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# Frobenius' Theorem and Gauß-Kummer's Formula

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## § 1. Introduction

We are concerned with global problems for the hypergeometric system

$$(t-B)\frac{dX}{dt}=AX \qquad (t \in C), \tag{1.1}$$

where $B=\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ and $A \in M_n(C)$. This system of linear differential equations has only regular singularities at $t=\lambda_j$ $(j=1, 2, \cdots, n)$ and $t=\infty$ in the whole complex $t$-plane and hence is Fuchsian. By an appropriate transformation, evrey single Fuchsian differential equation always can be reduced to (1.1). In general, $B$ has multiple eigenvalues, i.e.,

$$B=\mathrm{diag}(\overbrace{\lambda_1, \cdots, \lambda_1}^{n_1}, \overbrace{\lambda_2, \cdots, \lambda_2}^{n_2}, \overbrace{\lambda_p, \cdots, \lambda_p}^{n_p}) \tag{1.2}$$

$$(\lambda_i \neq \lambda_j \ (i\neq j;\ i, j=1, 2, \cdots, p),\ n_i \geqq 1\ (i=1, 2, \cdots, p),\ n_1+n_2+\cdots+n_p=n).$$

In this case, since (1.1) is invariant under the linear transformation $X=DY$, where $D$ is a block-diagonal constant matrix of the form

$$D=\mathrm{diag}(D_1 \oplus D_2 \oplus \cdots \oplus D_p),$$

the $D_i$ being $n_i$ by $n_i$ matrices, we may assume that when we denote $A=(A_{ij};\ i, j=1, 2, \cdots, p)$, where the $A_{ij}$ are $n_i$ by $n_j$ matrices, the diagonal blocks $A_{ii}$ $(i=1, 2, \cdots, p)$ are of Jordan canonical form.

We here assume that $A$ is similar to a diagonal matrix, i.e.,

$$A \sim \mathrm{diag}(\nu_1, \nu_2, \cdots, \nu_n) \tag{1.3}$$

together with the condition:

$$\nu_i \not\equiv 0, \quad \nu_i \not\equiv \nu_j \ (\mathrm{mod}\ Z) \quad (i\neq j;\ i, j=1, 2, \cdots, n). \tag{1.4}$$

Now, if all $A_{ii}$ $(i=1, 2, \cdots, p)$ are diagonal and their diagonal elements $\rho_{ij}$ $(j=1, 2, \cdots, n_i)$ are not negative integers and not congruent to each other modulo

$Z$, then there appear no logarithmic solutions for (1.1). Near each regular singularity $t=\lambda_i$ there exist $n_i$ non-holomorphic solutions of the form

$$x_{ij}(t)=(t-\lambda_i)^{\rho_{ij}}\sum_{m=0}^{\infty}G_{ij}(m)\,(t-\lambda_i)^m \qquad (j=1,2,\cdots,n_i) \tag{1.5}$$

and $n-n_i$ holomorphic solutions given by putting $\rho_{ij}=0$ in the above, and near $t=\infty$ there exist $n$ non-holomorphic solutions of the form

$$y^k(t)=t^{\nu_k}\sum_{s=0}^{\infty}H_k(s)t^{-s} \qquad (k=1,2,\cdots,n). \tag{1.6}$$

In the paper [1] we have solved the connection problem between these fundamental sets of solutions near $t=\lambda_i$ and $t=\infty$, that is, we have clarified how to determine the connection coefficients $T_{ij}^k(\rho_{ij})$ depending on $\rho_{ij}$ in the expression of linear combinations

$$x_{ij}(t)=\sum_{k=1}^{n}T_{ij}^k(\rho_{ij})y^k(t) \qquad (i=1,2,\cdots,p). \tag{1.7}$$

On the other hand, K. Okubo [3, 4] proved that there holds the extended Gauß formula for the non-holomorphic solutions $x_{ij}(t)$ $(j=1,2,\cdots,n_i;\ i=1,2,\cdots,p)$:

$$\begin{aligned} W(t)&=\det\,[x_{11}(t),\cdots,x_{1n_1}(t),\cdots,x_{p1}(t),\cdots,x_{pn_p}(t)] \\ &=\frac{\prod_{i=1}^{p}\prod_{j=1}^{n_i}(t-\lambda_i)^{\rho_{ij}}\Gamma(\rho_{ij}+1)}{\prod_{k=1}^{n}\Gamma(\nu_k+1)}, \end{aligned} \tag{1.8}$$

which played an important role in the systematic calculation of monodromy groups for (1.1).

Now, in the case when one of $A_{ij}$ $(i=1,2,\cdots,p)$ is not diagonal or when for some $i$, $\rho_{ij}\equiv\rho_{ik}$ (mod $Z$)$(j\neq k; j, k=1,2,\cdots,n_i)$, there appear logarithmic solutions. As is well-known as the Frobenius theorem, the logarithmic solutions are given by derivatives of the non-logarithmic solution (1.5) with respect to the parameter (characteristic exponent) $\rho_{ij}$. The purpose of this paper is to show that the Frobenius theorem holds in the large, that is, the connection coefficients of the logarithmic solutions also can be given by derivatives of those $T_{ij}^k(\rho_{ij})$ of the non-logarithmic solution with respect to the parameter $\rho_{ij}$. Moreover in §3 we shall prove a formula similar to (1.8) for a fundamental set of non-holomorphic logarithmic solutions.

Hereafter we shall consider the hypergeometric system (1.1) with (1.2) and, for simplicity, we shall treat of the case where $A_{ij}$ $(i=1,2,\cdots,p)$ are consisting of one Jordan canonical block of the form

$$(1.9)\qquad A_{ii}=\rho_i+J_i^*,\quad J_i=\begin{bmatrix}0 & 1 & & & 0\\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1\\ 0 & & & & 0\end{bmatrix}\qquad (i=1,2,\cdots,p),$$

where the superscript * denotes the transposition of a matrix. However, in more general cases where the $A_{ii}$ consist of a finite number of Jordan canonical blocks, the diagonal elements of $A_{ii}$ are negative integers or are congruent to each other modulo $\boldsymbol{Z}$ and so on, we only need a slight modification of the consideration which will be stated below.

Throughout this paper, together with the assumptions (1.3) and (1.4), we assume that $\rho_i$ $(i=1,2,\cdots,p)$ are not negative integers, and $\rho_i\not\equiv\nu_j$ $(\mathrm{mod}\,\boldsymbol{Z})$ $(i=1,2,\cdots,p;$ $j=1,2,\cdots,n)$. It is also assumed that the singularities $\lambda_i$ $(i=1,2,\cdots,p)$ lie on the apexes of a convex polygon. As to a detailed investigation on weakening the above assumption for the location of singularities, see [5].

## § 2. Frobenius' theorem

One can easily verify that near each regular singular point $t=\lambda_i^{\#)}$ $(i=1,2,\cdots,p)$ there exist $n-n_i$ column vectorial holomorphic solutions of the form (1.5) with $\rho_{ij}=0$ and an $n$ by $n_i$ matrix solution involving logarithmic terms:

$$(2.1)\qquad X_i(t)=\hat{X}_i(t)(t-\lambda_i)^{J_i}\qquad (i=1,2,\cdots,p),$$

where $\hat{X}_i(t)$ is convergent power series of the form

$$(2.2)\qquad \hat{X}_i(t)=(t-\lambda_i)^{\rho_i}\sum_{m=0}^{\infty}\boldsymbol{G}_i(m)(t-\lambda_i)^m\qquad (i=1,2,\cdots,p).$$

The coefficient matrix $\boldsymbol{G}_i(m)$ satisfies the system of linear difference equations

$$(2.3)\qquad \begin{cases}(B-\lambda_i)\{(m+\rho_i)\boldsymbol{G}_i(m)+\boldsymbol{G}_i(m)J_i\}=(m-1+\rho_i-A)\boldsymbol{G}_i(m-1)+\boldsymbol{G}_i(m-1)J_i,\\ (B-\lambda_i)\{\rho_i\boldsymbol{G}_i(0)+\boldsymbol{G}_i(0)J_i\}=0,\quad \boldsymbol{G}_i(-r)=0\qquad (r=1,2,\cdots).\end{cases}$$

From now on we consider an $n$ by $N$ matrix solution $X(t)$ of the form (2.1) with (2.2), where the suffix $i$ is dropped, and we regard $\rho$ as a parameter. So when we put $\rho=\rho_i$ and $N=n_i$ in the last stage of the analysis to follow, $X(t)$ gives the matrix solution $X_i(t)$.

Putting

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[#] Since the hypergeometric systems are invariant under the translation of the variable $(t-a)$, we may assume that $\lambda_i\neq 0$ $(i=1,2,\cdots,p)$. This assumption is needed only in the process of proof.

$$X(t)=(x(t),\ x^{(1)}(t),\ \cdots,\ x^{(N-1)}(t))$$
$$=(x(t),\ \hat{x}^{(1)}(t),\ \cdots,\ \hat{x}^{(N-1)}(t))(t-\lambda)^{J},$$

$J$ being an $N$ by $N$ shifting matrix, we shall rewrite the expression (2.1) with (2.2) in the following column vectorial form:

$$(2.4)\quad \begin{cases} x(t)=(t-\lambda)^{\rho}\sum_{m=0}^{\infty}G(m)(t-\lambda)^{m}, \\ x^{(j)}(t)=\sum_{l=0}^{j}\dfrac{1}{(j-l)!}(\log(t-\lambda))^{j-l}\hat{x}^{(l)}(t) \qquad (j=1,2,\cdots,N-1), \end{cases}$$

where

$$(2.5)\quad \hat{x}^{(0)}(t)=x(t),\quad \hat{x}^{(l)}(t)=(t-\lambda)^{\rho}\sum_{m=0}^{\infty}G^{(l)}(m)(t-\lambda)^{m} \qquad (l=1,2,\cdots,N-1).$$

The column vectors of the matrix $\boldsymbol{G}(m)=(G(m),\ G^{(1)}(m),\ \cdots,\ G^{(N-1)}(m))$ satisfy the systems of linear difference equations

$$(2.6)_0\qquad (B-\lambda)(m+\rho)G(m)=(m-1+\rho-A)G(m-1),$$

$$(2.6)_j\qquad (B-\lambda)\{(m+\rho)G^{(j)}(m)+G^{(j-1)}(m)\}$$
$$=(m-1+\rho-A)G^{(j)}(m-1)+G^{(j-1)}(m-1),$$

where

$$(2.7)\qquad G^{(0)}(m)=G(m).$$

From those difference equations we can observe that $G^{(j)}(m)$ $(j=0,1,\cdots,N-1)$ are functions of $m$ and $\rho$, in particular, of $m+\rho$.

Now let us define the differential operators $\partial^{l}$ $(l=1,2,\cdots)$ by

$$(2.8)\qquad \partial^{l}=\frac{1}{l!}\frac{\partial^{l}}{\partial\rho^{l}} \qquad (l=1,2,\cdots)$$

and put

$$(2.9)\qquad G^{(j)}(m)=\partial^{j}[G(m)] \qquad (j=1,2,\cdots,N-1).$$

Then, combining the relation

$$\frac{\partial}{\partial\rho}[G^{(j-1)}(m)]=jG^{(j)}(m)$$

with the differential of $(2.6)_j$ with respect to $\rho$, one can see by induction that the $G^{(j)}(m)$ are particular solutions of the non-homogeneous linear difference equations $(2.6)_j$ $(j=1,2,\cdots,N-1)$, i.e.,

$$G(m)=(G(m), \partial[G(m)], \cdots, \partial^{N-1}[G(m)]) \tag{2.10}$$

is a particular matrix solution of (2.3).

The above choice of the coefficient $G(m)$ implies that

$$\begin{aligned}\partial^j[x(t)]&=\sum_{m=0}^{\infty}\partial^j[(t-\lambda)^\rho G(m)](t-\lambda)^m\\&=\sum_{l=0}^{j}\partial^{j-l}[(t-\lambda)^\rho]\sum_{m=0}^{\infty}\partial^l[G(m)](t-\lambda)^m\\&=x^{(j)}(t) \qquad (j=1, 2, \cdots, N-1).\end{aligned} \tag{2.11}$$

This is just the Frobenius method.

We here make some remarks.

*Remark* 1. Since $G(m)$ is a function of $m+\rho$, the differential operators $\partial^l$ may be replaced by

$$\partial^l_m=\frac{1}{l!}\frac{\partial^l}{\partial m^l} \qquad (l=1, 2, \cdots)$$

in the above.

*Remark* 2. One can see that

$$\hat{G}^{(j)}(m)=G^{(j)}(m)+\sum_{l=1}^{j}c_l G^{(j-l)}(m) \qquad (j=1, 2, \cdots, N-1)$$

also become particular solutions of $(2.6)_j$, i.e.,

$$\hat{G}(m)=G(m)C, \tag{2.12}$$

where

$$C=1+c_1J+c_2J^2+\cdots+c_{N-1}J^{N-1},$$

satisfies (2.3). This fact is also seen directly from the commutative property $CJ=JC$. By the choice (2.12) we have a matrix solution $X(t)C$.

From now on we regard $G(m)$ and $G^{(j)}(m)$ $(j=1, 2, \cdots, N-1)$ as functions of the complex variable $m$. Then, by the theory of linear difference equations in the complex domain, we take $G(m)$ as the particular solution of $(2.6)_0$, which is holomorphic in the right half-plane and meromorphic in the whole complex $m$-plane, having simple poles at $m=-\rho+\mu_k-r$ $(r=0, 1, \cdots; k=1, 2, \cdots, n)$ which are the poles of $(m+\rho-A)^{-1}, (m+1+\rho-A)^{-1}, \cdots$. In particular, for $\rho=\rho_i$ $(i=1, 2, \cdots, p)$ $G(m)$ has zeros of $n_i$-th order at $m=-r$ $(r=1, 2, \cdots)$.

In the paper [1] we have considered the connection problem between any non-logarithmic solution near $t=\lambda_i$ and a fundamental set of solutions $y^k(t)$ $(k=1, 2, \cdots, n)$ near $t=\infty$. Here we shall cite the results to be needed from [1].

The coefficients $H_k(s)$ $(k=1, 2, \cdots, n)$ of (1.6) satisfy the systems of linear difference equations

$$\text{(2.13)} \quad \begin{cases} (s-\nu_k+A)H_k(s)=B(s-1-\nu_k)H_k(s-1), \\ (A-\nu_k)H_k(0)=0, \quad H_k(-r)=0 \quad (r=1, 2, \cdots), \quad (k=1, 2, \cdots, n). \end{cases}$$

Let us introduce the functions depending on $m+\rho$ and $s$:

$$\text{(2.14)} \qquad g_k(m, s)=\frac{e^{\pi i(m+\rho)}\Gamma(m+\rho+s-\nu_k)\lambda^{-m-\rho-s+\nu_k}}{\Gamma(m+\rho+1)\Gamma(s-\nu_k)} \qquad (k=1, 2, \cdots, n)$$

which satisfy the equations

$$\text{(2.15)} \qquad \begin{cases} \text{(i)} \quad (m+\rho)g_k(m, s-1)=(\nu_k-s+1)g_k(m-1, s), \\ \text{(ii)} \quad \lambda(m+\rho)g_k(m, s)=(\nu_k-s-\rho-m+1)g_k(m-1, s). \end{cases}$$

Then we proved that the series

$$\text{(2.16)} \qquad F_k(m)=\sum_{s=0}^{\infty} H_k(s)g_k(m, s) \qquad (k=1, 2, \cdots, n)$$

are convergent in the left half-plane $\mathrm{Re}(m+\rho-\nu_k)<\alpha$, $\alpha$ being a constant, and satisfy $(2.6)_o$. Let $F_k(m)$ be analytically continued to the right half-plane through $(2.6)_o$, and then the $F_k(m)$ are meromorphic functions with simple poles at $m=-\rho+\nu_k-r$ $(r=0, 1, \cdots)$.

We have obtained the following result of the Mittag-Leffler type.

**Proposition 1.** *In the left half-plane $G(m)$ can be expressed in the form*

$$\text{(2.17)} \qquad G(m)=\sum_{k=1}^{n} T_k F_k(m)+E(m),$$

*where $E(m)$ is a holomorphic solution of $(2.6)_o$. The constants $T_k$ are given by the residue of $G(m)$ as follows:*

$$\text{(2.18)} \qquad \begin{aligned} \mathrm{Res}[G(m)\colon -\rho+\nu_k] &= T_k \,\mathrm{Res}[F_k(m)\colon -\rho+\nu_k] \\ &= T_k H_k(0)\, e^{\pi i\nu_k}\left(\frac{\sin \pi\nu_k}{\pi}\right) \qquad (k=1, 2, \cdots, n). \end{aligned}$$

It should be noted that since $G(m)$ is a function of $(m+\rho)$ and hence in the left-hand side of (2.18) the residue of $G(m)$ at $m+\rho=\nu_k$ is not depending on $\rho$, the $T_k$ are constants independent of the parameter $\rho$.

Now, as in [1], we here assume that $G(m)$ has the asymptotic behavior

$$G(m) \sim m^{\gamma}(\lambda-\lambda_0)^{-m} \qquad \left(|\arg m| \leqq \frac{\pi}{2}\right), \tag{2.19}$$

$\gamma$ being a constant, where $\lambda_0$ is the singularity nearest to $\lambda$, i.e., $|\lambda-\lambda_0| = \min_j \{|\lambda-\lambda_j|; \lambda \neq \lambda_j\}$. Then we have for $|t-\lambda| < |\lambda-\lambda_0|$

$$\begin{aligned} x(t) &= (t-\lambda)^{\rho} \sum_{m=0}^{\infty} G(m)(t-\lambda)^m \\ &= -\frac{(t-\lambda)^{\rho}}{2\pi i} \int_{\mathscr{C}} G(z)\left(\frac{\pi e^{-\pi i z}}{\sin \pi z}\right)(t-\lambda)^z dz, \end{aligned} \tag{2.20}$$

where the path of integration $\mathscr{C}$ is a Barnes-contour running from $\infty - ia$ to $\infty + ia$ ($a > \max_k |\mathrm{Im}(\nu_k - \rho)|$) such that $z = m$ ($m = 0, 1, \cdots$) lie to the right of $\mathscr{C}$ and $z = -\rho + \nu_k - r$ ($r = 0, 1, \cdots; k = 1, 2, \cdots, n$) lie to the left of $\mathscr{C}$. Then, according to the theory of Barnes-integrals, we slip the path $\mathscr{C}$ far to the left and then deform the path derived to the so-called Mellin-Barns' line-contour running from $-\infty i$ to $\infty i$, obtaining the analytic continuation of $x(t)$ in the domain $|t-\lambda| \geqq |\lambda-\lambda_0|$ and $0 < \arg(\lambda - t/\lambda - \lambda_0) < 2\pi$.

In that analysis, we must calculate the residues of the integrand in the left half-plane. For $\rho = \rho_i$ ($i = 1, 2 \cdots, p$), $G(z)$ has zeros at $z = -1, -2, \cdots$, and hence these points are no longer poles of the integrand. So, by Proposition 1, we have only to seek the following residues at $z = -\rho + \nu_k - r$ ($r = 0, 1, \cdots$):

$$\sum H_k(s) \operatorname{Res}[g_k(z, s)(t-\lambda)^z p(z)], \tag{2.21}$$

where we put

$$p(z) = \frac{\pi e^{-\pi i z}}{\sin \pi z},$$

which is a periodic function with period 1. The sum (2.21) becomes $y^k(t)$. And consequently, for $\rho = \rho_i$ ($i = 1, 2, \cdots, p$) we have the connection formula

$$x(t) = \sum_{k=1}^{n} \hat{T}_k(\rho) y^k(t) \qquad (0 < \arg(\lambda - t/\lambda - \lambda_0) < 2\pi), \tag{2.22}$$

where

$$\hat{T}_k(\rho) = \frac{\sin \pi \nu_k e^{\pi i \rho}}{\sin \pi(\nu_k - \rho)} T_k \qquad (k = 1, 2, \cdots, n). \tag{2.23}$$

See §3 in [1] for the detailed proof.

Now we consider the connection formulas for logarithmic solutions $x^{(j)}(t)$

$(j=1, 2, \cdots, N-1)$. From the asymptotic behavior (2.19) we can also see that the $G^{(j)}(z)$ are of the exponential order and hence there hold

$$\begin{aligned} \hat{x}^{(j)}(t) &= (t-\lambda)^{\rho} \sum_{m=0}^{\infty} G^{(j)}(m)(t-\lambda)^{m} \\ &= -\frac{(t-\lambda)^{\rho}}{2\pi i} \int_{\mathscr{C}} G^{(j)}(z)p(z)(t-\lambda)^{z}dz \\ & (|t-\lambda|<|\lambda-\lambda_0|; j=1, 2, \cdots, N-1). \end{aligned} \tag{2.24}$$

Following the consideration stated above, we now have only to calculate the residues of the integrand in the left half-plane. From (2.9) and (2.17) we have

$$G^{(j)}(z) = \sum_{k=1}^{n} T_k \partial^j [F_k(z)] + \partial^j [E(z)] \tag{2.25}$$

and moreover, since we can see from (2.13) that the $H_k(s)$ are independent of the parameter $\rho$, we have

$$\partial^j [F_k(z)] = \sum_{s=0}^{\infty} H_k(s) \partial^j [g_k(z, s)] \qquad (k=1, 2, \cdots, n). \tag{2.26}$$

*Remark* 3. For each $j$ $(j=1, 2, \cdots, N-1)$, $\partial^j [F_k(z)]$ is a particular solution of $(2.6)_j$. This fact is verified by combining (2.13) with the differential of (2.15) with respect to $\rho$.

Now we substitute (2.25) with (2.26) into (2.24). For $\rho=\rho_i$ $(i=1, 2, \cdots, p)$, because of the initial condition $G^{(j)}(-r)=0$ $(r=1, 2, \cdots,)$, we can observe that the residues of the integrand of (2.24) in the left half-plane are given only by

$$\sum_{k=1}^{n} T_k \sum H_k(s) \operatorname{Res} [\partial^j [g_k(z, s)] p(z) (t-\lambda)^z]. \tag{2.27}$$

Now, since the $g_k(z, s)$ are also functions of $z+\rho$, as stated in Remark 1, we may replace $\partial^l$ by $\partial_z^l$, and taking account of the fact that each $g_k(z, s)$ has simple poles at $z=-\rho+\nu_k-s-r$ $(r=0, 1, \cdots)$, we can carry out the calculation as follows:

$$\begin{aligned} & \operatorname{Res} [\partial_z^j [g_k(z, s)] p(z) (t-\lambda)^z] \\ & \quad = \operatorname{Res} [(-1)^j g_k(z, s) \partial_z^j [p(z)(t-\lambda)^z]] \\ & \quad = \operatorname{Res} \left[ g_k(z, s) (t-\lambda)^z (-1)^j \sum_{i=0}^{j} \frac{1}{i!} (\log (t-\lambda))^i \partial_z^{j-i} [p(z)] \right]. \end{aligned} \tag{2.28}$$

Moreover, since from periodicity we have

$$\partial_z^{j-i} [p(z)]_{z=-\rho+\nu_k-s-r} = \partial_z^{j-i} [p(z)]_{z=-\rho+\nu_k} = (-1)^{j-i} \partial^{j-i} [p(\nu_k-\rho)], \tag{2.29}$$

we consequently obtain

$$(2.30)\qquad \hat{x}^{(j)}(t)=\sum_{k=1}^{n} y^k(t)\left\{T_k\sum_{i=0}^{j}\frac{(-1)^i}{i!}(\log(t-\lambda))^i\partial^{j-i}\left[\frac{\sin\pi\nu_k e^{\pi i\rho}}{\sin\pi(\nu_k-\rho)}\right]\right\}$$

$$=\sum_{k=1}^{n} y^k(t)\left\{\sum_{i=0}^{j}\frac{(-1)^i}{i!}(\log(t-\lambda))^i\partial^{j-i}[\hat{T}_k(\rho)]\right\}$$

$$(0<\arg(\lambda-t/\lambda-\lambda_0)<2\pi;\ j=1,2,\cdots,N-1).$$

Substituting these into (2.4), we have

$$(2.31)\qquad x^{(j)}(t)=\partial^j[x(t)]=\sum_{k=1}^{n}\partial^j[\hat{T}_k(\rho)]y^k(t)$$

$$(0<\arg(\lambda-t/\lambda-\lambda_0)<2\pi;\ j=1,2,\cdots,N-1).$$

This is easily verified as follows: Putting

$$\hat{T}(\rho)=\begin{bmatrix}\hat{T}_1(\rho) & \partial[\hat{T}_1(\rho)]\cdots\partial^{N-1}[\hat{T}_1(\rho)]\\ \hat{T}_2(\rho) & \partial[\hat{T}_2(\rho)]\cdots\partial^{N-1}[\hat{T}_2(\rho)]\\ \vdots & \vdots \qquad\qquad \vdots\\ \hat{T}_n(\rho) & \partial[\hat{T}_n(\rho)]\cdots\partial^{N-1}[\hat{T}_n(\rho)]\end{bmatrix},$$

we can rewrite (2.30) in the form

$$(x(t),\hat{x}^{(1)}(t),\cdots,\hat{x}^{(N-1)}(t))=(y^1(t),y^2(t),\cdots,y^n(t))\hat{T}(\rho)(t-\lambda)^{-J}$$

and substitute this into

$$X(t)=(x(t),\hat{x}^{(1)}(t),\cdots,\hat{x}^{(N-1)}(t))(t-\lambda)^J,$$

obtaining

$$(2.32)\qquad X(t)=(y^1(t),y^2(t),\cdots,y^n(t))\hat{T}(\rho).$$

(2.31) just implies the Frobenius method in the global sense, that is, in order to obtain the connection formulas for the logarithmic solutions, we may merely differentiate both sides of the connection formula for the non-logarithmic solution with respect to the parameter $\rho$ and we put $\rho=\rho_i$.

We shall summarize above results as those for the hypergeometric system (1.1) with (1.2) and (1.9) as follows:

**Theorem 1** (Global Frobenius Theorem). *Let* $y^k(t)$ $(k=1,2,\cdots,n)$ *be a fundamental set of solutions near* $t=\infty$. *For each* $i$ $(i=1,2,\cdots,p)$, *let* $x_i(t)$ *be the non-logarithmic solution and* $x_i^{(j)}(t)$ $(j=1,2,\cdots,n_i-1)$ *the logarithmic solutions associated with* $x_i(t)$ *near* $t=\lambda_i$.

*Then there hold*

$$x_i(t)=\sum_{k=1}^{n} T_{ik}\frac{\sin\pi\nu_k e^{\pi i\rho_i}}{\sin\pi(\nu_k-\rho_i)}y^k(t) \qquad (0<\arg(\lambda_i-t/\lambda_i-\lambda_{i0})<2\pi), \tag{2.33}$$

*and*

$$x_i^{(j)}(t)=\sum_{k=1}^{n} T_{ik}\frac{1}{j!}\frac{\partial^j}{\partial\rho_i^j}\left[\frac{\sin\pi\nu_k e^{\pi i\rho_i}}{\sin\pi(\nu_k-\rho_i)}\right]y^k(t) \quad (0<\arg(\lambda_i-t/\lambda_i-\lambda_{i0})<2\pi), \tag{2.34}$$

$$(j=1, 2, \cdots, n_i-1),$$

*where the* $T_{ik}$ *are given by* (2.18) *in Proposition* 1 *and* $\lambda_{i0}$ *is the singularity nearest to* $\lambda_i$.

As to the similar result for the Birkhoff system of linear differential equations, see [2].

## § 3. Gauß-Kummer's formula

In this section we shall prove the extended Gauß-Kummer formula similar to (1.8) for suitably chosen logarithmic solutions $X_i(t)$ $(j=1, 2, \cdots, p)$. To this end, we had better introduce one more parameter $\mu$ into (1.1) as follows:

$$(t-B)\frac{dX}{dt}=(A+\mu)X. \tag*{(1.1)$_\mu$}$$

Then, as seen in §2, we have $p$ logarithmic matrix solutions depending on

$$X_i(t, \mu)=\hat{X}_i(t, \mu)(t-\lambda_i)^{J_i} \qquad (i=1, 2, \cdots, p), \tag{3.1}$$

where

$$\hat{X}_i(t, \mu)=(t-\lambda_i)^{\rho_i+\mu}\sum_{m=0}^{\infty} G_i(m, \mu)(t-\lambda_i)^m \qquad (i=1, 2, \cdots, p). \tag{3.2}$$

Obviously,

$$X_i(t)=X_i(t, 0) \qquad (i=1, 2, \cdots, p). \tag{3.3}$$

Now the differentiation of $(1.1)_\mu$ with respect to $t$ leads to

$$(t-B)X''(t, \mu)=(A+\mu-1)X'(t, \mu),$$

whence it can be easily seen that the derivative of $X_i(t, \mu)$ becomes the non-holomorphic matrix solution near $t=\lambda_i$ of $(1.1)_{\mu-1}$. So we shall investigate the relations between $X_i'(t, \mu)$ and $X_i(t, \mu-1)$, and thereby between $X_i(t, \mu)$ and $X_i(t, \mu-1)$ $(i=1, 2, \cdots, p)$.

The coefficient $G(m, \mu)$, where the suffix $i$ is dropped again, satisfies the system of linear difference equations

$$(3.4)\quad \begin{cases} (B-\lambda)\{(m+\rho+\mu)\boldsymbol{G}(m,\mu)+\boldsymbol{G}(m,\mu)J\} \\ \qquad =(m-1+\rho-A)\boldsymbol{G}(m-1,\mu)+\boldsymbol{G}(m-1,\mu)J, \\ (B-\lambda)\{(\rho+\mu)\boldsymbol{G}(0,\mu)+\boldsymbol{G}(0,\mu)J\}=0, \quad \boldsymbol{G}(-r,\mu)=0 \qquad (r=1,2,\cdots). \end{cases}$$

According to §2, we can immediately obtain a particular matrix solution $\boldsymbol{G}(m,\mu)=(G(m,\mu), G^{(1)}(m,\mu), \cdots, G^{(N-1)}(m,\mu))$ of (3.4), where $G(m,\mu)$ is a solution of the system of linear difference equations

$$(3.5)\qquad (B-\lambda)(m+\rho+\mu)G(m,\mu)=(m-1+\rho-A)G(m-1,\mu)$$

subject to the condition $G(-r,\mu)=0$ $(r=1,2,\cdots)$, and $G^{(j)}(m,\mu)$ $(j=1,2,\cdots,N-1)$ are given by

$$(3.6)\qquad G^{(j)}(m,\mu)=\partial^j[G(m,\mu)] \qquad (j=1,2,\cdots,N-1).$$

We here consider the relation $\boldsymbol{G}(m,\mu)$ and $\boldsymbol{G}(m,\mu-1)$. For that purpose, we put

$$(3.7)\qquad G(m,\mu)=\frac{\Gamma(\rho+\mu+1)}{\Gamma(m+\rho+\mu+1)}K(m)$$

and substituting this into (3.5), we have

$$(3.8)\qquad (B-\lambda)K(m)=(m-1+\rho-A)K(m-1),$$

which is now independent of the parameter $\mu$. So we take the solution $K(m)$ of (3.8) subject to the conditions $K(0)\equiv G(0,\mu)\neq 0$, $K(-r)=0$ $(r=1,2,\cdots)$ and then define $G(m,\mu)$ by (3.7). From the above determination (3.7) we have

$$(m+\rho+\mu)G(m,\mu)=(\rho+\mu)G(m,\mu-1),$$

whence by the Leibniz rule

$$(m+\rho+\mu)G^{(j)}(m,\mu)+G^{(j-1)}(m,\mu)=(\rho+\mu)G^{(j)}(m,\mu-1)$$
$$(j=1,2,\cdots,N-1).$$

Consequently, we have

$$(3.9)\qquad (m+\rho+\mu)\boldsymbol{G}(m,\mu)+\boldsymbol{G}(m,\mu)J=(\rho+\mu)\boldsymbol{G}(m,\mu-1).$$

We are now in a position to prove the following

**Proposition 2.** *Let $X_i(t,\mu)$ $(i=1,2,\cdots,p)$ be the logarithmic matrix solutions of* $(1.1)_\mu$. *Then we have*

$$(3.10)\qquad \begin{cases} \dfrac{d}{dt}X_i(t,\mu)=X_i(t,\mu-1)(\rho_i+\mu), \\ (A+\mu)X_i(t,\mu)=(t-B)X_i(t,\mu-1)(\rho_i+\mu) \qquad (i=1,2,\cdots,p). \end{cases}$$

*Proof.* We have already explained that $X_i'(t,\mu)$ is a non-holomorphic matrix solution of $(1.1)_{\mu-1}$ near $t=\lambda_i$. Since

$$\begin{aligned} X_i'(t,\mu) &= \{\hat{X}_i'(t,\mu)+\hat{X}_i(t,\mu)(t-\lambda_i)^{-1}J_i\}(t-\lambda_i)^{J_i} \\ &= \sum_{m=0}^{\infty}\{(m+\rho_i+\mu)G_i(m,\mu)+G_i(m,\mu)J_i\}(t-\lambda_i)^{m+\rho_i+\mu-1+J_i}, \end{aligned}$$

from (3.9) we immediately obtain

$$X_i'(t,\mu)=X_i(t,\mu-1)(\rho_i+\mu).$$

Combining this with $(1.1)_\mu$, we obtain the second formula of (3.10). Thus the proof is completed.

Now we shall consider the Wronskian of the form

$$\begin{aligned} (3.11)\qquad W(t,\mu) &= \det[(X_1(t,\mu), X_2(t,\mu), \cdots, X_p(t,\mu))] \\ &= \det[(\hat{X}_1(t,\mu), \hat{X}_2(t,\mu), \cdots, \hat{X}_p(t,\mu)) \\ &\qquad \times \operatorname{diag}((t-\lambda_1)^{J_1}\oplus(t-\lambda_2)^{J_2}\oplus\cdots\oplus(t-\lambda_p)^{J_p})] \\ &= \det[(\hat{X}_1(t,\mu), \hat{X}_2(t,\mu), \cdots, \hat{X}_p(t,\mu))], \end{aligned}$$

which is, as easily verified, a solution of the Jacobi equation

$$\begin{aligned} (3.12)_\mu\qquad W'(t,\mu) &= \{\operatorname{trace}[(t-B)^{-1}(A+\mu)]\}W(t,\mu) \\ &= \left\{\sum_{i=1}^{p}\frac{n_i(\rho_i+\mu)}{(t-\lambda_i)}\right\}W(t,\mu). \end{aligned}$$

Using the second formula of (3.10), we obtain the recurrence relation

$$\left[\prod_{k=1}^{n}(\nu_k+\mu)\right]W(t,\mu)=\left[\prod_{i=1}^{p}(t-\lambda_i)^{n_i}(\rho_i+\mu)^{n_i}\right]W(t,\mu-1),$$

whence

$$(3.13)\qquad W(t,0)=W(t,\mu)\prod_{i=1}^{p}(t-\lambda_i)^{-\mu n_i}\frac{\prod_{k=1}^{n}\Gamma(\mu+\nu_k+1)\prod_{i=1}^{p}(\Gamma(\rho_i+1))^{n_i}}{\prod_{i=1}^{p}(\Gamma(\mu+\rho_i+1))^{n_i}\prod_{k=1}^{n}\Gamma(\nu_k+1)}.$$

Then, letting $\mu$ tend to infinity in (3.13), we can derive the explicit value of $W(t,0)$, which is just the required formula. For that purpose, we must investigate the asymptotic behavior of $W(t,\mu)$, i.e., $\hat{X}_i(t,\mu)$ for sufficiently large values of $\mu$. Taking account of (3.7), we see that the $\hat{X}_i(t, \mu)$ are considered as the generalized factorial series expansions in $\mu$. From the Stirling formula it follows that

$$\begin{aligned} G^{(j)}(m, \mu) &= \sum_{l=0}^{j} \partial^l \left[ \frac{\Gamma(\mu+\rho+1)}{\Gamma(\mu+\rho+m+1)} \right] \partial^{j-l}[K(m)] \\ &= \mu^{-m}\{\partial^j[K(m)]+o(1)\} \qquad (j=1, 2, \cdots, N-1) \end{aligned}$$

and hence

$$(3.14) \qquad G(m, \mu)=\mu^{-m}\{(K(m), \partial[K(m)], \cdots, \partial^{N-1}[K(m)])+o(1)\}$$

for sufficiently large values of $\mu$. In particular, $G(0, \mu)$ is independent of $\mu$ and is equal to

$$(3.15) \qquad G(0, \mu)=(K(0), \partial[K(0)], \cdots, \partial^{N-1}[K(0)]).$$

For later use, we here calculate the value of (3.15). It is easy to see from (3.8) that $K(0)$ is of the form

$$K(0)=(\overset{n_1}{\widehat{0}}, \cdots, \overset{n_{i-1}}{\widehat{0}}, \overset{N}{\widehat{k(0)}}, \overset{n_{i+1}}{\widehat{0}}, \cdots, \overset{n_p}{\widehat{0}})^*,$$

where $k(0)$ must satisfy $-J^*k(0)=0$. Hence we take $k(0)=(0, \cdots, 0, 1)^*$. The differentiation of (3.8) with respect to $\rho$ leads to

$$(B-\lambda)\partial^l[K(m)]=(m-1+\rho-A)\partial^l[K(m-1)]+\partial^{l-1}[K(m-1)] \qquad (l=1, 2, \cdots),$$

whence it is again observed that

$$\partial^l[K(0)]=(\overset{n_1}{\widehat{0}}, \cdots, \overset{n_{i-1}}{\widehat{0}}, \overset{N}{\widehat{\partial^l[k(0)]}}, \overset{n_{i+1}}{\widehat{0}}, \cdots, \overset{n_p}{\widehat{0}})^*$$

and the $\partial^l[k(0)]$ must satisfy $-J^*\partial^l[k(0)]+\partial^{l-1}[k(0)]=0$ $(l=1, 2, \cdots)$. Hence we obtain

$$(3.16) \qquad \left\{ \begin{aligned} k(0)&=(0, \cdots, 0, 1)^*, \\ \partial[k(0)]&=(0, \cdots, 0, 1, 1)^*, \\ &\vdots \\ \partial^{N-1}[k(0)]&=(1, 1, \cdots, 1)^*. \end{aligned} \right.$$

Now, from (3.14) and (3.15) we have

$$(3.17) \qquad \begin{gathered} \hat{X}_i(t, \mu)=(t-\lambda_i)^{\rho_i+\mu}\{(K_i(0), \partial[K_i(0)], \cdots, \partial^{N-1}[K_i(0)])+o(1)\} \\ (i=1, 2, \cdots, p) \end{gathered}$$

as $\mu\to\infty$ for $|t-\lambda_i|\leqq|\lambda_i-\lambda_{i0}|-\varepsilon$, $\varepsilon$ being an arbitrarily small number.

We shall return to the investigation on the asymptotic behavior of the Wronskian $W(t, \mu)$ as $\mu\to\infty$. Let $D_0$ be the intersection of circles of convergence of $\hat{X}_i(t, \mu)$ $(i=1, 2, \cdots, p)$. We here assume that $D_0$ is not empty. For instance, if $|\lambda_i-\lambda_j|>|\lambda_i|>0$ $(i\neq j;\ i, j=1, 2, \cdots, p)$, then $D_0$ is certainly non-empty. Then from (3.17) we have

$$
\begin{aligned}
(3.18)\qquad W(t, \mu)&=\left[\prod_{i=1}^{p}(t-\lambda_i)^{(\rho_i+\mu)n_i}\right]\\
&\quad\times\det\{(K_i(0), \partial[K_i(0)], \cdots, \partial^{n_i-1}[K_i(0)]);\ i=1, 2, \cdots, p\}\{1+o(1)\}\\
&=\left[\prod_{i=1}^{p}(t-\lambda_i)^{(\rho_i+\mu)n_i}\right]\prod_{i=1}^{p}(-1)^{n_i(n_i-1)/2}\{1+o(1)\}
\end{aligned}
$$

as $\mu\to\infty$ in any compact set of $D_0$. Substituting this into (3.13) and letting $\mu\to\infty$, we consequently obtain

$$
(3.19)\qquad W(t, 0)=\frac{\prod_{i=1}^{p}(-1)^{n_i(n_i-1)/2}(\Gamma(\rho_i+1))^{n_i}}{\prod_{k=1}^{n}\Gamma(\nu_k+1)}\prod_{i=1}^{p}(t-\lambda_i)^{n_i\rho_i}
$$

in any compact set of $D_0$. In the above calculation we have used the Fuchs relation (the invariance of trace)

$$
\sum_{i=1}^{p}n_i\rho_i=\sum_{k=1}^{n}\nu_k.
$$

Moreover, taking account of the fact that $W(t, 0)$ is a solution of the Jacobi equation $(3.12)_0$, and by the analytic continuation, we can observe that the identity (3.19) holds in the so-called star domain.

Lastly we make a remark on the assumption that $D_0$ is not empty. In the above proof we only used the local expression (3.2). However, by more detailed investigations, one can obtain the expression of $X_i(t, \mu)$ and its analytic continuation in the star domain in terms of factorial series in $\mu$ under the more weak assumption. Such detailed investigations will be referred to the paper [5].

We write the result derived so far in the form

**Theorem 2** (Gauß-Kummer's Formula). *For each* $i$ $(i=1, 2, \cdots, p)$, *let* $X_i(t)$ *be the logarithmic matrix solution defined by* (2.1) *and* (2.2) *with the initial value* (3.15) *near* $t=\lambda_i$ *and also let us denote its analytic continuation by* $X_i(t)$. *Assume that* $D_0\neq\phi$. *Then we have the identity*

$$
(3.20)\qquad \det[X_1(t), X_2(t), \cdots, X_p(t)]=\frac{\prod_{i=1}^{p}(-1)^{n_i(n_i-1)/2}(\Gamma(\rho_i+1))^{n_i}}{\prod_{k=1}^{n}\Gamma(\nu_k+1)}\prod_{i=1}^{p}(t-\lambda_i)^{n_i\rho_i},
$$

*which holds in the whole plane except for the half-lines running from* $\lambda_i$ $(i=1,2,\cdots,p)$ *to infinity. The identity* (3.20) *implies that* $X_i(t)$ $(i=1,2,\cdots,p)$ *form a fundamental set of solutions of* (1.1) *in the large.*

As an example of the application of our theory, we shall deal with the connection problem for the Gauß equation

$$\left(t-\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)\frac{dX}{dt}=\begin{pmatrix}-p&1\\a_2&\rho_2\end{pmatrix}X, \tag{3.21}$$

where $p=\gamma-1$ is a positive integer, $\rho_2=\gamma-\alpha-\beta-1$ and $a_2=-(\gamma-\alpha-1)(\gamma-\beta-1)$. Since the characteristic exponents at $t=0$ are 0 and $-p$, there certainly exists a logarithmic solution near $t=0$.

Let $X_0(t)$ be a matrix solution near $t=0$ of the form

$$\begin{aligned}X_0(t)&=(x_0(t),\ x_0^{(1)}(t))\\&=(x_0(t),\ \hat{x}_0^{(1)}(t))t^J,\end{aligned}$$

where

$$\begin{cases}x_0(t)=\sum\limits_{m=0}^{\infty}G_0(m)t^m,\\ \hat{x}_0^{(1)}(t)=t^{-p}\sum\limits_{m=0}^{\infty}G_0^{(1)}(m)t^m.\end{cases} \tag{3.22}$$

The coefficients $G_0(m)$ and $G_0^{(1)}(m)$ are given by solutions subject to the conditions $G_0(-r)=G_0^{(1)}(-r)=0$ $(r=1,2,\cdots)$ of the following linear difference equations for $\rho=0$;

$$B(m+\rho)G(m)=(m-1+\rho-A)G(m-1), \tag{3.23}$$

and

$$\begin{aligned}&B(m+\rho-p)G^{(1)}(m)+BG(m-p)\\&\quad=(m-1+\rho-p-A)G^{(1)}(m-1)+G(m-1-p),\end{aligned} \tag{3.24}$$

respectively.

On the other hand, we have a fundamental set of solutions near $t=\infty$ of the form

$$\begin{cases}y^1(t)=t^{-\alpha}\begin{pmatrix}F(\alpha,\alpha-\gamma+1;\ \alpha-\beta+1;\ t^{-1})\\(\gamma-\alpha-1)F(\alpha,\alpha-\gamma+2;\alpha-\beta+1;t^{-1})\end{pmatrix},\\ y^2(t)=t^{-\beta}\begin{pmatrix}F(\beta,\beta-\gamma+1;\beta-\alpha+1;t^{-1})\\(\gamma-\beta-1)F(\beta,\beta-\gamma+2;\beta-\alpha+1;t^{-1})\end{pmatrix},\end{cases} \tag{3.25}$$

where $F(a, b; c; t)$ denotes the hypergeometric function.

Now we seek the connection formulas between $X_0(t)$ and (3.25). For that purpose, according to our theory, we first solve (3.23), obtaining

$$G(m)=\frac{\Gamma(m+\rho-\nu_1)\Gamma(m+\rho-\nu_2)}{\Gamma(m+\rho+p+1)\Gamma(m+\rho+1)}\binom{1}{m+\rho+p} \tag{3.26}$$

$$(\nu_1=-\alpha,\ \nu_2=-\beta).$$

Then the relation (2.18) becomes

$$\frac{\Gamma(\nu_k-\nu_{3-k})}{\Gamma(\nu_k+p+1)\Gamma(\nu_k+1)}\binom{1}{\nu_k+p}=T_k\binom{1}{\nu_k+p}e^{\pi i(\nu_k+1)}\frac{\sin \pi\nu_k}{\pi} \qquad (k=1, 2),$$

thereby, together with (2.23), we obtain

$$\hat{T}_k(\rho)=\frac{e^{\pi i\rho}}{\sin \pi(\nu_k-\rho)}\,\frac{\pi\Gamma(\nu_k-\nu_{3-k})e^{-\pi i(\nu_k+1)}}{\Gamma(\nu_k+p+1)\Gamma(\nu_k+1)} \qquad (k=1, 2). \tag{3.27}$$

Putting $\rho=0$ in (3.27), we immediately obtain the connection coefficients between the holomorphic solution $x_0(t)$ and (3.25).

We now investigate the form of $x_0^{(1)}(t)$. From (3.24) we can observe that

$$\begin{aligned} x_0^{(1)}(t)&=t^{-p}\sum_{m=0}^{p-1}G_0^{(1)}(m)t^m+\sum_{m=p}^{\infty}G_0^{(1)}(m)t^{m-p}\\ &=t^{-p}\sum_{m=0}^{p-1}G_0^{(1)}(m)t^m+\sum_{m=0}^{\infty}\partial[G_0(m)]t^m, \end{aligned} \tag{3.28}$$

where the first $p$ coefficients $G_0^{(1)}(m)$ are determined by

$$B(m-p)G_0^{(1)}(m)=(m-1-p-A)G_0^{(1)}(m-1) \qquad (0\leqq m\leqq p-1)$$

subject to the initial condition

$$-(1+A)G_0^{(1)}(p-1)=BG_0(0).$$

When we rewrite the first sum in the right hand side of (3.28) in the form

$$t^{-p}\sum_{m=0}^{p-1}G_0^{(1)}(m)t^m=\sum_{m=0}^{p-1}\hat{G}_0^{(1)}(m)t^{-m-1}, \tag{3.29}$$

the coefficients $\hat{G}_0^{(1)}(m)$ are given by solving

$$\begin{cases} (m+1+A)\hat{G}_0^{(1)}(m)=B_m\hat{G}_0^{(1)}(m-1),\\ -(1+A)\hat{G}_0^{(1)}(0)=BG_0(0), \end{cases}$$

whence

$$\hat{G}_0^{(1)}(m)=\frac{\Gamma(m+1)(-1)^m \sin\pi(-\nu_1)\sin\pi(-\nu_2)}{\Gamma(m+2+\nu_1)\Gamma(m+2+\nu_2)\Gamma(p-m)\pi^2}\begin{pmatrix}1\\ p-m-1\end{pmatrix}. \tag{3.30}$$

From this we have the Barnes-integral representation

$$x_0^{(1)}(t)=-\frac{1}{2\pi i}\int_{\mathscr{C}}\hat{G}_0^{(1)}(z)\left(\frac{\pi}{\sin\pi z}\right)(-t)^{-z-1}dz$$

$$-\frac{1}{2\pi i}\int_{\mathscr{C}}\partial[G_0(z)]\left(\frac{\pi}{\sin\pi z}\right)(-t)^z dz,$$

where it can be easily seen from (3.30) that in the left half-plane

$$\operatorname{Res}\left[\hat{G}_0^{(1)}(z)\left(\frac{\pi}{\sin\pi z}\right)(-t)^{-z-1}\right]=0.$$

Consequently, we can reduce the above problem, where the characteristic exponents differ from the other by integers, to that considered in §2. From Theorem 1, we have the connection formula

$$x_0^{(1)}(t)=\sum_{m=1}^{p-1}\hat{G}_0^{(1)}(m)t^{-m-1}+\sum_{m=0}^{\infty}\partial[G(m)]t^m \tag{3.31}$$

$$=\partial[\hat{T}_1(\rho)]y^1(t)+\partial[\hat{T}_2(\rho)]y^2(t),$$

where we put $\rho=0$. The first element of (3.31) just implies that

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(p+1)}g(\alpha,\beta;\gamma;t) \tag{3.32}$$

$$=\left(\frac{-\pi}{\sin\pi\alpha}\right)\frac{\Gamma(\beta-\alpha)\Gamma(\alpha)}{\Gamma(-\alpha+p+1)}t^{-\alpha}F(\alpha,\alpha-\gamma+1;\alpha-\beta+1;t^{-1})$$

$$+\left(\frac{-\pi}{\sin\pi\beta}\right)\frac{\Gamma(\alpha-\beta)\Gamma(\beta)}{\Gamma(-\beta+p+1)}t^{-\beta}F(\beta,\beta-\gamma+1;\beta-\alpha+1;t^{-1}),$$

where

$$g(\alpha,\beta;\gamma;t)=-(-t)^{-p}\sum_{m=0}^{p-1}\frac{p!\,(p-1-m)!\,\Gamma(\alpha-p+m)\Gamma(\beta-p+m)}{m!\,\Gamma(\alpha)\Gamma(\beta)}(-t)^m \tag{3.33}$$

$$+\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m m!}t^m\{\log t+\psi(\alpha+m)+\psi(\beta+m)-\psi(\gamma+m)-\psi(1+m)\},$$

$\psi(z)$ denoting the logarithmic derivative of $\Gamma(z)$, i.e., $\psi(z)=\Gamma'(z)/\Gamma(z)$. Let $\operatorname{Re}(\gamma-\alpha-\beta)=p+1-\operatorname{Re}(\alpha+\beta)>0$. Then, letting $t$ tend to 1 in (3.32) and using the Gauß formula

$$F(a, b; c; 1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (\mathrm{Re}(c-a-b)>0),$$

we have

$$(3.34) \qquad g(\alpha, \beta; \gamma; 1)$$
$$=\left(-\frac{\sin\pi\beta}{\sin\pi\alpha}\frac{\pi}{\sin\pi(\beta-\alpha)}-\frac{\sin\pi\alpha}{\sin\pi\beta}\frac{\pi}{\sin\pi(\alpha-\beta)}\right)\frac{\Gamma(p+1)\Gamma(p+1-\alpha-\beta)}{\Gamma(-\alpha+p+1)\Gamma(-\beta+p+1)}$$
$$=\frac{(-1)^{p+1}\Gamma(p+1)\Gamma(\alpha-p)\Gamma(\beta-p)}{\Gamma(\alpha+\beta-p)},$$

which is also derived from the Gauß-Kummer formula (3.20) slightly modified for this case.

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