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Spectral Matrices for First and Second Order Selfadjoint Ordinary Differential Operators with Diverging Potentials

By

Hitoshi KANETA

(Tokushima University, Japan)

§ 0. Introduction.

Let Λ be a real diagomal matrix $[\lambda_1, \cdots, \lambda_n]$ and let $V(t)$ be a locally integrable $n\times n$-Hermitian matrix-valued function on $\boldsymbol{R}$. Then the differential operator

$$L=i\Lambda d/dt+V(t) \qquad \text{in } L^2(\boldsymbol{R})^n$$

with domain $\{f\in L^2(\boldsymbol{R})^n;\ f$ is absolutely continuous with $i\Lambda f'+Vf\in L^2(\boldsymbol{R})^n\}$ is selfadjoint. Let, further, $\Phi(t, l)$ be a solution of

$$(1) \qquad (i\Lambda d/dt+V-l)\Phi=0$$

with intial value $\Phi(0, l)=E_n$ (the unit matrix in M_n), and let $\rho(\lambda)$ be the spectral matrix for L with respect to $\Phi(t, \lambda)$. In [6] the spectral matrix has been discussed under the condition that, roughly speaking, $V(t)$ tends to zero as $t\rightarrow\pm\infty$. In this paper two cases will be considered; the first one where $V(t)$ tends to zero as $t\rightarrow-\infty$ while $|V(t)|$ diverges as $t\rightarrow\infty$ and the second one where $|V(t)|$ diverges as $t\rightarrow\pm\infty$. As a variation of the first case, the condition that $V(t)$ tends to zero as $t\rightarrow-\infty$ and $|V(t)|$ diverges as $t\rightarrow 1$ is also treated(Theorems 1, 2 and 3).

Let $v(t)$ be a locally integrable real-valued function on $\boldsymbol{R}$ such that, again roughly speaking, $v(t)$ tends to zero as $t\rightarrow-\infty$ and $v(t)$ diverges to infinity as $t\rightarrow\infty$. Then the differential operator

$$M=-d^2/dt^2+v(t) \qquad \text{in } L^2(\boldsymbol{R})$$

is selfadjonit with domain $\{u\in L^2(\boldsymbol{R});\ u$ and u' are absolutely continuous with $-u''+vu\in L^2(\boldsymbol{R})\}$. Let $\phi_j(t)=\phi_j(t, l)$ $(j=1, 2)$ be solutions of the equation

$$(2) \qquad (-d^2/dt^2+v(t)-l)\phi=0$$

with initial values $(\phi_1, \phi_2)_{t=0}=E_2$. The spectral matrix for M with respect to ϕ_j $(j=1, 2)$ can be described satisfactorily (Theorem 4). Our result is better than Theorem A 1 2 [4]

Theorems and examples are collected in § 1 and § 5 respectively. In § 2

Theorems 1 through 3 are proved by the aid of a fundamental Lemma 4, whose proof is given in § 3. The § 4 is devoted to the proof of Theorem 4.

Finally we shall explain our notation. $\boldsymbol{R}$ is the set of real numbers and $\boldsymbol{R}_-=(-\infty, 0)$, $\boldsymbol{R}_+=(0, \infty)$. $\boldsymbol{C}$ is the set of complex numbers and $\boldsymbol{C}_\pm=\{l \in \boldsymbol{C};\ \pm\mathrm{Im}\ l>0\}$, $\bar{\boldsymbol{C}}_\pm=\{l \in \boldsymbol{C};\ \mathrm{Im}\ l\geqq 0\}$, where Im l denotes the imaginary of l. M_n stands for the set of $n\times n$-matrices and $GL_n=\{X \in M_n;\ \det X\neq 0\}$. For $X=(x_{jk})$ in M_n ${}^tX$ and X^* denote the transposed and the conjugate matrix of X respectively. The norm $|X|$ is equal to $\max_j (\Sigma_k|x_{jk}|)$, while rank X means the rank of X. E_n is the unit matrix in M_n. Let V be a vector space. V^n stands for the vector space $\{{}^t(v_1, \cdots, v_n);\ v_j \in V\}$. For $\xi=(\xi_j) \in \boldsymbol{C}^n$ the norm $|\xi|$ is equal to $\max_j (|\xi_j|)$. If an f is an absolutely continuous function on some interval of $\boldsymbol{R}$, f' means the derivative of f.

§ 1. Theorems.

To begin with we shall assume the following conditions on Λ and $V(t)$.

C-1) There exist real d, a locally integrable real-valued function $q(t)$ on $[d, \infty)$ and an Hermitian matrix $H \in M_n$ such that $q(t)>0$, $\lim_{t\to\infty} q(t)=\infty$ and $V(t)=q(t)H$ on $[d, \infty)$.

C-2) All characteristic roots of $(i\Lambda)^{-1}H$ have non-zero real parts.

C-3) $V(t)$ is integrable on $(-\infty, d]$.

The condition C-2) implies that n is even (Lemma 2).

Theorem 1. *Under the conditions* C-1) *through* C-3) *the spectral matrix* $\rho(\lambda)$ *for the operator introduced in* § 0 *is an absolutely continuous function having a continuous density* ρ' *with rank* $\rho'(\lambda)=n/2$.

In place of the condition C-3) we now assume the following ones.

C-4) There exist real c, a locally integrable real-valued function on $(-\infty, 0]$ and an Hermitian matrix $G \in M_n$ such that $q(s)>0$, $\lim_{s\to-\infty} q(s)=\infty$ and $V(s)=q(s)G$ on $(-\infty, c]$.

C-5) The characteristic roots of $(i\Lambda)^{-1} G$ have non-zero real parts.

Theorem 2. *Under the conditions* C-1), C-2), C-4) *and* C-5) *the spectral matrix* $\rho(\lambda)$ *for the operator L is a step function with* rank $(\rho(\lambda)-\rho(\lambda-))\leqq n/2$, *where* $\rho(\lambda-)$ *denotes the left limit of* ρ *at* λ.

As a variation of C-1), consider the following condition.

C-1′) There exist real $d(<1)$, a locally integrable realvalued function $q(t)$ on $[d, 1)$ and an Hermitian matrix $H \in M_n$ such that $q(t)>0$, $\lim_{t\to 1} q(t)=\infty$ and $V(t)=q(t)H$ on $[d, 1)$.

Let $\tilde{L}=i\Lambda d/dt+V(t)$ be a differential operator in $L^2(-\infty, 1)^n$ with domain $\{f \in L^2(-\infty, 1)^n;\ f$ is absolutely continuous with $i\Lambda f'+Vf \in L^2(-\infty, 1)^n\}$.

Theorem 3. *If the operator defined above is selfadjoint under the conditions* C-1′), C-2) *and* C-3), *then the spectral matrix* $\rho(\lambda)$ *for* $\tilde{L}$ *with respect to* $\Phi(t, \lambda)$ *is an absolutely continuous function having a continuous density* ρ' *with rank* $\rho'(\lambda)=n/2$.

Let $v(t)$ be a locally integrable real-valued function on $\boldsymbol{R}$ with the following properties.

C-6) There exists real c such that $v(t)\geqq c$ for $t\geqq 0$.

C-7) The function $v(t)$ tends to ∞ as $t\to\infty$.

C-8) There exist real-valued functions v_1 and v_2 on $(-\infty, 0]$ such that $v=v_1+v_2$ and that, v_1 being absolutely continuous, v_1' and v_2 are integrable on $(-\infty, 0]$. In particular, we may assume that $v_1(t)$ tends to some real v_- as $t\to-\infty$.

As is well known, $-d^2/dt^2+v$ is in the limit case at $t=\pm\infty$ ([3, p. 231 and p. 255] and [11, Theorem 4.2]). Define the differential operator M in $L^2(\boldsymbol{R})$ and functions $\phi_j(j=1, 2)$ for this v as in §0. Then M is selfadjoint. We refer to [3, p. 250] for the definition of the spectral matrix for M with respect to the generalized eigenfunctions $\phi_j(t, \lambda)$ $(j=1, 2)$.

Theorem 4. *Let* $\rho(\lambda)$ *be the spectral matrix for the operator* M *with respect to* $\phi_j(j=1, 2)$.

(i) ρ *is an absolutely continuous function on* (v_-, ∞) *having a continuous density* ρ' *with* rank $\rho'(\lambda)=1$

(ii) ρ *is a step function on* $(-\infty, v_-)$ *with* rank $(\rho(\lambda)-\rho(\lambda-))\leqq 1$. *The possible discontinuous points are bounded from below, and the accumulation points, if any, is equal to* v_-.

Remark. The statement (ii) of Theorem 4 is known. We cite it above for the sake of completeness. The condition C-7) can be replaced by Molchanov's one [1, p. 528].

§ 2. Proof of Theorems 1,2 and 3.

To begin with we remark on the condition C-2). Let H be an Hermitian matrix in M_n and set $B=-(i\Lambda)^{-1}H$.

Lemma 1. *If* Λ *is positive definite, then* Re $\alpha=0$ *for any characteristic root* α *of* B.

Proof. Let $\phi(t)$ be an arbitrary solution of the equation $i\Lambda\phi'+H\phi=0$. Then $(\phi^*\Lambda\phi)'=0$, which results in the boundedness of ϕ on $\boldsymbol{R}$. This is possible only if Re $\alpha=0$.

Lemma 2. *Assume* C-2). *If* α *is a characteristic root of* B *of multiplicity* m, *then* $-\bar{\alpha}$ *is also a characteristic root of* B *of multiplicity* m. *In particular,* n *is even.*

Proof. The characteristic polynomial $P(x)$ of B is proportional to $Q(y)=\det(i\Lambda x+H)$ $(y=ix)$. Clearly Q is a real polynomial. Thus, if $i\alpha$ is a root of Q of multiplicity m, then $\overline{i\alpha}(\neq i\alpha)$ is a root of Q of multiplicity m.

Assume the conditions C-1) through C-3), until the proof of Theorem 1 completes. Without loss of generality we may assume $\lambda_1^{-1}\geqq\cdots\geqq\lambda_p^{-1}>0>\cdots\geqq\lambda_n^{-1}$ for some $1<p<n$. Set

$$P_+=\mathrm{diag}[1,\cdots,\overset{p}{1},0,\cdots,0],\qquad Q_+=\mathrm{diag}[1,\cdots,\overset{n/2}{1},0,\cdots,0]$$

and $P_-=E_n-P_+$, $Q_-=E_n-Q_+$. It turns out later on that $P_+=Q_+$ under the condition C-2) (Lemma 3). The equation (1) can be rewritten as

$$\phi'=(q(t)B+l(i\Lambda)^{-1})\phi,\qquad B=-(i\Lambda)^{-1}H$$

on $[d,\infty)$. By the substitution $\tau=\int_d^t q(s)ds$ and $\psi(\tau)=\phi(t)$ the above equation takes the form

$$\psi'=(B+lQ(\tau))\psi,\qquad Q(\tau)=(i\Lambda)^{-1}/q(t). \tag{3}$$

Let α_j $(1\leqq j\leqq k)$ be all characteristic roots of B with positive real part. Then there exists a $T\in GL_n$ such that

$$T^{-1}BT=\begin{bmatrix} B_1(\alpha_1) & & & & & 0\\ & \ddots & & & & \\ & & B_k(\alpha_k) & & & \\ & & & B_{-1}(-\bar\alpha_1) & & \\ & & & & \ddots & \\ 0 & & & & & B_{-k}(-\bar\alpha_k)\end{bmatrix}=\begin{bmatrix} B_+ & 0\\ 0 & B_-\end{bmatrix},$$

where $B_i(\alpha)$ is of the form

$$B_i(\alpha)=\begin{bmatrix} J(\alpha,m_1) & & 0\\ & \ddots & \\ 0 & & J(\alpha,m_{n(i)})\end{bmatrix}$$

with

$$J(\alpha,m)=\begin{bmatrix} \alpha & \mu & & 0\\ & \ddots & \ddots & \\ 0 & & \alpha & \mu\\ & & & \alpha\end{bmatrix}\in M_m.$$

Under the similarlity transformation by a diagonal matrix $\mu(\neq 0)$ can be arbitrarily small. We may assume that every characteristic root of the transformation $U \to B_+ U - UB_-$ in $M_{n/2}$ has positive real part, because it is nearly equal to $\alpha_i + \bar{\alpha}_j$ for some $1 \leqq i, j \leqq k$ provided μ is small enough.

Lemma 3. *Assume* C-2). *The number of positive eigenvalues of Λ is equal to $n/2$. In particular, $P_\pm = Q_\pm$.*

Proof. We shall show that the matrix $X = Q_+ T^{-1} P_+ + Q_- T^{-1} P_-$ is non-singular. Let $X\xi = 0$ for some $\xi \in \boldsymbol{C}^n$. Denote by $\phi(t)$ a solution of the equation $i\Lambda\phi' + H\phi = 0$ with $\phi(0) = P_+\xi$. Since $\phi^*\Lambda\phi$ does not depend on t, it is euqal to $(P_+\xi)^*\Lambda \times (P_+\xi)$. On the other hand, $\chi = T^{-1}\phi$ satisfies $\chi' = T^{-1}BT\chi$ with $\chi(0) = T^{-1}P_+\xi$ and $Q_+\chi(0) = 0$. Thus $\chi(t)$ tends to zero as $t \to \infty$, which results in $(P_+\xi)^*\Lambda(P_+\xi) = 0$. Namely $P_+\xi = 0$. In a similar manner we can show that $P_-\xi = 0$. Consequently the matrix X is non-singular. Now that rank $X = n$, we conclude that $P_+ = Q_+$.

By the substitution $\psi(\tau) = Ty(\tau)$ in (3), we obtain

$$y' = (T^{-1}BT + lT^{-1}Q(\tau)T)y. \tag{4}$$

Fix now $l_0 \in \boldsymbol{R}$ and set $U(l_0) = \{l \in \boldsymbol{C};\ |l - l_0| < 1\}$.

Lemma 4. *There exist $\tau_0 > 0$ and a fundamental solution $Y(\tau) = Y(\tau, l)$ of* (4) *with the following properties.*

(i) *$Y(\tau, l)$ is defined on $[\tau_0, \infty) \times U(l_0)$ and holomorphic in l.*

(ii) *Define $Y_{ij}(\tau) \in M_{n/2}$ $(i, j = 1, 2)$ by $Y(\tau) = (Y_{ij}(\tau))$. Then $Y_{ii}(\tau)$ $(i = 1, 2)$ are non-singular, and $Y_{21}Y_{11}^{-1}$, $Y_{12}Y_{22}^{-1}$, Y_{11}^{-1}, Y_{22} and Y_{12} convere to zero as $\tau \to \infty$.*

We defer the proof until § 3. As to asymptotic behavior of a fundamental solution of (1) near $t = -\infty$ we have

Lemma 5. [6, Lemma 2] *There exist real c, a bounded neighborhood $U_\pm(l_0) \subset \bar{\boldsymbol{C}}_\pm$ of l_0 and a fundamental solution $Y_\pm(t) = Y_\pm(t, l)$ of* (1) *with the following properties.*

i) *$Y_\pm(t, l)$ is defined on $(-\infty, c] \times U_\pm(l_0)$ and continuous in l.*

ii) *For $l \in U_\pm(l_0) \cap \boldsymbol{C}_\pm$*

$$\lim_{t \to -\infty} Y_\pm(t, l)\,[\exp il\Lambda^{-1}(t+c)] = E_n,$$

while for $l \in U_\pm(l_0) \cap \boldsymbol{R}$

$$\lim_{t \to -\infty} [\exp il\Lambda^{-1}(t+c)]\,Y_\pm(t, l) = W_\pm(l),$$

where W_+ (resp. W_-) is of the form

$$\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \quad \left(resp. \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix} \right).$$

The fundamental matrix $\Phi(t, l)$ of (1) can be expressed as

$$(5) \qquad \Phi(t, l) = \begin{cases} Y_\pm(t, l)C_\pm(l) & (t \sim -\infty, \ \pm \operatorname{Im} l \geqq 0) \\ TY(\tau, l)D(l) & (t \sim \infty). \end{cases}$$

Clearly D is holomorphic in $U(l_0)$ and $C_\pm$ is continuous in $U_\pm(l_0)$.

Lemma 6. *The function $Q_+DC_\pm^{-1}P_\pm + Q_-T^{-1}P_\mp$ is GL_n-valued and continous in $U_\pm(l_0) \cap U(l_0)$.*

Proof. Denote by $X_\pm(l)$ the value of the function. We shall prove the lemma for $\operatorname{Im} l \geqq 0$. The another case can be treated similarly. Assume $X_+(l)\xi = 0$ for some $\xi \in C^n$, and set $\xi_\pm = P_\pm \xi$. Let ϕ satisfy $i\Lambda\phi' + H\phi = 0$ with $\phi(0) = \xi_-$. Then $\chi = T^{-1}\phi$ satisfies $\chi' = T^{-1}BT\chi$ with $Q_-\chi(0) = 0$. Consequently $\chi(t) \to 0$ as $t \to -\infty$. Since $\phi^*\Lambda\phi$ is constant, $\phi^*\Lambda\phi = 0$. Thus $\xi_- = 0$. It remains to show that $\xi_+ = 0$. Since $Q_+DC_+^{-1}\xi_+ = 0$, $Y(\tau)\ DC_+^{-1}\xi_+$ tends to zero as $\tau \to \infty$ by virtue of Lemma 4. Consequently $\phi(t) = TY(\tau)DC_+^{-1}\xi_+$ tends to zero as $t \to \infty$. When t is near $-\infty$, $\phi(t) = Y_+(t)\xi_+$ in view of (5). In case $\operatorname{Im} l > 0$, $\phi(t)$ tends to zero as $t \to -\infty$ by virtue of Lemma 5. Observing that $(\phi^*\Lambda\phi)' = 2 \operatorname{Im} l\phi^*\phi$, we conclude that a monotone function $\phi^*\Lambda\phi$ (hence ϕ as well) is identically equal to zero. Thus $\xi_+ = 0$. In case $\operatorname{Im} l = 0$, $\phi^*\Lambda\phi$ is equal to a constant, which is equal to zero because $\phi(t) \to 0$ as $t \to \infty$. If t is near $-\infty$, $\phi^*\Lambda\phi$ is equal to

$$\xi_+^* Y_+(t)^* \left[\exp -il\Lambda^{-1}(t+c)\right] \Lambda \left[\exp il\Lambda^{-1}(t+c)\right] Y_+(t)\xi_+,$$

which tends to $\xi_+^* W_+^* \Lambda W_+ \xi_+$ as $t \to -\infty$ by virtue of Lemma 5. Thus $\xi_+ = 0$.

Lemma 7. *The function $C_+^{-1}P_+\{Q_+DC_+^{-1}P_+ + Q_-T^{-1}P_-\}^{-1} - C_-^{-1}P_-\{Q_+DC_-^{-1}P_- + Q_-T^{-1}P_+\}^{-1}$ is GL_n-valued and continuous in $U_+(l_0) \cap U_-(l_0) \cap U(l_0)$.*

Proof. Denote by $X(l)$ the value of the function, and assume $X(l)\xi = 0$ for some $\xi \in C^n$. It suffices to show that $\xi = 0$. Consider a solution

$$\phi(t) = \Phi(t)C_+^{-1}P_+\{Q_+DC_+^{-1}P_+ + Q_-T^{-1}P_-\}^{-1}\xi$$

of (1). We shall show that $\phi = 0$. Put $\xi_\pm = P_\pm\{Q_+DC_\pm^{-1}P_\pm + Q_-T^{-1}P_\mp\}^{-1}\xi$. Since $X(l)\xi = 0$, ϕ can be rewitten as $Y_\pm(t)\xi_\pm$ near $-\infty$ in veiw of (5). As in the proof of Lemma 6, we obtain

$$(6) \qquad \pm\phi^*\Lambda\phi = \pm \lim_{t \to -\infty} (Y_\pm\xi_\pm)^*\Lambda Y_\pm\xi = (W_\pm\xi_\pm)^*\Lambda(W_\pm\xi_\pm) \geqq 0.$$

Thus $\phi^*\Lambda\phi = 0$, which implies $\xi_\pm = 0$ by (6). Since $P_\pm = Q_\pm$ and $Q_+DC_\pm^{-1}P_\pm +$

$Q_- T^{-1} P_\mp$ is non-singular, it follows that $\xi = 0$.

Proof of Theorem 1. Let s and t be near $-\infty$ and ∞ respectively. Using (5), we obtain for $\pm \mathrm{Im}\, l > 0$

$$(7)\qquad (\Phi(s)+\Phi(t))^{-1}\Phi(t) = (E_n + \Phi(t)^{-1}\Phi(s))^{-1} = C_\pm^{-1}\{DC_\pm^{-1} + (TY(\tau))^{-1} Y_\pm(s)\}^{-1} D.$$

We shall show that the above matrix converges to

$$(8)\qquad N_\pm = C_\pm^{-1} P_\pm \{Q_+ D C_\pm^{-1} P_+ + Q_- T^{-1} P_\mp\} Q_+ \qquad \text{for } \pm \mathrm{Im}\, l > 0$$

as $s \to -\infty$ and t (or τ) $\to \infty$. Assume $\mathrm{Im}\, l > 0$. Recalling the definition of Y_{ij} (see Lemma 4), we rewrite $\{DC_+^{-1} + (TY(\tau))^{-1} Y_+(s)\}^{-1}$ as

$$\begin{aligned} &\{DC_+^{-1} + (TY(\tau) Y_d(\tau)^{-1} Y_d(\tau))^{-1} Y_+(s) e^{il\Lambda^{-1}(s+c)} e^{-il\Lambda^{-1}(s+c)}\}^{-1} \\ &= e^{ilP_-\Lambda^{-1}(s+c)} \left\{ \begin{bmatrix} E_{n/2} & 0 \\ 0 & Y_{22}(\tau) \end{bmatrix} DC_+^{-1} e^{ilP_-\Lambda^{-1}(s+c)} \right. \\ &\quad \left. + \begin{bmatrix} Y_{11}(\tau)^{-1} & 0 \\ 0 & E_n/2 \end{bmatrix} T(\tau)^{-1} \tilde{Y}_+(s) e^{-ilP_+\Lambda^{-1}(s+c)} \right\}^{-1} \begin{bmatrix} E_{n/2} & 0 \\ 0 & Y_{22}(\tau) \end{bmatrix}, \end{aligned}$$

where $Y_d(\tau) = \begin{bmatrix} Y_{11}(\tau) & 0 \\ 0 & Y_{22}(\tau) \end{bmatrix}$, $T(\tau) = TY(\tau) Y_d(\tau)^{-1}$ and $\tilde{Y}_+(s) = Y_+(s)[\exp il\Lambda^{-1}(s+c)]$. By virtue of Lemmas 3 and 4, $T(\tau)$ and $\tilde{Y}_+(s)$ tend to T and E_n as $\tau \to \infty$ and $s \to -\infty$ respectively. It is now clear in view of Lemmas 4 and 6 that $(\Phi(s)+\Phi(t))^{-1}\Phi(t)$ tends to N_+ as $s \to -\infty$ and $t \to \infty$. The case $\mathrm{Im}\, l < 0$ can be treated similarly. Let μ, $\lambda (\mu < \lambda)$ be continuous points of the spectral matrix ρ and let $[\mu, \lambda]$ be in $U_+(l_0) \cap U_-(l_0) \cap U(l_0)$. Then

$$\rho(\lambda) - \rho(\mu) = \frac{1}{2\pi i} \lim_{\varepsilon \to +0} \int_\mu^\lambda -(N_+(\nu + i\varepsilon) - N_-(\nu - i\varepsilon))(i\Lambda)^{-1} d\nu$$

(cf. [6, Proposition 2]). On account of Lemma 6, the right-hand side is equal to

$$\frac{1}{2\pi i} \int_\mu^\lambda -(N_+(\nu) - N_-(\nu))(i\Lambda)^{-1} d\nu,$$

where $N_\pm(\nu \pm i\varepsilon) \to N_\pm(\nu)$ as $\varepsilon \to +0$. By Lemma 7 the integrand $\rho'(\nu)$ is a continuous function with rank $\rho'(\nu) = n/2$. Theorem 1 has been proved.

We now turn to the proof of Theorem 2. By the substitution $\sigma = -\int_t^c q(u)\, du$ $(t \leq c)$ and $\psi(\sigma) = \phi(t)$, the equation (1) reduces to

$$(9)\qquad \psi' = (A + lQ(\sigma))\psi, \quad A = -(i\Lambda)^{-1}G, \quad Q(\sigma) = (i\Lambda)^{-1}/q(t).$$

Let $S \in GL_n$ be as T for $B = -(i\Lambda)^{-1}H$, and set

$$S^{-1}AS = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}.$$

In particular, real parts of the characteristic roots of $\pm A_\pm$ are positive, and the characteristic roots of the linear transformation $U \to A_+U - UA_-$ ($U \in M_{n/2}$) have positive real parts. Substituting $Sy(\sigma)$ for $\psi(\sigma)$ in (9), we obtain

$$(10)\qquad y' = (S^{-1}AS + lS^{-1}Q(\sigma)S)y.$$

Fix now $l_0 \in \boldsymbol{R}$ and set $U(l_0) = \{l \in \boldsymbol{C};\ |l - l_0| < 1\}$. The analogy to Lemma 4 runs as follows.

Lemma 8. *There exist $\sigma_0 < 0$ and a fundamental solution $Y(\sigma) = Y(\sigma, l)$ of* (9) *with the following properties.*

i) *$Y(\sigma, l)$ is defined on $(-\infty, \sigma_0] \times U(l_0)$ and holomorphic in l.*

ii) *Define $Y_{ij}(\sigma) \in M_{n/2}$ $(i, j = 1, 2)$ by $Y(\sigma) = (Y_{ij}(\sigma))$. Then $Y_{ij}(\sigma)$ $(i = 1, 2)$ is non-singular, and $Y_{21}Y_{11}^{-1}, Y_{12}Y_{12}^{-1}$, Y_{22}^{-1}, Y_{11} and Y_{21} converge to zero as $\sigma \to -\infty$.*

The generalized eigenfunction $\Phi(t, l)$ of the operator L can be written as

$$(11)\qquad \Phi(t, l) = \begin{cases} SY(\sigma, l)\, C(l) & (t \sim -\infty) \\ TY(\tau, l)\, D(l) & (t \sim \infty). \end{cases}$$

Clearly C and D are holomorphic in $U(l_0)$. Define $X_{ij}(l) \in M_{n/2}$ $(i, j = 1,\ 2)$ by $DC^{-1} = (X_{ij})$.

Lemma 9. *The matrix $X_{11}(l)$ defined above is non-singular for non-real l.*

Proof. Assuming $Q_+DC^{-1}Q_+\xi = 0$ for some $\xi \in \boldsymbol{C}^n$, we shall show that $Q_+\xi = 0$. By the assumption and Lemma 4 $\phi(t) = TY(\tau)$ $(DC^{-1}Q_+\xi)$ is a solution of (1) tending to zero as $t \to \infty$. Since $\phi(t) = \Phi(t)C^{-1}Q_+\xi = ST(\sigma)Q_+\xi$, $\phi(t)$ converges to zero as $t \to -\infty$ by Lemma 8. Now a monotone function $\phi^*\Lambda\phi$ on $\boldsymbol{R}$ must vanish identically, which yiels $\phi^*\phi = 0$ (recall that $(\phi^*\Lambda\phi)' = 2\,\mathrm{Im}\, l\ \phi^*\phi$). Thus $\phi = 0$ and $Q_+\xi = 0$.

Proof of Theorem 2. It suffices to show that for non-real l

$$(12)\qquad (\Phi(s, l) + \Phi(t, l))^{-1}\Phi(t, l) \to C(l)^{-1}\begin{pmatrix} X_{11}(l)^{-1} & 0 \\ 0 & 0 \end{pmatrix} D(l)$$

as $s \to -\infty$ and $t \to \infty$. Indeed, denote by $N(l)$ the right-hand side of (12), and let μ and λ $(\mu < \lambda)$ be continuous points of the spectral matrix ρ with $[\mu, \lambda] \subset U(l_0)$. Then

$$\rho(\lambda)-\rho(\lambda)=\frac{1}{2\pi i}\lim_{\varepsilon\to+0}\int_{\mu}^{\lambda}-(N(\nu+i\varepsilon)-N(\nu-i\varepsilon))(iA)^{-1}d\nu.$$

Since $N(l)$ is meromorphic in $U(l_0)$, Theorem 2 follows. We shall prove (12). Using (11), we can rewrite the left-hand side of (12) as

$$\begin{aligned}(13)\qquad & C^{-1}\{DC^{-1}+(TY(\tau))^{-1}SY(\sigma)\}^{-1}D\\ &=C^{-1}\{DC^{-1}+Y_d(\tau)^{-1}Z(\sigma,\tau)Y_d(\sigma)\}^{-1}D,\end{aligned}$$

where $Y_d=\begin{pmatrix}Y_{11} & 0\\ 0 & Y_{22}\end{pmatrix}$ and $Z(\sigma,\tau)=Y_d(\tau)Y(\tau)^{-1}T^{-1}SY(\sigma)Y_d(\sigma)^{-1}$. It now suffices to show that the matrix $\{\ \}^{-1}$ in (13) converges to $\begin{pmatrix}X_{11}^{-1} & 0\\ 0 & 0\end{pmatrix}$. Set $Z(\sigma,\tau)=(Z_{ij}(\sigma,\tau))$ ($Z_{ij}\in M_{n/2}$, $i,j=1,2$) and $\eta(\sigma,\tau)=\{DC^{-1}+(TY(\tau))^{-1}SY(\sigma)\}^{-1}\xi$ ($\xi\in C^n/\{0\}$). As $\sigma\to-\infty$ and $\tau\to\infty$, $Z(\sigma,\tau)$ converges to $T^{-1}S$ by virtue of Lemmas 4 and 8 while $\eta(\sigma,\tau)$ tends to some $\eta\in C^n$ because $(\Phi(s)+\Phi(t))^{-1}\Phi(t)$ converges. Note that $Q_-\eta=0$. Otherwise, $\{DC^{-1}+(TY(\tau))^{-1}SY(\sigma)\}\eta(\sigma,\tau)$, which is equal to ξ, turns out to be unbounded as $\sigma\to-\infty$. In fact, let P be a diagonal matrix in M_n such that $P^2=P$ and rank $P=$rank $PT^{-1}SQ_-=n/2$. The existence of P is clear, since $T^{-1}S$ is non-singular. There exists positive c_0 such that rank $PZ(\sigma,\tau)Q_-=n/2$ for $-\sigma,\tau>c_0$. Denote by $Z_0(\sigma,\tau)$ the matrix in $GL_{n/2}$ obtained by deletting zero column vectors and zero row vectors of $PZ(\sigma,\tau)Q_-$. In the equality

$$\begin{aligned}PZ(\sigma,\tau)Y_d(\sigma)\eta(\sigma,\tau)&=PZ(\sigma,\tau)Q_+Y_d(\sigma)Q_+\eta(\sigma,\tau)\\ &\quad+PZ(\sigma,\tau)Q_-Y_d(\sigma)Q_-\eta(\sigma,\tau),\end{aligned}$$

the first term on the right-hand side converges to zero as $\sigma\to-\infty$ because of Lemma 8, while the norm of the second term is greater than $|Y_{22}(\sigma)^{-1}Z_0(\sigma,\tau)^{-1}|^{-1}|Q_-\eta(\sigma,\tau)|$, which tends to ∞ as $\sigma\to-\infty$ since $Y_{22}(\sigma)^{-1}\to0$ as $\sigma\to-\infty$. Summing up, the assumption $Q_-\eta\neq0$ yields the unboundedness of $Y_d(\tau)(TY(\tau))^{-1}SY(\sigma)\eta(\sigma,\tau)$ on $(-\infty, c_0)$, a contradiction. To complete the proof of (12), we shall next show that $Q_+\eta=0$ provided $Q_+\xi=0$. Indeed, expressing ξ in terms of $\eta(\sigma,\tau)$, we obtain

$$\begin{aligned}Q_+\xi=(&Q_+DC^{-1}+Q_+Y_d(\tau)^{-1}Q_+Z(\sigma,\tau)Q_+Y_d(\sigma)Q_+\\ &+Q_+Y_d(\tau)^{-1}Q_+Z(\sigma,\tau)Q_-Y_d(\sigma)Q_-)\eta(\sigma,\tau).\end{aligned}$$

The second and the third terms on the right-hand side converge to zero as $\tau\to\infty$ since $Y_{11}(\tau)^{-1}\to0$ as $\tau\to\infty$. Consequently $Q_+\xi=Q_+DC^{-1}\eta$, which is assumed to equal zero. Since $Q_-\eta=0$ and X_{11} is non-singular (see the passage proceeding Lemma 9 for the definition of X_{ij}), $Q_+\eta=0$ as desired. Now the proof of (12) is complete.

We now turn to the proof of Theorem 3. Since Proposition 2 in [6] holds for the operator $\tilde{L}$ and since the analogy to Lemma 4 also holds, we can argue as in the proof of Theorem 1.

§ 3. Proof of Lemma 4.

We shall prove two auxiliary lemmas first. Let A be a matrix in M_m and let Ω be a bounded region in C. Let, further, $f(t, x, l)$ and $g(t, l)$ be C^m-valued functions on $[0, \infty)\times C^m\times\Omega$ with the following properties.

i) f and g are holomorphic in l and $f(t, 0, l)=0$.

ii) There exist positive δ and a locally integrable function $r(t)$ on $[0, \infty)$ such that

$$|f(t, x, l)-f(t, y, l)|\leqq r(t)|x-y|, \qquad |g(t, l)|\leqq r(t),$$
$$\lim_{t\to\infty} r(t)=0$$

provided $|x|$, $|y|\leqq\delta$.

Consider a differential equation of the form

$$x'=Ax+f(t, x, l)+g(t, l). \tag{14}$$

Lemma 10. *If the real parts of all the characteristic roots of A are negative, there exist positive T_1 and δ_1 such that there exists uniquely a solution $x(t)$ of (14) defined on $[T_1, \infty)$ with*

$$\lim_{t\to\infty} x(t)=0, \qquad x(T_1)=\xi$$

for any ξ ($|\xi|\leqq\delta_1$). The solution x is holomorphic in l.

Proof. Arguing as in the proof of Theorem 3.1 [3, p. 327], there exist positive T_0 and $\delta_1(<\delta)$ such that every solution $\phi(t)$ of (14) $|\phi(T_1)|\leqq\delta_1$ $(T_1\geqq T_0)$ satisfies $|\phi(t)|<\delta$ $(t\geqq T_1)$ and $\phi(t)\to 0$ as $t\to\infty$. Let K, N and α be positive constants such that $|e^{At}|<Ke^{-\alpha t}$ $(t\geqq 0)$ and $0<(\delta/2+1)/(N-1)<\delta/2$. Let T_1 be chosen so that $\sup r(t)\leqq\alpha/(NK)$ $(t\geqq T_1)$. Defining inductively $x_k(t, l)$ $(k\geqq 0)$ by $x_0=e^{A(t-T_1)}\xi$ ($|\xi|\leqq\delta_1$) and

$$x_k(t, l)=e^{A(t-T_1)}\xi+\int_{T_1}^{t} e^{A(t-s)}(f(s, x_{k-1}, l)+g(s, l))ds \qquad (k\geqq 1)$$

we can show by induction that $|x_k|<\delta$ and

$$\sup_{t\geqq T_1}|x_k(t)-x_{k-1}(t)|\leqq N^{-k+1}\sup_{t\geqq T_1}|x_1(t)-x_0(t)|.$$

From this estimate follows the uniform convergence of x_k on $[T_1, \infty)\times\Omega$. The limit function $x(t, l)$ is holomorphic in Ω and a solution of (14) with $x(T_1, l)=\xi$.

Let $y(t)$ is a solution of (14) with $y(T_1)=\xi$. Since x and y satisfy the same integral equation, we can show that $\sup|x(t)-y(t)|\leqq N^{-1}\sup|x(t)-y(t)|(t\geqq T_1)$. This means $x=y$.

The following lemma is less obvious.

Lemma 11. *If the real parts of all the characteristic roots of A are positive, there exists positive T_1 such that a solution $x(t)$ of* (14) *on $[T_1, \infty)$ satisfying $\lim_{t\to\infty}x(t)=0$ uniquely exists. The solution $x(t, l)$ is holomorphic in Ω.*

Proof. Let K, N and α be positive constants such that $|e^{-At}|\leqq Ke^{-\alpha t}$ $(t\geqq 0)$ and $\delta+1<N$. There exists positive T_1 for which $\sup r(t)<\alpha\delta/(2NK)$ $(t\geqq T_1)$. We can define $x_k(t)=x_k(t, l)$ $(k\geqq 0)$ on $[T_1, \infty)\times\Omega$ recursively as follows; $x_0=0$ and

$$x_k(t)=-\int_t^\infty e^{A(t-s)}(f(s, x_{k-1}, l)+g(s, l))ds \qquad (k\geqq 1).$$

Indeed, it can be easily seen by induction that $|x_1(t)|\leqq\delta/2$, $|x_k(t)|\leqq\delta$ and

$$\sup_{t\geqq T_1}|x_k(t)-x_{k-1}(t)|\leqq\left(\frac{\delta}{2N}\right)^{k-2}\sup_{t\geqq T_1}|x_2(t)-x_1(t)|$$

for $k\geqq 2$. Thus the sequence x_k converges uniformly on $[T_1, l)\times\Omega$. The limit function $x(t, l)$ is holomorphic in Ω and a solution of (14). Let $y(t)$ be a solution of (14) tending to zero as $t\to\infty$. As long as $|y(s)|\leqq\delta$ $(s\geqq t)$

$$y(t)=-\int_t^\infty e^{A(t-s)}(f(s, y, l)+g(s, l))ds. \tag{15}$$

We denote $\sup|y(s)|$ $(s\geqq t)$ by $M(t)$. From (15) we obtain $M(t)\leqq(M(t)+1)\delta/(2N)$, equivalently $M(t)\leqq\delta/(2N-\delta)<\delta$. By this observation we conclude that $|y(t)|\leqq\delta$ for $t\geqq T_1$. Since $x(t)$ also satisfies (15), it follows that

$$\sup_{t\geqq T_1}|x(t)-y(t)|\leqq\frac{\delta}{2N}\sup_{t\geqq T_1}|x(t)-y(t)|,$$

which implies $x=y$.

Proof of Lemma 4. Let $Y(\tau)=(Y_{ij}(\tau))$ $(Y_{ij}\in M_{n/2}, i, j=1, 2)$ be an M_n-valued solution of (4), and denote by $F(\tau, l)=(F_{ij}(\tau, l))$ the matrix $T^{-1}BT+T^{-1}Q(\tau)T(F_{ij}\in M_{n/2}, i, j=1, 2)$. If $Y_{ii}(\tau)$ $(i=1, 2)$ are non-singular, then

$$Z_{11}=Y_{11},\ Z_{21}=Y_{21}Y_{11}^{-1},\ Z_{12}=Y_{12}Y_{22}^{-1} \text{ and } Z_{22}=Y_{22} \tag{16}$$

satisfy the following equations.

$$(17)\qquad Z_{11}'=(F_{11}+F_{12}Z_{21})Z_{11}$$

$$(18)\qquad Z_{21}'=F_{21}+F_{22}Z_{21}-Z_{21}F_{11}-Z_{21}F_{12}Z_{21}$$

$$(19)\qquad Z_{12}'=F_{12}+F_{11}Z_{12}-Z_{12}F_{22}-Z_{12}F_{21}Z_{12}$$

$$(20)\qquad Z_{22}'=(F_{22}+F_{21}Z_{12})Z_{22}.$$

On account of Lemma 11, there exist positive τ_1 and a solution $Z_{12}(\tau, l)$ of (19) such that $Z_{12}(\tau, l)$, being defined on $[\tau_1, \infty)\times U(l_0)$ and holomorphic in l, tends to zero as $\tau\to\infty$. This Z_{12} substituting for Z_{12} on the right-hand side of (20), it follows by virture of Lemma 10 the existence of positive $\tau_2\geqq\tau_1$ and a solution $Z_{22}(\tau, l)$ of (20) such that $Z_{22}(\tau, l)$, being defined on $[\tau_2, l)\times U(l_0)$, holomorphic in l and non-singular, tends to zero as $\tau\to\infty$. There exist positive $\tau_3\geqq\tau_2$ and a solution $Z_{21}(\tau, l)$ of (18) such that $Z_{21}(\tau, l)$, being defined on $[\tau_3, \infty)\times U(l_0)$ and holomorphic in l, tends to zero as $\tau\to\infty$. Clearly a $GL_{n/2}$-valued function Z_{11} is a solution of (17) if and only if

$$(21)\qquad ({}^tZ_{11}^{-1})'=-{}^t(F_{11}+F_{12}Z_{21}){}^tZ_{11}^{-1}.$$

Substituting $Z_{21}(\tau, l)$ defined above for Z_{21} in (21), and applying Lemma 10 to the equation (21), we obtain a solution $W(\tau, l)$ of (21) such that $W(\tau, l)$, being defined on $[\tau_0, \infty)\times U(l_0)$ $(\tau_0\geqq\tau_3)$, holomorphic in l and non-singular, converges to zero as $\tau\to\infty$. Put $Z_{11}(\tau, l)={}^tW(\tau, l)^{-1}$ and define $Y_{ij}(\tau, l)$ $(i, j=1, 2)$ by (16) for $\tau\geqq\tau_0$ and $l\in U(l_0)$.

§ 4. Proof of Theorem 4.

Let $\phi_i(t)=\phi_i(t, l)$ $(i=1, 2)$ be the solutions of (2) defined in § 0. Denote by $m_\pm(l)$ for $l\in \boldsymbol{C}_+$ the limit

$$-\lim_{t\to\pm\infty}\frac{\phi_1(t, l)}{\phi_2(t, l)}.$$

As is well-known, $m_\pm$ is holomorphic in $\boldsymbol{C}_+$ with $\pm\operatorname{Im} m_\pm(l)>0$.

Lemma 12. (i) *There exist an infinite sequence $\lambda_1<\lambda_2<\cdots$ tending to ∞ such that m_+ is a meromorphic function on $\boldsymbol{C}$ whose singular points are simple poles at λ_j $(j\geqq 1)$.*
(ii) *If $l_0>v_-$, the limit*

$$m_-(l_0)=\lim_{l\to l_0, l\in C_+} m_-(l)$$

exists and $\operatorname{Im} m_-(l_0)<0$. *In particular $m_-(\lambda)$ is continuous on (v_-, ∞).*

Proof. The statement (i) is a consequence of Theorems 1.13, 4.3 [12] and Theorem 2 [16, p. 210]. The existence of $m_-(l_0)$ with $\operatorname{Im} m_-(l_0)<0$ has been established (see (39) and Lemma 10 [6]). It is an easy excercise to show that m_- in continuous on $\boldsymbol{C}_+ \cup (v_-, \infty)$.

Proof of Theorem 4. Define a square matrix $M=(m_{jk})$ $(j, k=1, 2)$ by

$$m_{11}=(m_- - m_+)^{-1},\ m_{12}=m_{21}=(m_- + m_+)m_{11}/2,\ m_{22}=m_- m_+ m_{11}. \tag{22}$$

If the spectral matrix ρ for M is continuous at λ and μ $(\lambda>\mu)$, then

$$\rho(\lambda)-\rho(\mu)=\frac{1}{\pi}\lim_{\varepsilon\to+0}\int_\mu^\lambda \operatorname{Im} M(\nu+i\varepsilon)d\nu \tag{23}$$

by virtue of Theorem 5.1 [3, p. 251]. The limit

$$\operatorname{Im} M(\nu)=\lim_{l\to\nu,\, l\in C} \operatorname{Im} M(l) \qquad (\nu>v_-)$$

exists on account of Lemma 12, even at $\nu=\lambda_n>v_-$, where

$$\operatorname{Im} M(\nu)=\begin{bmatrix} 0 & 0 \\ 0 & -\operatorname{Im} \mathrm{m}_-(\nu)\end{bmatrix}.$$

Thus $\operatorname{Im} M(l)$ is continuous on $\boldsymbol{C}_+ \cup (v_-, \infty)$, which resutls in the absolutely continuity of the spectral matrix ρ on (v_-, ∞) with $\rho'(\nu)=\operatorname{Im} M(\nu)/\pi$. Applying Lemma 11 [6] to $\operatorname{Im} M(\nu)$ $(\nu\neq\lambda_n)$, we conclude that rank $\rho'(\nu)=1$ $(\nu>v_-)$, since evidently rank $\operatorname{Im} M(\lambda_n)=1$. The assertion (i) of Theorem 4 has been proved. We apply the splitting method ([16, pp. 208–209] or [1, p. 520]) to the operator M. In view of Theorems 2 and 5 [16, p. 210 and p. 214] as well as Theorem 3 [1, p. 522], it follows that $\rho(\lambda)$ is a step function on $(-\infty, v_-)$. Besides, the jump point of ρ, if any, are bounded from below and they can accumulate only at v_-. To see the simplicity of M's eigenvalues, it suffices to recall the domain of M (cf. Remark [6, p. 42]).

§ 5. Examples.

Example 1. Let k be a positive half-integer and denote by A_k the diagonal matrix $2[k, k-1, \cdots, \nu, \cdots, -k]$ in M_{2k+1}. Let H_k be an Hermitian matrix in M_{2k+1} with $(\nu, \nu+1)$ and $(\nu, \nu-1)$ components $(\nu=k, k-1, \cdots, -k)$

$$-i\sqrt{(k-\nu)(k+\nu+1)} \quad \text{and} \quad i\sqrt{(k-\nu+1)(k+\nu)}$$

respectively, other components being vanishing. Then Theorem 1 is applicable to the operator $iA_k d/dt+e^t H_k$ in $L^2(\boldsymbol{R})^{2k+1}$ [cf. 8].

Example 2. Let k and A_k be as above. Denote by V_k the Hermitian matrix in M_{2k+1} with $(\nu, \nu+1)$ and $(\nu, \nu-1)$ components $(\nu=k, k-1, \cdots, -k)$

$$-\sqrt{(l-\nu)(l+\nu+1)(k-\nu)(k+\nu+1)}$$

and

$$-\sqrt{(l-\nu+1)(l+\nu)(k-\nu+1)(k+\nu)}$$

respectively, other components being vanishing. Here l is also a half-integer not less than k. Then Theorem 3 can be applied to the opeator $iA_k d/dt+V_k/\mathrm{sh}\, t$ in $L^2(\boldsymbol{R}_+)^{2k+1}$, because we have

Proposition 1. (i) *The characteristic roots of* $(iA_k)^{-1}V_k$ *are*

$$\pm(l+k), \quad \pm(l+k-2), \cdots, \quad \pm(l-k+1).$$

(ii) *The operator* $\dot{L}_k=iA_k d/dt+V_k/\mathrm{sh}\, t$ *in* $L^2(\boldsymbol{R}_+)^{2k+1}$ *with domain* $C_0^\infty(\boldsymbol{R}_+)^{2k+1}$ *is essentially selfadjoint.*

Proof. Let $F_{+,k}$ be the differential operator sending $C^\infty(\boldsymbol{R}_+)^{2k+1}$ into $C^\infty(\boldsymbol{R}_+)^{2k+3}$ of the form

$$F_{+,k}=iB_k\{d/dt-(k+1)\coth t\}+Y_k/\mathrm{sh}\, t,$$

where $B_k \in M_{2k+3}$ is a diagonal matrix with (ν, ν) component $2\sqrt{(k-\nu+1)(k+\nu+1)}$ $(\nu=k+1, \cdots, -k-1)$ and Y_k's $(\nu, \nu+1)$ and $(\nu, \nu-1)$ components are equal to

$$-\sqrt{(l-\nu)(l+\nu+1)(k-\nu)(k-\nu+1)}$$

and

$$\sqrt{(l-\nu+1)(l+\nu)(k+\nu)(k+\nu+1)}$$

respectively $(\nu=k+1. \cdots, -k-1)$, whereas other components of Y_k vanish. It is convenient to identify $f={}^t(f_k, \cdots, f_{-k}) \in C^\infty(\boldsymbol{R}_+)^{2k+1}$ with ${}^t(0, f_k, \cdots, f_{-k}, 0) \in C^\infty(\boldsymbol{R}_+)^{2k+3}$ and $A \in M_{2k+1}$ with an element of M_{2k+3} accordingly. By this convention the following equalities hold on $\boldsymbol{C}^{2k+1}$.

$$\begin{aligned} &A_{k+1}Y_{k+1}+V_{k+1}B_k=Y_kA_k+B_kV_k, \\ &-(k+1)A_{k+1}B_k+V_{k+1}Y_k=Y_kV_k, \\ &A_{k+1}Y_k+(k+1)V_{k+1}B_k=(k+2)B_kV_k. \end{aligned}$$

It is now immediate that $\hat{L}_{k+1}F_{+,k}=F_{+,k}\hat{L}_k$, where $\hat{L}_k$ is the differential operator $iA_k d/dt+V_k/\mathrm{sh}\, t$ with domain $C^\infty(\boldsymbol{R}_+)^{2k+1}$. Let $Q_k(x)$ be the characteristic polynomial of $-(iA_k)^{-1}V_k$. We shall show by induction on k that

$$Q_k(x)=\{x^2-(l+k)^2\}\cdots\{x^2-(l-k+1)^2\}. \tag{24}$$

Denote by I the matrix in M_{2k+1} corresponding to the isometry of $\boldsymbol{C}^{2k+1}$ assigning

${}^t(\xi_\nu)$ to ${}^t(\xi_{-\nu})$. Clearly $IA_kI=-A_k$ and $IV_kI=V_k$, which results in the equality $Q_k(x)=Q_k(-x)$.
In case $k=1/2$, simple calculation yields (24). Assume that (24) is valid up to $k<l$. To complete the induction procedure it suffices to show that $Q_k(x+1)$ divides $Q_{k+1}(x)$, since Q_{k+1} is even. If $F_{+,k}f=0$ for an $f\in C^\infty(\boldsymbol{R}_+)^{2k+1}$, $f=0$, since $(k+1, k)$-compoment of Y_k is nonzero. Let α_j $(1\leqq j\leqq 2k+1)$ be roots of the polynomial $Q_k(x)$. Since $t=0$ is a regular singular point for the equation

$$(iA_k d/dt+V_k/\operatorname{sh} t-z)=0, \tag{25}$$

there exist linearly independent solutions ϕ_j $(1\leqq j\leqq 2k+1)$ of (25) of the forms

$$\phi_j(t)=t^\alpha j(s_j+th_j(t, \log t))$$

in a neighborhood of $t=0$, where $s_j\in C^{2k+1}$ and h_j $(t, \log t)$ denotes a polynomial in $\log t$ with holomorphic coefficients ([3, p. 136] or [10, Lemma A.2]). Since $(iA_k{}^{-1}V_k-\alpha_j)s_j=0$, the k-th component of s_j does not vanish. It is now immediate that $\psi_j=F_{+,k}\phi_j$ takes the form $t^{\alpha_j-1}(\tilde{s}_j+t\tilde{h}_j(t, \log t))$ in a neighborhood of $t=0$ and that ψ_j satisfies (25) provided A_k and V_k are replaced by A_{k+1} and V_{k+1} respectively. Since $(iA_{k+1}{}^{-1}V_{k+1}-\alpha_j+1)\tilde{s}_j=0$, $Q_k(x+1)$ divides $Q_{k+1}(x)$. We now turn to the proof of (ii). Let an $f\in L^2(\boldsymbol{R}_+)^{2k+1}$ be orthogonal to the image $(\dot{L}_k-\bar{z})C_0^\infty(\boldsymbol{R}_+)^{2k+1}$ with Im $z\neq 0$. We must show that $f=0$. Since f turns out to be a solution of (25), $(f^*A_kf)'=2\operatorname{Im} z f^*f$. In particular, f^*A_kf is a monotone function on $\boldsymbol{R}_+$ which is integrable. Since f belongs to $L^2(\boldsymbol{R}_+)^{2k+1}$, f is a linear combination of $\phi_j(\alpha_j>0)$, hence, tends to zero as $t\to 0$. Now a monotone and integrable function f^*A_kf on $\boldsymbol{R}_+$ must vanish identically, which implies $f=0$, as desired.

Example 3. Consider a differential operator $M_k=-d^2/dt^2+2k\eta e^{-t}+\eta^2e^{-2t}$ in $L^2(\boldsymbol{R})$ with $2k\in\boldsymbol{Z}$ and $\eta>0$. Theorem 4 can be applied to M_k. As to the eigenvalues, we maintain

Proposition 2. *The above M_k has an eigenvalue if and only if $k<-1/2$. In case $k<-1/2$ the eigenvalues are $-(k+1/2)^2$, $-(k+3/2)^2$, $\cdots$, $-1/4$ (or -1) according as k is an integer or a half-integer.*

Proof. It is convenient to introduce operators

$$F_{\pm,k}=-d/dt+1/2\pm(\eta e^{-t}+k).$$

We can easily verify

$$F_{-,k+1}F_{+,k}=-M_k-(k+1/2)^2, \qquad F_{+,k-1}F_{-,k}=-M_k-(k-1/2)^2.$$

From the two equalities it follows immediately that

$$F_{+,k}M_k = M_{k+1}F_{+,k}, \qquad F_{-,k}M_k = M_{k-1}F_{-,k}.$$

If u is an eigenvector of M_k corresponding to an eigenvalue λ, then $u_\pm = F_{\pm,k}u$ is an eigenvector of $M_{k\pm1}$ corresponding to an eigenvalue λ provided $u_\pm \neq 0$. Fo see this, note that u' and $e^{-t}u$ belong to $L^2(\boldsymbol{R})$ since $|u'_\pm|^2$ and $(2k\eta e^{-t}+\eta^2 e^{-2t})|u_\pm|^2$ are integrable on $\boldsymbol{R}$ [11, p. 344]. Clearly $u_\pm$ is smooth and belongs to $L^2(\boldsymbol{R})$. Thus $u_\pm$ belongs to the domain of $M_{k\pm1}$ and is an eigenvector of $M_{k\pm1}$ corresponding to eigenvalue λ, provided $u_\pm \neq 0$. M_k ($k \geqq 0$) has no eigenvalues. Any non-zero solution of $F_{+,-1/2}u=0$ does not belong to $L^2(\boldsymbol{R})$. Thus $M_{-1/2}$ has no eigenvalues either. Assume $k<-1/2$ and let $l-k$ be a non-negative integer. Let u_l be a nonzero solution of $F_{+,l}u=0$. It can be easily seen that u_l is an eigenvector of M_l corresponding to eigenvalue $-(l+1/2)^2$. It is not difficult to show that u_k, $F_{-,k-1}u_{k-1}$, $\cdots$, $F_{-,k-1}\cdots F_{-,-1}u_{-1}$ (or $F_{-,k-1}\cdots F_{-,-3/2}u_{-3/2}$) exhaust the eigenvectors of M_k.

Remark. The operators M_k above arise as restrictions of the Laplacian of the group $SU(1, 1)$ with respect to a unitary representation induced by one of three one-parameter subgroups of $SU(1, 1)$. In connection with other two subgroups of SU $(1, 1)$ we come across the following operators.

$$\tilde{M}_k = -d^2/dt^2 + (2kn \operatorname{ch} t + k^2 + n^2 - 4^{-1})/\operatorname{sh}^2 t \quad \text{in } L^2(\boldsymbol{R}_+) \quad (k, n \in \boldsymbol{Z}/2),$$
$$\hat{M}_k = -d^2/dt^2 + (2k\eta \operatorname{sh} t - k^2 + \eta^2 + 4^{-1})/\operatorname{ch}^2 t \quad \text{in } L^2(\boldsymbol{R}) \quad (k \in \boldsymbol{Z}/2, \eta \in \boldsymbol{R}).$$

Tatsuuma obtained an exact form of the density of the spectral matrix for $\tilde{M}_k$ and $\hat{M}_k$ when either $k=0$ or $|k|=1/2$ [17]. Qualitative description of the spectral matrices for $\tilde{M}_k$ and $\hat{M}_k$ can be found, for example, in [12, p. 940] and [6, Theorem 4] respectively. The operators

$$\tilde{F}_{\pm,k} = -d/dt + 2^{-1}\coth t \pm (k \operatorname{ch} T + n)/\operatorname{sh} t,$$
$$\hat{F}_{\pm,k} = -d/dt + 2^{-1}\operatorname{th} t \quad \pm (k \operatorname{sh} t + \eta)/\operatorname{ch} t$$

play the same role as $F_{\pm,k}$.

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References

[1] Akhiezer, N. I. and Glazman, I. M., *Theory of linear operors in Hilbert space,* Vol. II, Pitman, 1981 (English translation).

[2] Atkinson, F. V., *Discrete and continuous boundary problems,* Academic Press, 1964.

[3] Coddington, E. A. and Levinson, N., *Theory of ordinary differential equations,* McGraw-Hill, 1955.

[4] Davies, E. B. and Simon, B., Scattering theory for systems with different asymptotics on the left and right, Comm. Math. Phys., **63** (1078), 277–301.

[5] Dunford, N. and Schwartz, J. T., *Linear operators,* Part II, Interscience, 1963.
[6] Itatsu, S. and Kaneta, H., Spectral matrices for first and second order selfadjoint ordinary differential operators with long range potentials, Funkcial. Ekvac., **24** (1981), 23–45.
[7] ——, Spectral matrices for first and second order selfadjoint ordinary differential operators with short range potentials, Funkcial. Ekvac., **24** (1981), 167–186.
[8] Kaneta, H., Decomposition of representation of SL(2, **C**) induced by the continuous series of E(2), Nagoya Math. J., **78** (1980), 95–111.
[9] ——, Irreducibility of some unitary representations of the Poincaré group with respect to the Poincaré subsemigroup I, Nagoya Math. J., **78** (1980), 113–136.
[10] ——, Irreducibility of some unitary representations of the Poincaré group with respect to the Poincaré subsemigroup II, Nagoya Math. J., **87** (1982), 147–174.
[11] Kato, T., *Perturbation theory for linear operators,* Springer, 1980.
[12] Kodaira, K., The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices, Amer, J. Math., **71** (1949), 92–945.
[13] Levitan, B. M. and Sargsjan, I. S., *Introduction to spectral theory: selfadjoint ordinary differential operators,* Translation of Mathematical Monographs, vol. 39, Providence, AMS, 1975.
[14] Martynov, V. A., Condition for discreteness and continuity of the spectrum of a selfadjoint system of first-order differential equations, Dokl. Akad. Nauk SSSR, **165** (1965), 996–999.
[15] Molchanov, A. M., Conditions for discretense of the spectrum of second-order selfadjoint differential operators, Trudy Moskov. Mat. Obšč., **2** (1953), 169–199.
[16] Naimark, M. A., *Linear differential operators,* Part II, Frederic Ungar, 1968 (English translation).
[17] Tatsuuma, N., Decomposition of representations of three dimentional Lorentz group, Proc. Japan Acad., **36** (1962), 12–14.
[18] Titchmarsh, E. C., *Eigenfunction expansions associated with second-order differential equation,* Part II, Oxford, Claredon Press, 1958, 1962.

nuna adreso:
Faculty of Education
Tokushima University
Tokushima 770
Japan

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