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On a System of Nonlinear Diffusion Equations

By

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§ 1. Introduction.

In this note we are concerned with the following system of nonlinear diffusion equations:

$$(1)\qquad u_t-\sum_{i=1}^{N}(|u_{x_i}|^{m-1}u_{x_i})_{x_i}+f(u, v)=0 \qquad \text{in } \Omega\times(0, \infty)$$

$$(2)\qquad v_t-\sum_{i=1}^{N}(|v_{x_i}|^{m-1}v_{x_i})_{x_i}-kf(u, v)=0 \qquad \text{in } \Omega\times(0, \infty)$$

with the initial-boundary conditions $u|_{\partial\Omega}=v|_{\partial\Omega}=0$ and

$$(3)\qquad u(x, 0)=u_0(x), \qquad v(x, 0)=v_0(x)$$

where Ω is a bounded domain in R^N, k is a constant with $k\leqq 1$ and

$$(4)\qquad f(u, v)=u|v|^{\alpha+1}, \qquad \alpha>0.$$

We are interested in the nonnegative solutions.

When $m=1$ our problem is sometimes called as Martin's problem and has been investigated by several authors (Conway & Smoller [4], Alikakos [1], Masuda [8] etc.). Our problem is also motivated by the single equation of the form:

$$u_t-\sum_{i=1}^{N}(|u_{x_i}|^{m-1}u_{x_i})_{x_i}+f(u)=0.$$

This is a prototype of nonlinear parabolic equation and has been treated from various points of view both in pure and applied Mathematics. Thus it is reasonable to expect that our system (1)–(2) is meaningful in applications.

Here we assume $m>1$ and, in addition to the existence-uniqueness theorem, we derive some regularizing and decay estimates. Although each step of our arguments is rather standard, the result obtained is new. Indeed, to my knowledge, no result has been published for the systems like (1)–(2).

For simplicity we restrict ourselves to the typical case $f(u, v)=uv^{\alpha+1}$. But, it is clear that our result can be generalized to a certain class of functions $f(u, v)$.

§ 2. Preliminaries.

Let Ω be a domain in R^N and A be the closure in $L^p \equiv L^p(\Omega)$ $(1 \leqq p \leqq \infty)$ of the operator

$$A_0 u = \sum_{i=1}^{N} (|u_{x_i}|^{m-1} u_{x_i})_{x_i} \text{ with } D(A_0) = C^2(\Omega) \cap C_0^1(\Omega)$$

$(m \geqq 1)$. Then it is known that A is a m-accretive operator in L^p for $1 \leqq {}^\forall p \leqq \infty$, and the problem

$$\frac{d}{dt} u + Au = f(t), \qquad u(0) = u_0 \in L^p \tag{5}$$

has a unique solution $u(t)$ (in the sense of nonlinear semi-group theory) for each $f \in L^1_{\mathrm{loc}}([0, \infty), L^p)$ (cf. Attouch-Damlamian [3]). Moreover we know (cf. Evans [6])

$$\|u(t)\|_p \leqq \|u_0\|_p + \int_0^t \|f(s)\|_p ds, \tag{6}$$

where $\|\cdot\|_p$ denotes L^p-norm. In this context, our problem is formulated as follows:

$$\text{(P)} \quad \begin{cases} \dfrac{d}{dt} u(t) + Au(t) + u|v|^{\alpha+1} = 0, & u(0) = u_0 \\ \dfrac{d}{dt} v(t) + Av(t) - ku|v|^{\alpha+1} = 0, & v(0) = v_0 \end{cases}$$

with the condition $u|v|^{\alpha+1} \in L^1_{\mathrm{loc}}([0, \infty); L^p)$ for some $p \geqq 1$.

The following lemma will play an essential role in our argument.

Lemma 2.1. *Let $u(t)$ be an appropriately smooth function satisfying:*

$$\|u(t)\|_{p_0} \leqq C_0 \|u_0\|_{p_0}^{q_0} t^{-\alpha_0}, \qquad 1 \leqq {}^\exists p_0 \leqq \infty$$

and

$$\frac{d}{dt} \|u(t)\|_{p+2}^{p+2} + C_1(1+p)^{-\nu} \|\nabla(|u|^{p/(m+1)} u)\|_2^{m+1} \leqq 0$$

for any $p \geqq 0$ and some $\nu > 0$ (C_i denote positive constants). Then,

$$\|u(t)\|_\infty \leqq C(m, N) \|u_0\|_{p_0}^{\gamma_0} t^{-\sigma_0}$$

where $C(m, N)$ denotes constants depending on m and N, and

$$\gamma_0 = q_0(m+1)\{(p_0-1)m+1\}/\{(m+1)p_0 + (m-1)N\}$$

and

$$\sigma_0=\{(m+1)\alpha_0+N\}/\{(m+1)p_0+(m-1)N\}.$$

Proof. The proof is given quite similarly as in Alikakos & Rostamian [2] or Herrero & Vazquez [7] (see also Veron [11]) and omitted. □

We have also:

Lemma 2.2. *Let Ω is bounded, and we assume*

$$f\in L^2_{\mathrm{loc}}([0,\infty);L^2) \quad \textit{and} \quad u_0\in W_0^{1,m+1}(\Omega).$$

Then, the solution $u(t)$ of the problem

$$\frac{d}{dt}u+Au=f(t), \qquad u(0)=u_0$$

satisfies

$$(7) \qquad \|u(t)\|_{1,m+1}^{m+1}\leqq C(m,N)\left\{(\|u_0\|_{1,m+1}^{-(m-1)/(m+1)}+t)^{-(m+1)/(m-1)}+\int_0^t\|f(s)\|_2^2ds\right\}$$

$$(8) \qquad \leqq C(m,N)\left\{t^{-(m+1)/(m-1)}+\int_0^t\|f(s)\|_2^2ds\right\}$$

where we set

$$\|u\|_{1,m+1}=\left(\int_\Omega\sum_{i=1}^N|u_{x_i}|^{m+1}dx\right)^{1/(m+1)}.$$

Proof. The proof is essentially included in [9] and omitted. (See also [10].) □

§ 3. Modified Problem.

To prove the existence of solution we first consider the modified problem:

$$(\mathrm{P}_M) \qquad \begin{cases} \dfrac{d}{dt}u+Au+f^M(u,v)=0, & u(0)=u_0 \\ \dfrac{d}{dt}v+Av-kf^M(u,v)=0, & v(0)=v_0 \end{cases}$$

with $u_0, v_0\in W_0^{1,m+1}\cap L^\infty$, where $f^M(u,v)$ is defined in such a way that

(i) $f^M(u,v)=u|v|^{\alpha+1}$ if $|u|, |v|\leqq M$

(ii) $|f^M(u,v)|\leqq C(M)$ and $|f^M(u,v)-f^M(\bar{u},\bar{v})|\leqq C(M)\{|u-\bar{u}|+|v-\bar{v}|\}$

with some positive constant function $C(M)$, and

(iii) $f^M(u,v)\,u\geqq 0$ and $f^M(0,v)=0$.

Let $T>0$ be fixed arbitrarily and set $X=L^2([0, T]; L^2)$. For any $(u, v) \in X^2= X\times X$ we can define $(U, V) \in C([0, T]; L^2)^2$ through the equations:

$$
(9) \qquad \begin{cases} \dfrac{d}{dt}U+AU+f^M(u, v)=0 \\ \dfrac{d}{dt}V+AV-kf^M(u, v)=0 \end{cases}
$$

with $U(0)=u_0$ and $V(0)=v_0$.

We shall show the mapping J: $(u, v)\to(U, V)$ has a fixed point in X^2.

Continuity of J. Let $(U_i, V_i)=J(u_i, v_i)$ $(i=1, 2)$. Then, by the theory of nonlinear semi-groups and (ii),

$$
\|U_1(t)-U_2(t)\|_2+\|V_1(t)-V_2(t)\|_2 \leqq C(M)(1+k)\int_0^t (\|u_1-u_2\|_2+\|v_1-v_2\|_2)\, ds
$$

which proves the continuity of J.

Boundedness. Similarly as in the above

$$
\|U(t)\|_2 \leqq \|u_0\|_2+C(M)T\equiv a_0(T)
$$

and

$$
\|V(t)\|_2 \leqq \|v_0\|_2+C(M)T\equiv b_0(T).
$$

Thus, setting

$$
B\equiv\{(u, v) \in X^2 \,|\, \|u\|_X \leqq a_0(T)\sqrt{T} \text{ and } \|v\|_X \leqq b_0(T)\sqrt{T}\},
$$

B is a closed, bounded, convex set in X, and it follows easily that $J(B)\subset B$.

Compactness. Multiplying the first equation of (9) by $U_t(t)$ and integrating, we have

$$
(10) \qquad \int_0^T \|U_t(s)\|_2^2\, ds+\frac{2}{m+1}\|U(t)\|_{1,m+1}^{m+1} \leqq C(M, T)+\frac{2}{m+1}\|u_0\|_{1,m+1}.
$$

Similar inequality is valid for $V(t)$. Since $W_0^{1,m+1}$ is compactly imbedded in L^2 we see (Aubin's compactness theorem) that $J(B)$ is compact in X^2.

Thus, by shauder's fixed point theorem, J has a fixed point in B, which is a solution of the modified problem (P_M). Uniqueness follows from the Lipshitz continuity of $f^M(u, v)$. Moreover, if u_0, v_0 are nonnegative the solutions u and v are also nonnegative. To see this we consider the problem (P_M) with $f^M(u, v)$ replaced by $f_+^M(u, v)$, where

$$
f_+^M(u, v)=\begin{cases} f^M(u, v) & \text{if } u, v\geqq 0 \\ 0 & \text{otherwise.} \end{cases}
$$

Let (u, v) be the solution. Setting $u^+=\sup(u, 0)$ and multiplying the first equation by u^+-u, we have

$$-\int_0^t \frac{d}{ds}\|u^+-u\|_2^2 ds-\int_0^t\int_\Omega \sum_{i=1}^N (|u_{x_i}^+|^{m-1}u_{x_i}^+-|u_{x_i}|^{m-1}u_{x_i})(u^+-u)_{x_i}dxds$$
$$+\int_0^t\int_\Omega \left\{\sum_{i=1}^N |u_{x_i}^+|^{m-1}u_{x_i}^+(u^+-u)_{x_i}+f_+^M(u, v)(u^+-u)\right\}dxds=0.$$

Since the last two terms equal to zero and the second one is nonpositive we have

$$\|u^+(t)-u(t)\|_2^2\leqq\|u_0^+-u_0\|_2^2=0$$

or $u(t)\geqq 0$. Similarly $v(t)\geqq 0$.

We summarize above result in the following:

Proposition 3.1. *Let $u_0, v_0 \in W_0^{1,m+1}$. Then the problem (P_M) has a unique solution (u, v) such that*

$$u, v \in C([0, \infty); L^2)\cap L_{loc}^\infty([0, \infty); W_0^{1,m+1})$$

and

$$u_t, v_t \in L_{loc}^2([0, \infty); L^2).$$

Moreover if $u_0, v_0\geqq 0$, $u(t)$ and $v(t)$ are nonnegative.

§ 4. Existence and decay.

The solution $(u(t), v(t))$ of the problem may depend on M. Here we shall derive L^∞ estimates for $u(t)$ and $v(t)$ under the assumption $u_0, v_0 \in W_0^{1,m+1}\cap L^\infty$ and $u_0, v_0\geqq 0$. From these estimates we shall see that they are infact independent of M if M is sufficiently large and consequently (u, v) is the nonnegative solution of the original problem.

Multiplying the first equation of (P_M) by u^{p+1} $(p\geqq 0)$ we have

$$\frac{d}{dt}\|u(t)\|_{p+2}^{p+2}+\frac{(p+2)^2(m+1)^{m+1}}{(p+m+1)^{m+1}}\int_\Omega |\nabla(|u|^{p(m+1)}u)|^{m+1}dx\leqq 0. \tag{11}$$

From (11) it follows immediately that

$$\|u(t)\|_1\leqq C\|u(t)\|_2\leqq Ct^{-1/(m-1)} \text{ and } \|u(t)\|_\infty\leqq\|u_0\|_\infty \tag{12}$$

where C denotes constants independent of u_0 and v_0. Thus, applying Lemma 2.1 with $p_0=1$, $\nu=m-1$, $q_0=0$ and $\alpha_0=1/(m-1)$, we have

$$\|u(t)\|_\infty\leqq C(m, N)t^{-1/(m-1)} \tag{13}$$

and also by (12)

$$\|u(t)\|_\infty \leqq C(m, N, \|u_0\|_\infty)(1+t)^{-1/(m-1)}. \tag{14}$$

Next, adding two equations of (P_M)

$$\frac{d}{dt}(u+v)+Au+Av\leqq 0. \tag{15}$$

Multiplying (15) by $(u+v)^{p+1}$ and using the inequality

$$(|x|^{m-1}x+|y|^{m-1}y)(x+y)\geqq C(m)(x+y)^{m+1}$$

we have, for $w=u+v$,

$$\frac{d}{dt}\|w\|_{p+2}^{p+2}+\frac{C(p+2)^2(m+1)^{m+1}}{(p+m+1)^{m+1}}\|\nabla(|w|^{p/(m+1)}w)\|_{m+1}^{m+1}\leqq 0 \tag{16}$$

and hence, as in the above,

$$\|w(t)\|_\infty \leqq Ct^{-1/(m-1)} \quad \text{and} \quad \|w(t)\|_\infty \leqq C(\|w_0\|_\infty)(1+t)^{-1/(m-1)}.$$

Now, we have proved the following theorem.

Theorem 4.1. *Let $u_0, v_0 \in L^\infty \cap W_0^{1,m+1}$ and $u_0, v_0 \geqq 0$. Then, the problem (P) has a unique solution $(u(t), v(t))$ such that*

$$u, v\geqq 0, \quad u, v \in C([0, \infty); L^2)\cap L^\infty([0, \infty); W_0^{1,m+1})\cap L^\infty([0, \infty)\times\Omega)$$
$$u_t, v_t \in L^2_{\text{loc}}([0, \infty); L^2)$$

and

$$\|u(t)\|_\infty \leqq C(\|u_0\|_\infty)(1+t)^{-1/(m-1)} \quad \textit{and} \quad \|v(t)\|_\infty \leqq C(\|u_0\|_\infty, \|v_0\|_\infty)(1+t)^{-1/(m-1)}.$$

Concerning $W_0^{1,m+1}$ estimate we have:

Theorem 4.2. *Let $(u(t), v(t))$ be the solution in Theorem 4.1. Then,*

$$\begin{aligned}\int_t^{2t}\|u_t(s)\|_2^2 ds+\|u(t)\|_{1,m+1}^{m+1}&\leqq C(\|u_0\|_{1,m+1})(1+t)^{-(m+1)/(m-1)}\\ &\quad +C(\|u_0\|_\infty, \|v_0\|_\infty)(1+t)^{-(2\alpha-m+5)/(m-1)}.\end{aligned} \tag{17}$$

The same estimate holds also for $v(t)$ provided that $C(\|u_0\|_{1,m+1})$ is replaced by $C(\|v_0\|_{1,m+1})$.

Proof. By the estimates in Theorem 4.1,

$$\|uv^{\alpha+1}\|_2 \leqq C(\|u_0\|_\infty, \|v_0\|_\infty)(1+t)^{-(\alpha+2)/(m-1)}.$$

Therefore, applying (8) of Lemma 2.2,

$$\|u(t)\|_{1,m+1}^{m+1} \leqq C\left\{t^{-(m+1)/(m-1)} + C(\|u_0\|_\infty, \|v_0\|_\infty)\int_{t/2}^{t}(1+s)^{-2(\alpha+2)/(m-1)}ds\right\}$$
$$\leqq Ct^{-(m+1)/(m-1)} + C(\|u_0\|_\infty, \|v_0\|_\infty)(1+t)^{-(2\alpha-m+5)/(m-1)}.$$

Next, multiplying the equation by u_t and integrating on $\Omega\times[t, 2t]$, we obtain the estimate for $\int_t^{2t}\|u_t(s)\|_2^2 ds$. □

§ 5. Regularizing effect.

In this section we consider the case $u_0, v_0 \in L^{p_0}$, $p_0 \geqq 1$. Let us begin with:

Lemma 5.1. *Let $u_0, v_0 \in L^{p_0}$ $(p_0 \geqq 1)$, u_0, v_0 be nonnegative and $(u(t), v(t))$ be a solution of the problem. Then,*

$$\|u(t)\|_{p_0} \leqq \|u_0\|_{p_0} \quad \textit{and} \quad \|v(t)\|_{p_0} \leqq C(\|u_0\|_{p_0} + \|v_0\|_{p_0})$$

for a certain $C > 0$.

Proof. If $P_0 \geqq 2$ the estimates follow from (11) and (16). To prove the case $1 \leqq p_0 < 2$ we take a sequence of functions $\theta_j(s)$, $j=1, 2, 3, \cdots$, such that

$$\lim_{j\to\infty}\theta_j(s) = |s|^{p_0-1}\operatorname{sign}_0 s \qquad \text{for } s \in R$$

and

$$|s|^{p_0-1} \geqq |\theta_j(s)| \qquad \text{(Cf. Crandall [5])}$$

where $\operatorname{sign}_0 x = 1$ if $x>0$, 0 if $x=0$ and -1 if $x<0$. We may assume θ_j are piecewise smooth, Lipshitz continuous, monotone increasing and odd. Multiplying (1) by $\theta_j(u)$, we have

$$\frac{d}{dt}\int_\Omega\int_0^{u(t)}\theta_j(s)dsdx + \int_\Omega \theta_j'(u)\sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}u\right|^{m+1}dx \leqq 0$$

and hence

$$\int_\Omega\int_0^{u(t)}\theta_j(s)dsdx \leqq \int_\Omega\int_0^{u_0(t)}\theta_j(s)dsdx.$$

Passing j to ∞ we get $\|u(t)\|_{p_0} \leqq \|u_0\|_{p_0}$. Similarly we can show $\|u(t)+v(t)\|_{p_0} \leqq \|u_0+v_0\|_{p_0}$. □

Using above we obtain:

Lemma 5.2. *Under the assumptions of Lemma* 5.1,

$$\|u(t)\|_\infty \leqq C\|u_0\|_{p_0}^{\gamma_1} t^{-\sigma_1} \tag{18}$$

and

$$\|v(t)\|_\infty \leqq C(\|u_0\|_{p_0}+\|v_0\|_{p_0})^{\gamma_1} t^{-\sigma_1} \tag{19}$$

where C is a constant independent of (u_0, v_0) *and we set*

$$\sigma_1 = \frac{N}{(m+1)p_0+(m-1)N} \quad \text{and} \quad \gamma_1 = \frac{(m+1)(p_0-1)m+1}{(m+1)p_0+(m-1)N}.$$

Proof. The proof follows immediately from Lemma 2.1, 5.1 and inequalities (11), (16). □

Now our main result in this section reads as follows.

Theorem 5.1. *Let* $u_0, v_0 \in L^{p_0}$ *with* $p_0>(\alpha-m+2)N/(m+1)$ *and* $p_0 \geqq 1$, *and* $u_0, v_0 \geqq 0$. *Then the problem* (P) *has a unique nonnegative solution* $(u(t), v(t))$, *satisfying the estimates* (18) *and* (19). *Moreover the estimate* (13) *holds for u and v.*

Proof. Let us take sequences $\{u_{0,n}\}, \{v_{0,n}\} \subset C_0^\infty(\Omega)$ such that $u_{0,n} \to u_0$ and $v_{0,n} \to v_0$ in L, and let (u_n, v_n) be nonnegative solutions with the initial data $(u_{0,n}, v_{0,n})$ (We may assume $u_{0,n} \geqq 0$ and $v_{0,n} \geqq 0$). Then, by the semi-group theory

$$\begin{aligned}\|u_n(t)-u_m(t)\|_{p_0} &\leqq \|u_n(0)-u_m(0)\|_{p_0} + \int_0^t \|u_n v_n^{\alpha+1} - u_m v_m^{\alpha+1}\|_{p_0} ds \\ &\leqq \|u_{0,n}-u_{0,m}\|_{p_0} + C\int_0^t (\|u_n(s)\|_\infty + \|v_n(s)\|_\infty)^{\alpha+1} \\ &\quad \times (\|u_n(s)-u_m(s)\|_{p_0} + \|v_n(s)-v_m(s)\|_{p_0}) ds.\end{aligned}$$

Similar inequality is valid for $\|v_n(t)-v_m(t)\|_{p_0}$. Thus, by Lemma 5.2,

$$x_{n,m}(t) \leqq x_{n,m}(0) + C\int_0^t s^{-(\alpha+1)\sigma_1} x_{n,m}(s) ds \tag{20}$$

where $x_{n,m}(t) = \|u_n(t)-u_m(t)\|_{p_0} + \|v_n(t)-v_m(t)\|_{p_0}$. From (20) we have

$$x_{n,m}(t) \leqq x_{n,m}(0) \exp\left\{C\int_0^t s^{-(\alpha+1)\sigma_1} ds\right\} \tag{21}$$

under the assumption $(\alpha+1)\sigma_1 < 1$ which is fulfilled if $p_0 > (\alpha-m+2)\cdot N/(m+1)$. Thus we can conclude that

$$u_n(t) \longrightarrow {}^\exists u(t) \quad \text{and} \quad v_n(t) \longrightarrow {}^\exists v(t) \text{ in } L_{\text{loc}}^\infty([0, \infty); L^{p_0})$$

as $n \to \infty$. Moreover we see

$$f(u_n(t), v_n(t)) \longrightarrow f(u(t), v(t)) \qquad \text{in } L^1_{loc}([0, \infty); L^{p_0}).$$

Therefore $(u(t), v(t))$ is a desired solution. Uniqueness follows from an inequality like (21). □

Concerning $W_0^{1,m+1}$ estimate (near $t=0$) we obtain:

Theorem 5.2. *Under the assumption of Theorem* 5.1 *the solutions satisfy*

$$u, v \in L^\infty_{loc}((0, \infty); W_0^{1,m+1}) \quad \text{and} \quad u_t, v_t \in L^2_{loc}((0, \infty); L^2)$$

and, for $w=u, v$,

$$\int_t^{2t} \|w_t(s)\|_2^2 ds + \|w(t)\|_{1,m+1}^{m+1} \leqq C\{t^{-(m+1)/(m-1)} + (\|u_0\|_{p_0} + \|v_0\|_{p_0})^{p_0+\gamma_1\theta} t^{-\sigma_1\theta+1}\}$$

where γ_1, σ_1 *are the ones appeared in Lemma* 5.2, $\theta = \max(2\alpha+4-p_0, 0)$ *and* C *is a constant independent of* (u_0, v_0).

Proof. Since, by (18) and (19),

$$\|uv^{\alpha+1}\|_2 \leqq \|u\|_{2\alpha+4} \|v\|_{2\alpha+4}^{\alpha+1} \leqq C(\|u\|_{p_0} + \|v\|_{p_0})^{(p_0+\gamma_1\theta)/2} t^{-\sigma_1\theta/2}$$

we have from Lemma 2.2

$$\|u(t)\|_{1,m+1}^{m+1} \leqq C\left\{t^{-(m+1)/(m-1)} + (\|u_0\|_{p_0} + \|v\|_{p_0})^{p_0+\gamma_1\theta} \int_{t/2}^{t} s^{-\sigma_1\theta} ds\right\}$$

which implies the estimate for $u(t)$ in $W_0^{1,m+1}$. The estimate for $\int_t^{2t} \|u_t(s)\|_2^2 ds$ follows immediately from above, and the same inequality holds for $v(t)$. □

Finally we state a corollary which shows decay properties of the solutions with initial data belonging to L^p.

Corollary 5.1. *Under the assumption of Theorem* 5.1 *we have that for* $\forall T > 0$, *there exists a constant* $\bar{C} \equiv C(\|u_0\|_{p_0}, \|v_0\|_{p_0}, T)$ *such that*

$$\int_t^{2t} \|w_t(s)\|_2^2 ds + \|w(t)\|_{1,m+1}^{m+1} \leqq \bar{C}\left\{(1+t)^{-(m+1)/(m-1)} + (1+t)^{-(2\alpha-m+5)/(m-1)}\right\}$$

and

$$\|w(t)\|_\infty \leqq \bar{C}(1+t)^{-1/(m-1)}$$

for $w=u, v$ *and* $t \geqq T$.

Proof. The proof is clear from Theorems 4.1, 4.2, 5.1 and 5.2. □

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