus $u \in \mathscr{P}$. Q.E.D.

Remark 3. As is understood from the proof of Theorem 3, the T-periodic solution of $E(\phi^t;f)$ is unique under condition (i), and the difference of two T-periodic solutions is constant under condition (ii) or (iii) with $\alpha_0(t)\equiv\alpha_0$.

Finally we prove

Proposition 1. *Suppose that* (H.1) *and* (H.2) *hold. Then* $\mathscr{P}$ *and* $\mathscr{A}\mathscr{P}$ *are convex and closed in the topology of pointwise convergence.*

Proof. Let u, $v \in \mathscr{P}$ (resp. $\mathscr{A}\mathscr{P}$). Then, since $|u(t)-v(t)|=\text{Const.}\,(=d)$ for all $t \in \boldsymbol{R}$, it follows from Lemma 7 that $w=\lambda u+(1-\lambda)v$ is also a solution of $E(\phi^t;f)$ on $\boldsymbol{R}$ for any $\lambda \in (0, 1)$. Clearly w is T-periodic (resp. almost periodic) on $\boldsymbol{R}$. Hence $w \in \mathscr{P}$ (resp. $\mathscr{A}\mathscr{P}$). Thus $\mathscr{P}$ (resp. $\mathscr{A}\mathscr{P}$) is convex. Next, in order to show the closedness of $\mathscr{P}$ (resp. $\mathscr{A}\mathscr{P}$), let $\{u_n\}$ be a sequence in $\mathscr{P}$ (resp. $\mathscr{A}\mathscr{P}$) such that $u_n(t_0)-u_m(t_0)\to 0$ strongly in H as n, $m\to+\infty$ for some $t_0 \in \boldsymbol{R}$. Then, since $|u_n(t)-u_m(t)|=|u_n(t_0)-u_m(t_0)|$ for all $t \in \boldsymbol{R}$, $u_n(t)$ converges strongly in H and uniformly in $t \in \boldsymbol{R}$ as $n\to+\infty$. It is clear that the limit function u of u_n is T-periodic (resp. almost periodic) on $\boldsymbol{R}$. Moreover, as in the proof of Lemma 5, we can show that u is a solution of $E(\phi^t;f)$ on $\boldsymbol{R}$. Thus $u \in \mathscr{P}$ (resp. $\mathscr{A}\mathscr{P}$). Q.E.D.

§ 4. Applications

Let Ω be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary $\Gamma=\partial\Omega$, and define $a_p(\cdot,\cdot)$: $W^{1,p}(\Omega)\times W^{1,p}(\Omega)\to\boldsymbol{R}$, $2\leqq p<\infty$, by

$$a_p(z_1, z_2)=\sum_{k=1}^{N}\int_\Omega |z_{1,x_k}(x)|^{p-2}z_{1,x_k}(x)z_{2,x_k}(x)dx \qquad \text{for } z_1,\ z_2 \in W^{1,p}(\Omega),$$

where $x=(x_1, x_2, \cdots, x_N) \in \boldsymbol{R}^N$ and z_{x_k} denotes the partial derivative of $z \in W^{1,p}(\Omega)$ with respect to x_k.

Given function $g_i(\cdot, t) \in W^{1,\infty}(\Omega)$, $i=0, 1$, $t \in \boldsymbol{R}$, we put

$$K(t)=\{z \in W^{1,p}(\Omega);\ g_0(x, t)\leqq z(x)\leqq g_1(x, t) \text{ for a.e. } x \in \Omega\}$$

for each $t \in \boldsymbol{R}$, which is closed and convex in $W^{1,p}(\Omega)$. Now, consider the following parabolic variational inequality on an interval J in $\boldsymbol{R}$:

$$u \in C(J;\ L^2(\Omega))\cap W^{1,2}_{\text{loc}}(J;\ L^2(\Omega))\cap L^p_{\text{loc}}(J;\ W^{1,p}(\Omega)), \tag{4.1}$$

$$u(\cdot, t) \in K(t) \qquad \text{for a.e. } t \in J, \tag{4.2}$$

and

$$\int_\Omega (u_t(x,t)-f(x,t))(u(x,t)-z(x))dx+a_p(u(\cdot,t),u(\cdot,t)-z)\geqq 0 \tag{4.3}$$
for any $z \in K(t)$ and for a.e. $t \in J$,

where $u_t=\partial u/\partial t$ and f is a given function in $L^2_{\mathrm{loc}}(\boldsymbol{R}; L^2(\Omega))$.

We are going to investigate the asymptotic behavior of solutions of the problem (4.1)–(4.3) by applying Theorems 2 and 3.

Now we suppose that the following conditions (g.1)–(g.4) hold for a fixed positive number T:

(g.1) $g_i(\cdot, t+T)=g_i(\cdot, t)$ in $W^{1,\infty}(\Omega)$ for any $t \in \boldsymbol{R}$ and $i=0, 1$.

(g.2) There are positive numbers c_0 and c_1 such that

$$g_1(x,t)-g_0(x,t)\geqq c_0 \qquad \text{for a.e. } x \in \Omega \text{ and any } t \in [0, T]$$

and

$$|g_i(x,t)|+|g_{i,x_k}(x,t)|\leqq c_1$$
$$\text{for a.e. } x \in \Omega, \text{ any } t \in [0,T],\ k=1,2,\cdots,N \text{ and } i=0,1.$$

(g.3) There exists a function $a(\cdot) \in W^{1,2}(0, T)$ such that

$$|g_i(x,t)-g_i(x,s)|\leqq|a(t)-a(s)|$$
$$\text{for a.e. } x \in \Omega \text{ any } s, t \in [0,T] \text{ and } i=0,1.$$

(g.4) There exists a function $b(\cdot) \in W^{1,1}(0, T)$ such that

$$|g_{i,x_k}(x,t)-g_{i,x_k}(x,s)|\leqq|b(t)-b(s)|$$
$$\text{for a.e. } x \in \Omega, \text{ any } s, t \in [0,T],\ k=1,2,\cdots,N \text{ and } i=0,1.$$

Also, we define a proper lower semicontinuous convex function ϕ^t on $L^2(\Omega)$ for each $t \in \boldsymbol{R}$ by putting

$$\phi^t(z)=\begin{cases}\dfrac{1}{p}a_p(z,z) & \text{if } z \in K(t),\\ +\infty & \text{otherwise.}\end{cases} \tag{4.4}$$

Clearly, by (g.1), ϕ^t is T-periodic, and the problem (4.1)–(4.3) can be rewritten in the form

$$u_t(t)+\partial\phi^t(u(t)) \ni f(t), \qquad t \in J, \tag{4.5}$$

in the Hilbert space $L^2(\Omega)$ (cf. [12]).

Lemma 9. *The family* $\{\phi^t\}$ *of convex functions on* $L^2(\Omega)$ *given by* (4.4) *satisfies hypothesis* (H.2).

Proof. Given $s, t \in [0, T]$ and $z \in D(\phi^s)=K(s)$, we take

$$z'(x)=(z(x)-g_0(x, s))\frac{g_1(x, t)-g_0(x, t)}{g_1(x, s)-g_0(x, s)}+g_0(x, t).$$

Then, by some elementary computations, we see that there is a positive constant C independent of s, t, z such that

$$|z'-z|_{L^2(\Omega)}\leqq C|a(t)-a(s)|(|z|_{L^2(\Omega)}+1)$$

and

$$|\phi^t(z')-\phi^s(z)|\leqq C\{|a(t)-a(s)|+|b(t)-b(s)|\}(1+\phi^s(z)).$$

This shows that (C.1) holds for

$$a_r(t)=C(r+1)a(t) \quad \text{and} \quad b_r(t)=C\int_0^t \{|a'(\tau)|+|b'(\tau)|\}d\tau.$$

Then, as was remarked in (1) of Remark 1, (H.2) is satisfied. Q.E.D.

Proposition 2. *Suppose that* (g.1)–(g.4) *hold and* $f \in L^2_{loc}(\boldsymbol{R}; L^2(\Omega))$ *and* $f(\cdot, t+T)=f(\cdot, t)$ *in* $L^2(\Omega)$ *for a.e.* $t \in \boldsymbol{R}$. *Let u be any solution of the problem* (4.1)–(4.3) *on* $[t_0, +\infty)$, $t_0 \in \boldsymbol{R}$. *Then there exists a T-periodic solution ω of the problem* (4.1)–(4.3) *on $\boldsymbol{R}$ such that $u(t)-\omega(t)\to 0$ strongly in $L^2(\Omega)$ as $t\to+\infty$.*

Proof. We have seen in Lemma 9 that the family $\{\phi^t\}$ given by (4.4) satisfies (H.2). Conditions (H.1) and (C.2) are clearly fulfilled. Also, by a result in [12; Theorem 2.3.1], there exists at least one T-periodic solution of (4.5). Furthermore, the statement (iii) of Theorem 3 holds. In fact, let $z_i^* \in \partial\phi^t(z_i)$, $i=1, 2$. Then it is easy to see

$$(z_1^*-z_2^*, z_1-z_2)\geqq a_p(z_1, z_1-z_2)-a_p(z_2, z_1-z_2)\geqq 0.$$

Therefore $(z_1^*-z_2^*, z_1-z_2)_{L^2(\Omega)}=0$ implies that $z_{1,x_k}=z_{2,x_k}$ for all $k=1, 2, \cdots, N$, that is, $z_1=z_2+\text{Const.}$, so that (iii) (with $\alpha_0(t)\equiv 1$) of Theorem 3 holds. Thus we can apply Theorem 3. Q.E.D.

Finally we notice that Theorems 2 and 3 are able to be applied to many other parabolic variational inequalities associated with various time-dependent convex constrains $K(t)$. As typical examples, we can consider the following families $\{K(t)\}$.

(A) Given functions g_i, $i=0, 1$, on $\Gamma\times\boldsymbol{R}$, we put

$$K(t)=\{z \in W^{1,p}(\Omega);\ g_0(x, t)\leqq z(x)\leqq g_1(x, t) \text{ for a.e. } x \in \Gamma\}$$

for each $t \in \boldsymbol{R}$. In this case, under suitable assumptions on g_i, we obtain the same conclusion as in Proposition 2.

(B) Let $\Omega(t)$, $t \in \boldsymbol{R}$, by a relatively compact domain in Ω with smooth boundary, and put

$$K(t)=\{z \in W^{1,p}(\Omega);\ z(x)=0 \text{ a.e. } x \in \Omega\backslash\overline{\Omega(t)}\}.$$

Then, the parabolic variational inequality (4.1)–(4.3) on J associated with this family $\{K(t)\}$ represents the problem

$$(4.6)\qquad \begin{cases} u_t(x,t)-\sum_{k=1}^{N}(|u_{x_k}|^{p-2}u_{x_k}(x,t))_{x_k}=f(x,t) & \text{on } \bigcup_{t\in J}\Omega(t)\times\{t\}, \\ u(x,t)=0 \quad \text{on } \bigcup_{t\in J}\partial\Omega(t)\times\{t\}. \end{cases}$$

If the mapping $t\mapsto\Omega(t)$ is smooth in a suitable sense and T-periodic, then the family $\{\phi^t\}$ given by (4.4) fulfills (i) of (H.1), (C.1) and (C.2) (cf. [12, 17, 18, 21, 22]) as well as (i) of Theorem 3. Therefore, any solution of (4.6) converges strongly in $L^2(\Omega)$ as $t\to+\infty$ to a unique T-periodic solution of (4.6).

(C) Let $\Gamma(t)$, $t \in \boldsymbol{R}$, be a closed subset of Γ with positive surface measure, and put

$$k(t)=\{z \in W^{1,p}(\Omega);\ z(x)=0 \text{ for a.e. } x \in \Gamma(t)\}.$$

Then the parabolic variational inequality (4.1)–(4.3) on J associated with this family $\{K(t)\}$ is formally equivalent to the problem:

$$(4.7)\qquad \begin{cases} u_t(x,t)-\sum_{k=1}^{N}(|u_{x_k}|^{p-2}u_{x_k}(x,t))_{x_k}=f(x,t) & \text{on } \Omega\times J, \\ u(x,t)=0 \quad \text{on } \bigcup_{t\in J}\Gamma(t)\times\{t\}, \\ \sum_{k=1}^{N}(|u_{x_k}|^{p-2}u_{x_k}(x,t))_{x_k}\nu_k=0 & \text{on } \bigcup_{t\in J}(\Gamma\backslash\Gamma(t))\times\{t\}, \end{cases}$$

where $(\nu_1(x), \nu_2(x), \cdots, \nu_N(x))$ is the unit outward normal vector at $x \in \Gamma$. If the mapping $t\mapsto\Gamma(t)$ is smooth in a suitable sense and T-periodic, then (i) of (H.1), (C.1) and (C.2) hold (cf. [12]). Furthermore, (i) of Theorem 3 holds. Therefore, (4.7) has a unique T-periodic solution and by Theorem 3, any solution of (4.7) converges strongly in $L^2(\Omega)$ as $t\to+\infty$ to a unique T-periodic solution.

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(Ricevita la 15-an de oktobro, 1984)
(Reviziita la 20-an de februaro, 1985)